SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE

SLMath Summer Graduate School

Stochastic quantisation

Week 1

[website of the school]

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[These notes are based on a graduate course given by M. Gubinelli at the University of Milan in 2021]

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1 Introduction

The aim of these lectures is to illustrate the idea of *stochastic quantization* using as motivating example the construction of the Φ^4_3 Euclidean quantum field theory (EQFT).

(Bosonic) EQFTs are certain (complicated and not very explicit) probability measures μ on spaces of Schwarz distributions $\mathscr{S}'(\mathbb{R}^d)$ over the Euclidean space $\mathbb{R}^d$ which have applications in mathematical physics, in particular in the construction of models of relativistic quantum mechanics, the so called quantum field theories (QFTs).

The stochastic quantization idea addresses the problem of EQFT via the construction of an auxiliary map F_μ which push forward a Gaussian measure γ (possibly defined on a very different space) into the measure μ describing the EQFT: $\mu = F_\mu \gamma$. The measure γ should be thought as a convenient source of randomness and will be chosen such as to facilitate the analysis: Gaussian measures are very well understood and several tools are available for them. The map F_μ on the other hand is expected to contain all the difficulties. However it turns out that tools and ideas from the analysis of PDE or even new analysis tools (like Hairer's regularity structures) can be put forward to obtain interesting results.

The leitmotif of these lectures will be to explore the possibility of obtaining as much information as possible on μ via the stochastic quantization map F_μ. It is therefore useful to fix our *partito preso* in a very general definition which seems to fit all the cases I care about:

Definition 1. *A stochastic quantization of a probability measure μ is a pair (F_μ, W) of a map F_μ and a Gaussian r.v. W (on a given probability space) such that the r.v.*

$$\phi = F_\mu(W)$$

has law μ.

There are various possibilities for the choice of F_μ, some of them quite equivalent in their usefulness. During these lectures we will concentrate on a specific measure μ, the Φ^4_3 Euclidean quantum field theory and on a specific way to construct F_μ first suggested in this context by Parisi and Wu: introducing a fictious time and considering a (over-damped) Langevin dynamics driven by a space–time white noise and with invariant measure μ.

Part of my goal is to introduce all these concepts.

For the moment I would like to concentrate on *motivating* why we are interested in such measures μ in the first place, i.e. what an EQFT has to do with quantum mechanics. The link between quantum mechanics and probability theory (and statistical mechanics) has been discovered and exploited in the '60-'70 by people like Segal, Nelson, Symanzik, Simon, Jaffe, Glimm, Osterwalder, Schrader, Rosen, Guerra, Gallavotti, Jona–Lasinio, Fröhlich,… and many others. It is not my aim here to provide a full account of the development of these ideas, the interested reader can refer e.g. to the book of Glimm and Jaffe (James Glimm and Arthur Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed. (New York: Springer-Verlag, 1987), //www.springer.com/gb/book/9780387964775.).

I want to introduce the basic mathematical structure of quantum mechanics and state the basic problems related to it: the construction of suitable representations of the algebra of observables with interesting dynamics. Very intuitively this comes down to solving certains ODE in non-commutative variables, the non-commutativity is imposed by the indetermination principle of Heisenberg (in one form or another). When trying to construct a quantum theory which satisfies also the requirements of special relativity (i.e. no physical influence can travel faster than light, covariance wrt. the Poincaré group of Minkowski space) one need to introduce the concepts of observables localized in certain space–time regions and impose constraints on space-like-separated observables. The so called Haag–Kastler axioms of local QFT formalize the basic structures one would like to obtain from these algebras of observables.

Constructing interesting examples of these algebras, especially for physical geometries, has reveled itself a very difficult task and naive attempts (i.e. perturbation theory around known solutions) run into serious problems of divergences: some of the correction to the zero-order behavior are ill-defined and formally infinitely-large. The main problems are related to the local non-commutativity of the algebras which "creates" such singular behavior. In order to cope with these difficulties it has been proposed, first by the physicist, then rigorously by Nelson and Osterwalder–Schräder, to consider the Euclidean "shadow" of the Minkowski theory, by performing an analytic continuation in the time variable $t \in \mathbb{R}$ to the imaginary axis $ix_0 \in i\mathbb{R}$. This transformation sends the Minkowski metric $x_1^2 + \cdots + x_n^2 - t^2$ of the n-dimensional Minkowski space $\mathbb{M}^n$ into the Euclidean metric $x_1^2 + \cdots + x_n^2 + x_0^2$ of the space $\mathbb{R}^{n+1}$. Technically this analytic continuation is performed at the level of Wightman functions: a system of correlation functions which allows the reconstruction of an algebra of local observables. The analytically continued Wightman functions become Schwinger functions. It turns out (a bit surprisingly) that Schwinger functions can be constructed via probability theory: they can be taken to be the moments of a measure μ on $\mathscr{S}'(\mathbb{R}^d)$ with $d = n + 1$. In order to allow to go back from Schwinger functions to Wightman functions and to the algebra of local observables one need to verify a series of conditions, i.e. axioms for Schwinger functions. The key conditions are reflection positivity, Euclidean covariance and certain moment estimates. Additionally one would like from the model to satisfy additional conditions but in these lectures we will not study them and concentrate on these three. For the scope of these lecture we can therefore preliminary define

Definition 2. *An EQFT μ is a probability measure on $\mathscr{S}'(\mathbb{R}^d)$ satisfying: reflection positivity, Euclidean invariance and certain moment estimates (to be specified later on).*

We need to define all the concepts entering this definition: this will be the goal of the first few lectures. The less natural is reflection positivity (RP) which is a however the key property which encodes the quantum theory within a probabilistic object. RP seems innocuous, but in connection with Euclidean invariance give hard constraints for μ to be an EQFT.

The main theorem we would like to prove in these lectures is

Theorem 3. *There exists a (two-parameter) family $\{\mu\}$ of EQFT on $\mathscr{S}'(\mathbb{R}^3)$. We call them the Φ_3^4 model.*

The name of the model contains two numbers: 3 refers to the dimension of $\mathbb{R}^3$ while the 4 is a reminder of the form of the approximations we will use to show Theorem 3.

A similar statement (and more general ones) are true in two (Euclidean) dimension, i.e. the Φ_2^4 model. We concentrate in dimension three since the proof is more difficult and showcase certain features of the stochastic quantization approach. Sometimes along the lectures we will refer to the Φ_2^4 for illustrative purpose.

The literature on the Euclidean approach is vast and again I will not make any attempt to present it completely, however it mainly exploits techniques and points of view others than the stochastic quantization approach (be it phase cell expansion, renormalization group, …). A lot is known of EQFT even beyond the Φ_d^4 models and many research produced profound and technically difficult pieces of work. Later on I might spend some time reviewing some the highlights.

The rigorous construction of Φ^4_3 via stochastic quantization has been developed very recently and I will try to mention all the relevant references so that the interested reader can have an overview of the current state of this line of research.

Let me go back for the moment to the Definition 1 of SQ and to its spirit. As I already mentioned, the idea is to use a convenient “universal” source of randomness to construct a random field with given law. This broad definition fits also to Ito's approach to the construction of diffusion processes with prescribed characteristics. Indeed consider the (strong) solution X of a stochastic differential equation (SDE) of the form

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \qquad X_0 = x$$

on $\mathbb{R}^k$ where W is a (finite-dimensional) Brownian motion and b, σ are prescribed vector-fields on $\mathbb{R}^k$. This strong solution is really a map from W to $X = F(W)$ and Ito's motivation to introduce SDEs was mainly to construct the Markov process associated to the solution X, i.e. a law μ on a space of continuous paths which satisfy the Markov property. Here we see a huge similarity with what we try to do: describe a measure μ which should satisfy a series of constraint (here Markovianity) in an useful and direct way. The solution of the SDE is such a map. Only incidentally it happens in this case that the Brownian motion lives in the same kind of space as the solution. Stochastic calculus is a tool to study the class of stochastic processes arising in this way (and many other related processes arising in more difficult ways). [see e.g. Daniel Revuz and Marc Yor, *Continuous Martingales and Brownian Motion*, 3rd ed. (Springer, 2004).]

Stochastic quantization is here the stochastic analysis of measures of the type encountered in EQFTs.

2 Quantum mechanics

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I want to start by giving hints on some aspects of quantum mechanics which are relevant to our discussion. For what concerns us here, quantum mechanics (QM) is the theory which describes certain kind of measurements which cannot be performed with inifite precision and which exhibit a certain complementarity, in the sense that an attempt to a precise measurement of one quantity will result in a larger uncertainty in the measure of another. These strange features concerns primarily with physical phenomena involving interactions at the scales of Planck's quantum of action $\hbar$. We will however happily ignore the precise value of this constants in what follows.

The basic structure is given by a set $\mathcal{O}$ of observable quantities (e.g. experimental setups, gauges, indicators) and a set $\mathcal{S}$ of states of the physical system. To every observable $A \in \mathcal{O}$ and state $\omega \in \mathcal{S}$ we can associate a real number $\omega(A) \in \mathbb{R}$ which is our representation of the measure of A in the state ω. Operationally this is interpreted in a frequentist way: it corresponds in performing N experiments, each time preparing the system in the fixed state ω and each time measuring the same quantity A in order to obtain the sequence $(x_i)_{i=1,\ldots,N}$ of numbers. It is then postulated that the empirical mean converges as $N \to \infty$ and the limit is $\omega(A)$:

$$\omega(A) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i.$$

Given two observables A, B I can measure both and sum the results, obtaining $A + B$ and it is natural to assume that $\omega(A + B) = \omega(A) + \omega(B)$, clearly $\mathcal{O}$ is a (real) vector space and states are linear on $\mathcal{O}$. Also if I can measure A then I can also measure any (bounded) function $f(A)$ of A by taking $f(x_i)$ each time I observe x_i. It is natural to require that $1 \in \mathcal{O}$ and that $\omega(1) = 1$ for all states ω, moreover states are positive, i.e. $\omega(A) \geqslant 0$ if the observable A is positive (e.g. if $A = B^2$ for some B). I can measure $A^2, B^2, (A + B)^2$ and define a (commutative) product of two observables $A * B = (A + B)^2 - A^2 - B^2$. The corresponding algebraic structure is called a Jordan algebra. If one requires also that $A_1^2 + \cdots + A_n^2 = 0$ implies that $A_1 = \cdots = A_n = 0$ then the finite dimensional Jordan algebras can be classified and essentially they are composed of basic blocks given by square self-adjoint matrices with either real, complex, quaternion coefficients, or a finite dimensional Clifford algebra, or the algebra of 3×3 octonionic matrices (the Albert algebra of dimension 27).

Nature seems to have chosen for us ony the obsevable algebras made of complex self-adjoint matrices. In the formalization then one take $\mathcal{O}$ to be the algebra of self-adjoint elements of a C^*-algebra $\mathcal{A}$, i.e. a Banach algebra endowed with an involution $a \in \mathcal{A} \mapsto a^* \in \mathcal{A}$ which satisfy the condition $\|a^* a\| = \|a\|^2$ (C^* condition). This is true for finite-dimensional complex matrices by taking the involution as Hermitian conjugate and the norm as the operator norm. So seems a good starting point. The theory of C^* algebras is particularly rich and in particular any state on $\mathcal{A}$ is automatically continuous and the norm can be expressed by duality $\|A\| = \sup_{\omega \in \mathcal{S}} |\omega(A)|$.

Any commutative C^* algebra is (up to isomorphism) the algebra of bounded continuous functions on a compact space X (Gelfand's theorem). This implies in particular that for any $x \in X$ we have a state δ_x given by the valution at the point $x \in X$ (the Dirac measure) such that $\delta_x(AB) = \delta_x(A)\delta_x(B) = A(x)B(x)$. Any other state is a convex combination of such states and we have essentially a probability theory. In particular there exists states which corresponds to infinitely precise measurements of all the observables in $\mathcal{O} \subseteq \mathcal{A}$.

This consequence is at odd with the experience and therefore we have to admit into our modelization non-commutative C^*-algebras. Even in this general case, as soon as we have a sub-algebra which is commutative, it can be essentially considered a system of random variables where any state gives a specific joint probability distribution of the system of observables, in the sense that for any system $A_1, \ldots, A_n$ of commutative self-adjoint elements of $\mathcal{A}$, there exists a measure μ on $\mathbb{R}^n$ such that for all continuous bounded functions $f_1, \ldots, f_n \colon \mathbb{R} \to \mathbb{R}$ we have

$$\omega(f_1(A_1) \cdots f_n(A_n)) = \int_{\mathbb{R}^n} f_1(x_1) \cdots f_n(x_n) \mathrm{d}\mu(x_1, \ldots, x_n).$$

From very easy finite-dimensional examples one can construct C^* algebras of observable which are maximally non-commutative, i.e. where there exists states in which one of the observable is precisely determined while another is maximally uncertain (i.e. has uniform distribution). The

Gelfand–Naimark theorem states that any C^* algebra can be faithfully represented in Hilbert space $\mathscr{H}$ as a subalgebra of the algebra of all bounded operators $\mathscr{B}(\mathscr{H})$. A more interesting (for us) result is how this representation is obtained, namely via a construction called the Gelfand–Naimark–Segal construction (GNS).

Theorem 4. *(GNS) For any state ω there exists a triplet $(\mathscr{H}_\omega, \pi_\omega, \varphi_\omega)$ where $\mathscr{H}_\omega$ is an Hilbert space, π_ω a representation of $\mathscr{A}$ on $\mathscr{H}_\omega$ via bounded operators and a vector $\varphi_\omega \in \mathscr{H}_\omega$ such that, for all $A \in \mathscr{A}$*

$$\omega(A) = \langle \varphi_\omega, \pi_\omega(A)\varphi_\omega \rangle.$$

The representation is irreducible iff the state is pure, i.e. if cannot be written as convex linear combination of other states.

A representation π is faithful if $\pi(A) = 0 \Rightarrow A = 0$ and this can be achieved by tensoring all the GNS representations for all the states (it is enough to use the pure states).

Physically the GNS construction is more interesting: it means that the way the quantum system is realized on Hilbert space could apriori depend on the state which we are considering and that different states could result in inequivalent representations, i.e. representations which cannot be unitarily identified one with the other. This is not a particular problem for the QM of finite-dimensional systems since a theorem of von Neumann guarantees that all the faithful representations of the Weyl algebra (the C^*-algebra of a quantum system satisfying certain simple commutation relations) are equivalent, so it is not very important which one we use and only matter of technical convenience. However, as we will see, Poincaré covariance forces us to consider QM with infinitely many degrees of freedom and the question of the inequivalence of representations is at the orgin of many problems, in particular of divergences appearing in perturbation theory.

In order to understand better the problem we need to discuss dynamics of a quantum system. So far we essentially only addressed the question of the "kinematics", i.e. of the description of the motions. For example a classical Newtonian system is completely described by a point in a phase space giving position and momentum of all the particles. The dynamics acts on the phase space as an Hamiltonian flow, in particular it sends the initial positions and momenta (Q_0, P_0) to those at time $t \in \mathbb{R}$: $(Q(Q_0, P_0, t), P(Q_0, P_{0,}, t))$. Any function $f : (Q, P) \mapsto \mathbb{R}$ on the phase space evolves as $f(t, Q_0, P_0) = \alpha_t(f)(Q_0, P_0) = f(Q(Q_0, P_0, t), P(Q_0, P_0, t))$. Note that α_t is an automorphism of the algebra of functions on the phase space which encodes completely the dynamics, moreover $\alpha_{t+s} = \alpha_t \circ \alpha_s$, i.e. $(\alpha_t)_{t\in\mathbb{R}}$ is a continuous one parameter group of automorphism.

In the quantum kinematic picture one has to give away the possibility to describe the motion of the points in the state space (no such points exists since the algebra is non-commutative) and consider the evolution of the observables to be a continuous one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ acting on the algebra of observables. Let us write $A(t) = \alpha_t(A)$ for the evolution of the observable A. This is called the Heisenberg picture.

The goal of quantum mechanics is therefore to establish which particular triplet $(\mathscr{A}, \omega, \alpha)$ of algebra, state and dynamics allow to predict the behaviour of a given quantum system in a given state. In particular, given the observables $A_1, \ldots, A_n$ and the sequence of times $t_1, \ldots, t_n$ one would like to compute

$$\omega(\alpha_{t_1}(A_1) \cdots \alpha_{t_n}(A_n)).$$

One of the interests of having a specific representation is that it is then possible to proceed to numerical approximations of such quantities.

The dynamics α is an automorphism, therefore can be made to act on states $(\alpha_t\omega)(A)=\omega(\alpha_t(A))$ since it preserves positivity and normalization, moreover it preserves also purity. It is not however clear if $\alpha_t\omega$ give rise to a GNS representation unitary equivalent to that of ω, i.e. lying on the same folium. Assuming this is the case then one can fix the representation $(\mathscr{H}_\omega,\pi_\omega,\varphi_\omega)$ and associate each state $\alpha_t\omega$ with a unitary transformation $U(t)$ such that $\pi_{\alpha_t\omega}(A)=U(t)^{-1}\pi_\omega(A)U(t)$ and correspondingly $\varphi_{\alpha_t\omega}=U(t)\varphi_\omega$. An important case is when the state is invariant, i.e. $\alpha_t\omega=\omega$ since then $U(t)\varphi_\omega=\varphi_\omega$ for all $t\in\mathbb{R}$ and one can prove that the group $(U(t))_{t\in\mathbb{R}}$ of unitary operators is weakly (and thus strongly) continuous.

The assumption of invariance of the state is important, one can easily construct counterexamples to strong continuity of the group U if it does not hold (see the lecture notes of QMFI, Lecture 17).

Provided we assume the existence of an invariant state ω, then the study of the dynamics α can be reduced to the study of a strongly continuous unitary group U on a fixed Hilbert space $\mathscr{H}=\mathscr{H}_\omega$ with a special vector φ_ω called the vaccuum vector or ground state. To each observable a it corresponds a bounded operator $A=\pi_\omega(a)$ and $\pi_\omega(\alpha_t(A))=U(t)AU(t)^{-1}$, so in particular

$$\omega(\alpha_{t_1}(a_1)\cdots\alpha_{t_n}(a_n))=\langle\varphi_\omega,A_1U(t_2-t_1)A_2\cdots U(t_n-t_{n-1})A_n\varphi_\omega\rangle=:W(\{A_k\},\{t_k\}) \tag{1}$$

where we take $t_n<t_{n-1}<\cdots<t_1$.

We say that φ_ω is cyclic if the span of vectors of the form

$$A_1U(t_2-t_1)A_2\cdots U(t_n-t_{n-1})A_n\varphi_\omega,$$

for arbitrary As and times, is dense in $\mathscr{H}$. Provided the vector φ_ω is cyclic the family of all *correlation functions* of the form (1) for all obsevables and all increasing sets of times form a complete description of the dynamics of a given quantum system. With our assumptions this system is stationary in time, i.e. the correlation functions depends only on the difference of the set of times.

3 Euclidean quantum mechanics

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EQM means to do QM in "imaginary time". What does this means precisely? We need to understand what is the unitary group $U(t)$ at imaginary times.

By Stone's theorem the strongly continuous unitary group $(U(t))_{t\in\mathbb{R}}$ corresponds to a homomorphism $X: C(\mathbb{R};\mathbb{C}) \to \mathscr{B}(\mathscr{H})$ of C^*-algebras such that $X(e^{it\cdot}) = U(t)$ for all $t \in \mathbb{R}$.

We cay that U is of positive energy if $X(f) = 0$ for all f with support in $\{x<0: x\in\mathbb{R}\}$. In this case one can define $K(s) = X(e^{-s\cdot}\mathbb{1}_{\mathbb{R}_+})$ for all $s \geqslant 0$ (with some care) and observe that $K(t+s) = K(t)K(s)$, $\|K(t)\| \leqslant 1$ and that $t \mapsto K(t)$ is strongly continuous. So $(K(t))_{t\geqslant 0}$ is a strongly continuous semigroup of contractions.

The notable fact is that, given $(K(t))_{t\geqslant 0}$ one can reconstructs X and then U so they corresponds each other one-to-one and express essentially the same object, in our case the Hilbert space realisation of the dynamics of the quantum system.

The idea now is to take correlation functions for a dynamics with positive energy and continue it to imaginary time differences to obtain correlation functions of the form

$$S(\{A_k\},\{t_k\}) := \langle \varphi_\omega, A_1 K(t_1) A_2 \cdots K(t_{n-1}) A_n \varphi_\omega \rangle \tag{2}$$

for arbitrary operators A_k and positive times t_k. The question is now whether we can recover the full structure of the quantum problem only knowing these functions. In particular we would like to reconstruct the Hilbert space (with all its vectors) and the scalar product, and then also the unitary dynamics U.

We need a way to decode from these functions the key properties which allow the reconstruction. Surely we need the fact that K is a contractive semigroup. A more subtle property comes from the positivity of the Hilbert scalar product, indeed let us observe that we always have, for example that the correlation function

$$S((A_3,A_2,A_1,A_1,A_2,A_3),(t_2,t_1,t_1,t_2)) := \langle \varphi_\omega, A_3 K(t_2) A_2 K(t_1) A_1 A_1 K(t_1) A_2 K(t_2) A_3 \varphi_\omega \rangle$$

$$= \langle A_1 K(t_1) A_2 K(t_2) A_3 \varphi_\omega, A_1 K(t_1) A_2 K(t_2) A_3 \varphi_\omega \rangle \geqslant 0$$

is positive. Of course similar considerations hold for similar functions and for certain linear combinations of such functions (guess their form). It could be formulated explicitly but the formula will not give any further insight, so we avoid it here. This property is called refection positivity of the family of correlation functions 2. Let us give also a name to such a family $\{S(\mathbb{A},\mathbb{T})\}_{\mathbb{A},\mathbb{T}}$ and call it Schwinger functions. Here we denote $\mathbb{A}$ an arbitrary vector of observables and $\mathbb{T}$ one of times. Of course here it is trivial that such a property holds, but the idea is that we will call Schwinger functions all systems of functions indexed by $\mathbb{A},\mathbb{T}$ and satisfying reflection positivity and some other conditions encoding the semigroup property of K (compatibility condition) and an analytic condition which encodes the contractivity of the semigroup K and which essentially says that Schwinger functions have estimates

$$\left|\int S((A_k),(t_1,\ldots,t_n)) g_1(t_1)\cdots g_n(t_n) \mathrm{d}t_1\cdots \mathrm{d}t_n\right| \leqslant \left(\prod_k \|A_k\|\right) \prod_k \left(\sup_{x\in\mathbb{R}_+} \left|\int e^{-tx} g_k(t) \mathrm{d}t\right|\right). \tag{3}$$

This condition implies that $(t_1,\ldots,t_n) \mapsto S((A_k),(t_1,\ldots,t_n))$ is the (multidimensional) Laplace transform of a tempered distribution (see e.g. B. Simon "The $P(\varphi)_2$ Euclidean Quantum Field Theory", Chap. 2, Sect. 2.2). (This is an unnecessarily strong bound and a weaker condition would be sufficient).

The baby version of the Osterwalder–Schrader reconstruction theorem says that from the family of Schwinger functions $\{S(\mathbb{A},\mathbb{T})\}_{\mathbb{A},\mathbb{T}}$ one can recover the QM data, in particular the Wightman functions $\{W(\mathbb{A},\mathbb{T})\}_{\mathbb{A},\mathbb{T}}$. The idea of the proof is to use a GNS construction on the Schwinger functions to obtain the Hilbert space $\mathscr{H}$, the vacuum, the representation π of $\mathscr{A}$ and a semigroup K. In order to prove the continuity and contractivity of the semigroup the analytic condition (3) is needed. From there one can then construct the positive energy group U and therefore the original QM dynamical problem.

So the problem of the construction of interesting QM dynamics is reduced to that of finding Schwinger functions, i.e. correlation functions which satisfy the three above mentioned conditions.

It is a remarkable fact that the correlation functions of certain probabilistic models are Schwinger functions. To show this in a simple setting we assume that the observables indexing the Schwinger functions belong to a commutative subalgebra $\mathscr{O}'$ isomorphic to $C(\mathbb{R})$ (i.e. we have only one continuous degree of freedom) so that we can identify an element A with the function $a\in C(\mathbb{R})$. Then we can associate to a real valued stochastic process $(X_t)_{t\in\mathbb{R}}$ the family of Schwinger functions

$$S(a_1,\ldots,a_n,t_1,\ldots,t_{n-1})=\mathbb{E}[a_1(X_{s_1})\cdots a_{n-1}(X_{s_{n-1}})a_n(X_T)]$$

with $s_k=t_k+\cdots+t_{n-1}+T$ for an arbitrary $T\in\mathbb{R}$. In order for this definition not to depend on T we require that the process X is stationary in time, i.e. the processes $(X_t)_{t\in\mathbb{R}}$ and $(X_{t+s})_{t\in\mathbb{R}}$ have the same law for all $s\in\mathbb{R}$. So the question now becomes: under which conditions on the law of X these Schwinger functions have the form (2) for some quantum data and contractive semigroup K?

It is not difficult to show that the family of such functions satisfy the compatibility conditions required by the structure of the (so far only putative) r.h.s. of (2).

Reflection positivity is trickier: the explicit form of the functions S give us a simpler way to write the RP condition. On functions F on on $C(\mathbb{R};\mathbb{R})$ we can introduce an operation Θ of time inversion such that $(\Theta F)(x)=F(\theta x)$ with $(\theta x)(t)=x(-t)$. Then for any complex function F on $C(\mathbb{R}_{\geqslant 0};\mathbb{R})$ of the form

$$F(x)=\sum_k c_k e^{i\lambda_k x(t_k)} \tag{4}$$

with coefficients $c_k\in\mathbb{C}$, $\lambda_k\in\mathbb{R}$, $t_k\in\mathbb{R}_{\geqslant 0}$ (note that the times are positive!), the RP of the Schwinger functions become the relation

$$\mathbb{E}[\overline{\Theta F(X)}\,F(X)]\geqslant 0. \tag{5}$$

Definition 5. *A measure on $C(\mathbb{R};\mathbb{R})$ is reflection positive iff eq. (5) holds for any cylinder function.*

This is already a nontrivial condition. For example, if X is a Gaussian process we can use cylinder functions to approximate any bounded continuous function and then also polynomials and therefore obtain that a reflection positive Gaussian measure should be centred and with covariance $C(t)=\mathbb{E}[X_t X_0]$ satisfying

$$0\leqslant\mathbb{E}\left[\left(\sum_k \bar{c}_k X_{-t_k}\right)\left(\sum_k c_k X_{t_k}\right)\right]=\sum_{k,k'} c_k\bar{c}_{k'}C(t_k+t_{k'})$$

for all choices of coefficients here. Functions satisfying this property are called totally monotone. By a theorem of Bernstein a bounded and totally monotone function $C: \mathbb{R}_{\geqslant 0} \to \mathbb{C}$ has the representation

$$C(t) = \int_{\mathbb{R}_{\geqslant 0}} e^{-tx} \nu(\mathrm{d}x)$$

for some bounded positive measure ν on $\mathbb{R}_+$. In this case, indeed,

$$\sum_{k,k'} c_k \bar{c}_{k'} C(t_k + t_{k'}) = \int_{\mathbb{R}_{\geqslant 0}} \left| \sum_k c_k e^{-t_k x} \right|^2 \nu(\mathrm{d}x) \geqslant 0.$$

It turns out that this property is sufficient to have reflection positivity of the Gaussian process X. The simplest case is when the measure ν is concentrated in a point $\alpha > 0$, then we have the covariance

$$C(t) = \frac{e^{-\alpha|t|}}{2\alpha}, \qquad t \in \mathbb{R},$$

with an arbitrary normalization (see below). The corresponding Gaussian process is called the Ornstein–Uhlenbeck process (OU) and we conclude that any scalar reflection positive Gaussian process in one dimension can be constructed by taking sums of independent OU processes (see the QMFI lecture notes).

3.1 Symmetric Markov processes

Another important strategy to obtain RP processes is to use Markovianity. Let X be a Markov process, i.e. such that

$$\mathbb{E}[f(X_t)|\mathscr{F}_s] = \mathbb{E}[f(X_t)|X_s]$$

where $(\mathscr{F}_s = \sigma(X_r : r \leqslant s))_{s \in \mathbb{R}}$ is the filtration generated by X. Recalling the support of the function F in the RP condition, we have

$$\mathbb{E}[\overline{\Theta F(X)} \quad F(X)] = \mathbb{E}[\overline{\Theta F(X)} \quad \mathbb{E}[F(X)|\mathscr{F}_0]] = \mathbb{E}[\overline{\Theta F(X)} \quad \mathbb{E}[F(X)|X_0]] = \mathbb{E}[\overline{\mathbb{E}[\Theta F(X)|X_0]}\, \mathbb{E}[F(X)|X_0]]$$

an easy way to force this quantity to be positive is then to take $\mathbb{E}[\Theta F|X_0] = \mathbb{E}[F|X_0]$ which is easily seen to imply that the law of X is invariant under the time reversal Θ. Stationary and time-reversal invariant Markov processes are then reflection positive. The converse is also true: RP Markov processes are time-reversal invariant and stationary.

Having a symmetric (i.e. time-reversal invariant) Markov process allows to represent the reconstruction procedure quite explicitly. The idea is to consider the complex vector space $\mathscr{E}_+$ of cylindrical function supported on positive times as introduced above and make a pre-Hilbert space out of it by considering the Hermitian scalar product

$$\langle F, G \rangle = \mathbb{E}[\overline{\Theta F} G]$$

which is positive definite thanks to the RP of the law of the process X. We take the quotient $\mathscr{E}_+ \setminus \mathscr{N}$ where $\mathscr{N}$ is the subspace of elements in $\mathscr{E}_+$ with zero norm and complete wrt. the scalar product to obtain an Hilbert space $\mathscr{H}$. On this Hilbert space the elements $a \in \mathscr{A}$ act as multiplication operators via $(af)(X) = a(X_0) f(X)$ which is a C^*-homomorphism. Moreover we have the natural action of time-translation via $t \geqslant 0$: T_t which acts on $\mathscr{E}_+$ (since for example $T_t X_s = X_{s+t}$ with $s+t \geqslant 0$ if $s, t \geqslant 0$). Moreover we have that elements in the form $F - \mathbb{E}[F|X_0]$ have zero norm, since by Markovianity

$$\langle F, \mathbb{E}[F|X_0] \rangle = \mathbb{E}[\overline{\Theta F}\, \mathbb{E}[F|X_0]] = \mathbb{E}[\overline{\Theta F}\, \mathbb{E}[F|\mathscr{F}_0]] = \langle F, F \rangle$$

so $\| F - \mathbb{E}[F|X_0] \| = 0$. By stationarity the operators T_t form a symmetric semigroup:

$$\langle T_t F, G \rangle = \mathbb{E}[\overline{\Theta T_t F}\, G] = \mathbb{E}[T_{-t} \overline{\Theta F}\, G] = \mathbb{E}[\overline{\Theta F}\, T_t G] = \langle F, T_t G \rangle$$

which preserves the null space $\mathscr{N}$ since if $F \approx 0$ then by this formula we see that $\langle T_t F, G \rangle = 0$ for all G, i.e. $T_t F \approx 0$. It is also a contractive operator for all $t \geqslant 0$:

$$\langle T_t F, T_t F \rangle = \langle F, T_{2t} F \rangle \leqslant \|F\|^{1/2} \|T_{2t} F\|^{1/2}$$

and by iterating this inequality we get

$$\|T_t F\|^2 \leqslant \|F\|^{1/2 + 1/2^2} \|T_{2^2 t} F\|^{1/2^2} \leqslant \cdots \leqslant \|F\|^{1/2 + \cdots + 1/2^n} \|T_{2^n t} F\|^{1/2^n} \leqslant \to \|F\|$$

since

$$\|T_{2^n t} F\|^2 = \mathbb{E}[T_{-2^n t} \overline{\Theta F}\, T_{2^n t} F] \leqslant (\mathbb{E}[|T_{-2^n t} \overline{\Theta F}|^2])^{1/2} (\mathbb{E}[|T_{2^n} F|^2])^{1/2} = (\mathbb{E}[|\overline{\Theta F}|^2])^{1/2} (\mathbb{E}[|F|^2])^{1/2}$$

which is independent of n.

Strong continuity of $(T_t)_t$ follows then via approximations from weak continuity and from the fact that the the process X is continuous in distribution (Exercise).

We have already shown that $F - \mathbb{E}[F|X_0] \in \mathscr{N}$ so in the completed, quotiented Hilbert space $\mathscr{H}$ we have that all the vectors can be represented as $\mathbb{E}[F|X_0]$ for some $F \in L^2(\mathbb{P})$ measurable wrt $\sigma(X_s : s \geqslant 0)$. Therefore really all these vectors are in $L^2(\mathbb{R}, \rho)$ where ρ is the law of X_0:

$$\langle F, G \rangle_{\mathscr{H}} = \mathbb{E}[\overline{\Theta F}\, G] = \mathbb{E}[\overline{\Theta F}\, \mathbb{E}[G|\mathscr{F}_0]] = \mathbb{E}[\overline{\mathbb{E}[F|X_0]}\, \mathbb{E}[G|X_0]]$$

$$= \mathbb{E}[\overline{f(X_0)}\, g(X_0)] = \int_{\mathbb{R}} \overline{f(x)}\, g(x) \rho(\mathrm{d}x) = \langle f, g \rangle_{L^2(\mathbb{R}, \rho)}$$

with $f, g : \mathbb{R} \to \mathbb{C}$ such that $f(X_0) = \mathbb{E}[f|X_0]$ and $g(X_0) = \mathbb{E}[g|X_0]$ $\mathbb{P}$-almost surely. This shows that $\mathscr{H}$ can be isometrically embedded in $L^2(\mathbb{R}, \rho)$, the converse is also true. In this representation the semigroup $(T_t)_{t \geqslant 0}$ give rise to a semigroup $(K(t))_{t \geqslant 0}$ on $L^2(\mathbb{R}, \rho)$, indeed for any $f \in L^2(\mathbb{R}, \rho)$ we can consider $T_t(f(X_0)) = f(X_t)$ and then condition again onto X_0 and define

$$(K(t) f) = h, \qquad h(X_0) = \mathbb{E}[f(X_t)|X_0].$$

so that $\langle F, T_t G\rangle_{\mathscr{H}} = \langle f, K(t)g\rangle_{L^2(\mathbb{R},\rho)}$.

From this contractive semigroup one then construct a unitary dynamics U and therefore a model of QM dynamics. These considerations can also be extended without essential changes to the case of infinite-dimensional Markov processes, generalisation needed in QFT. It has been Nelson which first suggested the use of Markov processes to construct models of QFT along the lines we described. Unfortunately the infinity of degrees of freedom makes difficult to prove Markovianity of interesting examples. And it was realised by Osterwalder and Schräder that the weaker property of RP is much easier to prove and enough for the reconstruction of the QM data.

3.2 An important example and some Gaussian analysis

I think now useful to specialize the above considerations to an important example given by the OU process already introduced before which is RP positive.

Notably the OU process is Markovian, indeed is the solution to a linear SDE involving a Brownian motion B:

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \mathrm{d}B_t, \qquad X_0 \sim \mathscr{N}(0, (2\alpha)^{-1})$$

indeed the explicit solution of this equation, via variation of constants is

$$X_t = e^{-\alpha(t+S)} X_{-S} + \int_{-S}^{t} e^{-\alpha(t-s)} \mathrm{d}B_s.$$

where the stochastic integral is a Wiener integral (i.e. the limit in $L^2(\mathbb{P})$ of Riemann sums). Taking $S \to \infty$ the first term disappear and one remains with the Gaussian process

$$X_t = \int_{-\infty}^{t} e^{-\alpha(t-s)} \mathrm{d}B_s$$

which can be checked to have the correct covariance, to be stationary and to have invariant measure $\mathscr{N}(0, (2\alpha)^{-1})$. By Ito's formula, for $t \geqslant t'$

$$\mathbb{E}[X_t X_{t'}] = \mathbb{E}\left[\left(\int_{-\infty}^{t} e^{-\alpha(t-s)} \mathrm{d}B_s\right)\left(\int_{-\infty}^{t'} e^{-\alpha(t-s)} \mathrm{d}B_s\right)\right] = e^{-\alpha(t-t')} \int_{-\infty}^{t'} e^{-2\alpha(t'-s)} \mathrm{d}s = \frac{e^{-\alpha(t-t')}}{2\alpha}.$$

Note that the time-reversal symmetry is not quite explicit in the SDE formulation since Ito calculus breaks the invariance.

From the SDE we have that

$$X_t = e^{-\alpha t} X_0 + \int_0^t e^{-\alpha(t-s)} \mathrm{d}B_s = e^{-\alpha t} X_0 + \mathscr{N}\left(0, \frac{1 - e^{-2\alpha t}}{2\alpha}\right)$$

with the Gaussian r.v. independent of $\mathscr{F}_0$, so in particular it is a Markov process and

$$(K(t)f)(x) = \mathbb{E}[f(X_t)|X_0 = x] = \mathbb{E}\left[f\left(e^{-\alpha t}x + \mathscr{N}\left(0, \frac{1 - e^{-2\alpha t}}{2\alpha}\right)\right)\right] = \mathbb{E}\left[f\left(e^{-\alpha t}x + \left(\frac{1 - e^{-2\alpha t}}{2\alpha}\right)^{1/2} Z\right)\right]$$

where $Z \sim \mathscr{N}(0,1)$. One can check explicitly that this is a semigroup on $L^2(\mathbb{R}, \rho)$ which is self-adjoint, positive (i.e. $f \geqslant 0 \Rightarrow K(t)f \geqslant 0$), contractive, strongly-continuous. It is an important ingredient in the analysis of Gaussian r.v. (both in finite and infinite dimensions). One has also

$$K(t)f \to \int f \mathrm{d}\rho$$

in $L^2(\rho)$ as $t \to \infty$ (the convergence holds also in stronger senses).

If we want to understand better the associated quantum dynamics in $L^2(\rho)$, we need a grasp of the unitary operator U associated to K via the procedure we described above. We could compute the generator L of K and prove that it is a positive self-adjoint operator and then consider the associated unitary group.

Another way to proceed in this very explicit situation is just to diagonalize K. Which is what we are going to do now. Let $e_\lambda(x) = \exp(i\lambda x)$ the trigonometric exponential, then

$$(K(t)e_\lambda)(x) = \mathbb{E}\left[e_\lambda\left(e^{-\alpha t}x + \left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)^{1/2} Z\right)\right] = e_\lambda(e^{-\alpha t}x)\mathbb{E}\left[e_\lambda\left(\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)^{1/2} Z\right)\right]$$

$$= e_\lambda(e^{-\alpha t}x)\exp\left(-\frac{\lambda^2}{2}\left(\frac{1-e^{-2\alpha t}}{2\alpha}\right)\right)$$

so if we let $\hat{e}_\lambda(x) = \exp(i\lambda x)\exp\left(\frac{\lambda^2}{2}\frac{1}{2\alpha}\right)$ we have

$$(K(t)\hat{e}_\lambda)(x) = \hat{e}_{\lambda e^{-\alpha t}}(x)$$

By expanding both the l.h.s. and r.h.s in powers of λ we obtain

$$\sum_{n\geqslant 0} \frac{(i\lambda)^n}{n!} K(t)H_n(x) = (K(t)\hat{e}_\lambda)(x) = \hat{e}_{\lambda e^{-\alpha t}}(x) = \sum_{n\geqslant 0} \frac{(i\lambda)^n}{n!} e^{-\alpha n t}H_n(x)$$

where we denoted $H_n(x)$ the coefficients in this expansion. Each H_n is a polynomial in x with α dependent coefficients and highest order term x^n, they are called Hermite polynomials. We conclude that

$$K(t)H_n(x) = e^{-\alpha n t}H_n(x), \qquad t \geqslant 0,$$

i.e. Hermite polynomials are eigenfunctions of each $K(t)$. Since $K(t)$ is symmetric this proves also that the $(H_n)_n$ are orthogonal (recall that we are here in $L^2(\rho)$) since they corresponds to different eigenvalues of $K(t)$. It is not difficult also to show that they are total in $L^2(\rho)$, i.e. every vector can be written as a convergent sum of Hermite polynomials, ideed assume that this is not the case for the non-null vector f, then $\langle \hat{e}_\lambda, f\rangle = 0$ for all $\lambda \in \mathbb{R}$, but this implies that $\langle e_\lambda, f\rangle = 0$ since the two functions differ by a strictly positive constant factor. The functions e_λ can approximate on any compact set any arbitrary continuous function in the uniform norm, so by an approximation procedure we can show that $\langle g, f\rangle = 0$ for all $g \in L^2(\rho)$ and we conclude that $f = 0$ (we could have used also the Fourier transform to conclude).

At this point it is easy to construct the unitary group $U(t)$ (since we can reason in a one dimensional Hilbert space), we have:

$$U(t)H_n = e^{i\alpha n t}H_n, \qquad t \in \mathbb{R}.$$

We have completely solved the QM problem for this model: we have an Hilbert space $L^2(\rho)$, a representation of $C(\mathbb{R})$ given by multiplication operators, a strongly continuous unitary dynamical group $(U(t))_{t\in\mathbb{R}}$ and a ground state given by the constant function 1.

From U (or K) we can produce the homeomorphism $C(\mathbb{R}_+) \to \mathscr{B}(L^2(\rho))$ explicitly

$$E(f)H_n = f(\alpha n)H_n$$

and one see that it is actually an homeomorphism from $C(\alpha\mathbb{N})$. This new commutative subalgebra of $\mathscr{B}(L^2(\rho))$ does not commute with the algebra $Q \approx C(\mathbb{R})$ (which acts as multipliers), morever it corresponds to an observable which can take only discrete integer values. The system is quantized. The observables $X(f)$ do not change in time, by construction $U(t)^{-1}X(f)U(t) = X(f)$: they are constants of motion (it is indeed the "energy" of the system).

Note that $\partial_x \hat{e}_\lambda(x) = i\lambda \hat{e}_\lambda(x)$ which means that $\partial_x H_n = nH_{n-1}$, on the other hand by an integration by parts:

$$\delta_{m,n}\int H_m H_n \mathrm{d}\rho = \frac{1}{n+1}\int H_m \partial_x H_{n+1}\mathrm{d}\rho = \frac{1}{n+1}\int H_{n+1}\left(-\frac{x}{2\alpha} - \partial_x\right)H_m \mathrm{d}\rho$$

so

$$\left(\frac{x}{2\alpha} - \partial_x\right)H_n = c_{n+1}H_{n+1} \qquad \int H_n H_n \mathrm{d}\rho = \frac{c_{n+1}}{n+1}\int H_{n+1}H_{n+1}\mathrm{d}\rho$$

to determine c_{n+1} we compute

$$c_{n+1}H_n = \frac{c_{n+1}}{n+1}\partial_x H_{n+1} = \frac{1}{n+1}\partial_x\left(\frac{x}{2\alpha} - \partial_x\right)H_n = \frac{1}{n+1}\left[\frac{1}{2\alpha}H_n + \left(\frac{x}{2\alpha} - \partial_x\right)\partial_x H_n\right] = \frac{1}{n+1}\left[\frac{1}{2\alpha}H_n + nc_nH_n\right]$$

which gives $nc_n = n/2\alpha$ since $c_1 = 1/2\alpha$. Let us register them here by giving names to these operators

$$A = \partial_x, \qquad C = \left(\frac{x}{2\alpha} - \partial_x\right)$$

are called annihilation and creation operators. (They are unbounded but with a very simple algebraic structure over the Hermite basis so not so difficult to analyze. We will not use them much here, apart from specific algebraic computations on the Hermite basis). Note that $C^* \supseteq A$. We discovered several relations for them

$$AH_n = nH_{n-1}, \qquad CH_n = \frac{1}{2\alpha}H_{n+1},$$

$$CAH_n = \left(\frac{x}{2\alpha} - \partial_x\right)\partial_x H_n = \frac{n}{2\alpha}H_n.$$

By putting them together we have also

$$\frac{x}{2\alpha}H_n = nH_{n-1} + \frac{1}{2\alpha}H_{n+1}$$

and in particular that

$$\left(\frac{x}{2\alpha} - \partial_x\right)^n 1 = \frac{1}{(2\alpha)^n}H_n.$$

Moreover

$$[A, C] = \left[\partial_x, \left(\frac{x}{2\alpha} - \partial_x\right)\right] = \partial_x\left(\frac{x}{2\alpha} - \partial_x\right) - \left(\frac{x}{2\alpha} - \partial_x\right)\partial_x = \frac{1}{2\alpha}$$

We have also $x = 2\alpha(C + A)$. Let's then define

$$P = (2\alpha i)(C - A) = i(x - 2(2\alpha)\partial_x), \qquad Q = 2\alpha(C + A),$$

and observe that P is symmetric and that

$$Q^2 = (2\alpha)^2 (C+A)^2 = (2\alpha)^2 (C^2 + CA + AC + A^2)$$

$$P^2 = -(2\alpha)^2 (C-A)^2 = -(2\alpha)^2 (C^2 + A^2 - AC - CA)$$

so

$$P^2 + Q^2 = 2(2\alpha)^2 (CA + AC) = 2(2\alpha^2)(CA + AC) = 2(2\alpha)^2 \left(2CA + \frac{1}{2\alpha} \right) = 2\,(2\alpha) + 4(2\alpha)^2 CA$$

From which one gets

$$\left[\frac{1}{8}(P^2 + Q^2) - \frac{1}{2}\alpha \right] H_n = (\alpha n) H_n$$

so we can identify the generator $E \geqslant 0$ of the dynamics of system with

$$E = \frac{1}{8}(P^2 + Q^2) - \frac{1}{2}\alpha \geqslant 0$$

which resembles the classical Hamiltonian for the Harmonic oscillator, note that

$$[P, Q] = [(2\alpha i)(C - A), 2\alpha (C + A)] = -i4\alpha$$

so α describes the degree of non-commutativity of the algebra.

The us wrap up a take-away message from all these computations. A first one is that even if we restricted our considerations to a particular commutative subalgebra Q of $\mathscr{A}$ the fact that we included the dynamics in the description carry on information on the non-commutative nature of our original problem in the time evolution. In particular we can construct other subalgebras (those of the bounded functions of P or E) which do not commute with the Q algebra. Note also that the algebra generated by P, Q contains that of E so also the dynamics and can be considered therefore a complete description of the quantum system.

This dynamics is really boring however. Every pure state ω (in the folium of the stationary state ω_0) of the system corresponds to a vector f of the Hilbert space (any other state can be written as a linear combination of pure states). Every vector f can be decomposed on the basis (H_n) on which the dynamics is linear, so for an observable $a \in C(\mathbb{R})$ we have

$$\omega_0(\alpha_t(a) * a) = 2\mathrm{Re}\langle 1, U(t)^{-1} Q(a)\, U(t) Q(a) 1 \rangle$$

$$= 2\mathrm{Re} \sum_n e^{i\alpha t n} \langle 1, Q(a)\, H_n \rangle \langle H_n, Q(a) 1 \rangle = 2\mathrm{Re} \sum_n e^{i\alpha t n} |\langle H_n, Q(a) 1 \rangle|^2,$$

which could be interpreted by saying that the first measurement of a produces *n particles* from the vaccum state which then evolve for time t via simple oscillations and are reabsorbed by the second measurement with exaclty the same amplitude.

By an approximation procedure, one can polarize the above expression with two observables a, b and by an approximation procedure take $Q(a) \approx H_n$ and $Q(b) \approx H_m$, in which case one gets

$$\omega_0(\alpha_t(a) * b) \approx 2\mathrm{Re} \sum_k e^{i\alpha t k} \langle 1, Q(a)\, H_k \rangle \langle H_k, Q(b) 1 \rangle = \delta_{n=m} 2\mathrm{Re}\,(e^{i\alpha t n}) |\langle H_n, H_n \rangle|^2$$

which can be interpreted by saying that if one creates n particles at time 0 and let them evolve up to time t then they remains n particles and one will never measure m particles if $m \neq n$.

Another important take-away for us is that these computations give us tools for the Gaussian analysis later on.

3.3 Perturbations of RP processes

References

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As we have said, it is useful to have a source of RP processes which do not rely on Markovianity and survive the generalisation to infinite dimensions which we are going to pursue later on.

A convenient way to construct a large class of RP processes is to take Gibbsian perturbations of a RP process. With this we mean consider a potential function $V: \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ and a new probability measure $\mathbb{Q}$ given by

$$\mathbb{Q}_T(\mathrm{d}\omega) = \frac{1}{Z_T} \exp\Big(-\int_{-T}^{T} V(X_s(\omega))\mathrm{d}s\Big)\mathbb{P}(\mathrm{d}\omega)$$

with a normalization factor Z_T.

Lemma 6. *The measure $\mathbb{Q}_T$ is reflection positive.*

Proof. Take

$$G = \exp\Big(-\int_0^T V(X_s)\mathrm{d}s\Big),$$

then

$$\Theta G = \exp\Big(-\int_0^T V(X_{-s})\mathrm{d}s\Big) = \exp\Big(-\int_{-T}^0 V(X_s)\mathrm{d}s\Big)$$

and

$$\exp\Big(-\int_{-T}^{T} V(X_s(\omega))\mathrm{d}s\Big) = (\Theta G)G = (\overline{\Theta G})G$$

then for any $F \in \mathscr{E}_+$ we have

$$\mathbb{E}_{\mathbb{Q}_T}[\overline{\Theta F}F] = \frac{1}{Z_T}\mathbb{E}_{\mathbb{P}}\Big[\overline{\Theta F}F \exp\Big(-\int_{-T}^{T} V(X_s(\omega))\mathrm{d}s\Big)\Big]$$

$$= \frac{1}{Z_T}\mathbb{E}_{\mathbb{P}}[\overline{\Theta F}F\,(\overline{\Theta G})G] = \frac{1}{Z_T}\mathbb{E}_{\mathbb{P}}[\overline{\Theta(FG)}(FG)\,] \geqslant 0$$

since $\mathbb{P}$ is RP. □

Remark. More “fancy” perturbations, like e.g.

$$\exp\Big(-\int_{-T}^{T}\int_{-T}^{T} W(X_s(\omega), X_{s'}(\omega))\mathrm{d}s\mathrm{d}s'\Big),$$

are not in general reflection positive. Note also that to prove RP we used essentially that there is an integral over time and the multiplicativity of the exponential.

The problem is that the measure $\mathbb{Q}_T$ is not stationary anymore and to obtain a stationary measure we would need to take the limit $T \to \infty$, indeed if T_t denote the time translation on functions on the path space $T_t F(x) = F(T_t x)$ with $(T_t x)(s) = x(t+s)$, then by stationarity of $\mathbb{P}$ we have

$$\mathbb{E}_{\mathbb{Q}_T}[T_t F] = \frac{1}{Z_T} \mathbb{E}_{\mathbb{P}}\Big[T_t F \exp\Big(-\int_{-T}^{T} V(X_s(\omega))\mathrm{d}s\Big)\Big] = \frac{1}{Z_T} \mathbb{E}_{\mathbb{P}}\Big[T_t F T_t \exp\Big(-\int_{-T}^{T} V(X_{s-t}(\omega))\mathrm{d}s\Big)\Big]$$

$$= \frac{1}{Z_T} \mathbb{E}_{\mathbb{P}}\Big[F \exp\Big(-\int_{-T}^{T} V(X_{s-t}(\omega))\mathrm{d}s\Big)\Big] = \frac{1}{Z_T} \mathbb{E}_{\mathbb{P}}\Big[F \exp\Big(-\int_{-T-t}^{T-t} V(X_s(\omega))\mathrm{d}s\Big)\Big]$$

which is not what we want and we would need to take the limit $T \to \infty$ in both sides to obtain that $\mathbb{E}_{\mathbb{Q}_\infty}[T_t F] = \mathbb{E}_{\mathbb{Q}_\infty}[F]$. The existence and uniqueness of such limits are nontrivial and depends on the specific form of the potential V (and on the law of X under $\mathbb{P}$ of course).

Let us make connection with the Markovian point of view. Let X be a RP Markov process with semigroup K and introduce a new semigroup Q^V (not necessarily contractive) via the Feynman–Kac–Nelson formula

$$(Q_t^V f)(x) = \mathbb{E}_{\mathbb{P}}\Big[f(X_t) \exp\Big(-\int_0^t V(X_s)\mathrm{d}s\Big)\Big| X_0 = x\Big],$$

where we assume we have a regular version of the conditional probability $\mathbb{P}(*|X_0)$ (this requires some mild assumptions on X of course). That it is a semigroup derive easily from the Markovianity of X (exercise). It is positive and in particular satisfies $|Q_t^V f(x)| \leqslant (Q_t^V |f|)(x)$, moreover it is symmetric

$$\langle g, Q_t^V f\rangle = \mathbb{E}_{\mathbb{P}}\Big[\overline{g(X_0)} \exp\Big(-\int_0^t V(X_s)\mathrm{d}s\Big) f(X_t)\Big] = \mathbb{E}_{\mathbb{P}}\Big[\overline{g(X_{-t})} \exp\Big(-\int_{-t}^0 V(X_s)\mathrm{d}s\Big) f(X_0)\Big] = \langle Q_t^V g, f\rangle.$$

Assume that $\|Q_s^V\| < \infty$ for some $s > 0$. Then

$$|\langle g, Q_{2s}^V f\rangle| = |\langle Q_s^V g, Q_s^V f\rangle| \leqslant \|Q_s^V\|^2 \|f\| \|g\| \Rightarrow \|Q_{2s}^V\| \leqslant \|Q_s^V\|^2$$

and conversely

$$|\langle Q_{s/2}^V g, Q_{s/2}^V f\rangle| = |\langle g, Q_s^V f\rangle| \leqslant \|Q_s^V\| \|f\| \|g\| \Rightarrow \|Q_{s/2}^V\|^2 \leqslant \|Q_s^V\|$$

so iterating these inequalities we have $\|Q_{2^n s}^V\| \leqslant \|Q_s^V\|^{2^n}$, and $\|Q_{s/2^n}^V\| \leqslant \|Q_s^V\|^{2^{-n}}$. For any $t \geqslant 0$ we can write down the dyadic decomposition of $p = t/s$ and get $\|Q_t^V\| \leqslant \|Q_s^V\|^p \leqslant \|Q_s^V\|^{t/s}$. We have proven that there exists a constant $c = \|Q_s^V\|^{1/s}$ such that $\|Q_t^V\| \leqslant e^{ct}$. By repeating the argument with t we have now $\|Q_s^V\|^{1/s} \leqslant \|Q_t^V\|^{1/t} \leqslant \|Q_s^V\|^{1/s}$ so we must have $\|Q_t^V\| = e^{ct}$.

We can then define $K^V(t) f := e^{-ct} Q_t^V$ and obtain a contractive semigroup $K^V(t)$ for which $\|K^V(t)\| = 1$ for all $t \geqslant 0$, in particular there must exists at least one vector ψ such that $K^V(t)\psi = \psi$ and we are back in the situation where we have a ground state ψ and a contractive semigroup. If we assume that this eigenvector is unique (see e.g. the Perron–Frobenius theorem) then one can prove that for any other vector f we have

$$K^V(t) f \to \langle \psi, f\rangle \psi$$

and in particular $K^V(t)1 \to \langle \psi, 1 \rangle \psi$. Note that we can assume ψ positive since $|\psi| = |K^V(t)\psi| \leqslant K^V(t)|\psi|$

$$\langle |\psi|, K^V(t)|\psi| \rangle \geqslant \langle |\psi|, |K^V(t)\psi| \rangle \geqslant |\langle \psi, K^V(t)\psi \rangle| = \langle \psi, \psi \rangle = 1$$

so $|\psi|$ is also an eigenvector with eigenvalue 1. (We have also to prove that does not depends on t, exercise).

One situation when one has a unique ground state is when the operator K^V is positivity improving, i.e. when for some $t > 0$ and for any $f \geqslant 0$ such that $f \neq 0$ we have $K^V(t) f(x) > 0$ for all x.

One can prove that in the situation where there is uniqueness of the groundstate the Gibbsian recipe above give exactly as a limit the measure $\mathbb{Q}$ corresponding to the stationary Markov process with transition operator given by K^V.

3.4 Euclidean Quantum Field Theory

References

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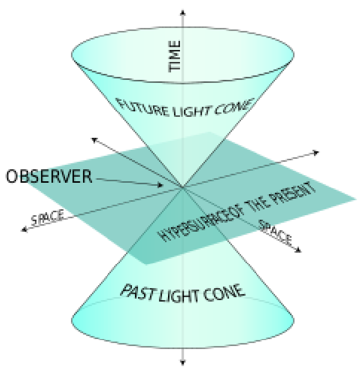
How we extend all these considerations to QFTs instead of just QM models? This will involve to consider QM with infinitely many degrees of freedom. (Recall that so far we gave only explicit examples with one degree of freedom, i.e. we were measuing only one quantity at any given time $\mathrm{Hom}(C(\mathbb{R}, \mathbb{C}), \mathscr{A})$).

In $(n+1)$-dimensional Minkowski space $\mathbb{M}^{n+1}$ the situation looks like this:



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So we need to be more specific on *where* we measure things, since measurements separated by space-like vectors should not interfere with each other.

We need to take into account special relativity: the Minkowski geometry of space–time and the limitation on the speed of allowed interactions among different regions of space–time. Accordingly our measurament algebra need to encode the fact that space-like separated experiments should not interfere with each other, i.e. the associated algebras commute (one should say this more precisely but we will skip over these fine points). Naively one could try to associate an observable algebra to each point of a equal time surface (because they are all space-like with respect to each other, i.e. outside the light-cone of each other). A more conservative approach is to consider each measuring apparatus to have a finite spatial precision and measure what happens in an extended (even if maybe small) region of space. More abstractly we could associate to any compactly supported smooth test function $f \in \mathscr{D}(\mathbb{R}^n)$ in n-dimensional space a one-dimensional obsevable algebra (for example), i.e. an homeomorphism $\Phi_0(f): C(\mathbb{R}; \mathbb{C}) \to \mathscr{A}$ into the C^*-algebra $\mathscr{A}$ generated by all observables. By writing the more suggestive notation $F(\Phi_0(f)) := \Phi_0(f)(F)$ for any $F \in C(\mathbb{R}; \mathbb{C})$ we would like to impose that

$$[F(\Phi_0(f)), G(\Phi_0(g))] = 0$$

for each $f, g \in \mathscr{D}(\mathbb{R}^n)$ with disjoint support and $F, G \in C(\mathbb{R}; \mathbb{C})$.

All together we then expect a representation of the full Poincaré group by automorphisms of $\mathscr{A}$.

For example if we assume that we can make measurements at a precise point, we can assume to have observables $F(\Phi(t,x))$ with measures a given quantity at time $t \in \mathbb{R}$ and in the point $x \in \mathbb{R}^n$, $(t,x) \in \mathbb{M}^n$. The an element g of the Poincaré group (i.e. the symmetry group of $\mathbb{M}^n$) acts via an automorshism α_g such that

$$\alpha_g(F(\Phi(t,x))) = F(\Phi(g.(t,x))), \qquad (t,x) \in \mathbb{M}^{n+1}$$

where $g.(t,x) \in \mathbb{M}^n$ denotes the action of g on the vector (t,x). Moreover $\alpha_g \circ \alpha_{g'} = \alpha_{gg'}$. (Implictly I'm considering a scalar field).

Accordingly is also natural to assume that there exists a state ω which is invariant under all the $(\alpha_g)_g$ and that the group $(\alpha_g)_g$ has some mild regularity properties.

With some additional mild assumptions one can then derive the existence of a Hilbert space $\mathscr{H}$, a state vector φ, an strongly continuous unitary representation $(U_g)_g$ of the Poincaré group which leaves the state vector φ invariant and a representation $\pi \in \mathrm{Hom}(\mathscr{A}, \mathscr{B}(\mathscr{H}))$ of the observable algebra on $\mathscr{H}$.

Note that $[F(\Phi(t,x)), G(\Phi(t,y))] = 0$ for all F, G and all $x \neq y$. So I can consider all the collection of all $(\Phi(0,x))_{x \in \mathbb{R}^n}$ for all the points at time zero. They form a commutative subalgebra of observables.

Let us now apply the Euclidean strategy and look for a measure which could deliver such data.

We expect the stochastic process X to be, at each point of time $t \in \mathbb{R}$ a random distribution, i.e. that for each $f \in \mathscr{D}(\mathbb{R}^n)$ we can construct a random variable $X_t(f)$ and

$$X_t(\alpha f + \beta g) = \alpha X_t(f) + \beta X_t(g).$$

That is: we have an infinite-dimensional random process, apart from this nothing much changes to the previous finite dimensional considerations above, in particular RP works just fine.

(However as we mentioned above the Markovian stategy is more subtle in infinite dimensions (i.e. conditioning wrt. $\mathscr{F}_0$ is a much stronger perturbation of the system and it is not clear that it is equivalent to just conditioning on X_0) and we adopt the strategy of perturbing a simple RP process.)

The goal is the to find RP processes in space-time which are invariant wrt. to time and space translations. (What about Poincaré rotations, i.e. rotations and boosts?) One realises that the Poincaré invariance corresponds to the Euclidean invariance of the stochastic process in $\mathbb{R}^{n+1} \approx \mathbb{R}^d$. That is we want the law of X to be invariant under the full Euclidean group acting on $(t, x) \in \mathbb{R}^d$.

► This justifies our definition of EQFT: i.e. probability measure on distributions over $\mathbb{R}^d$ that is RP and Euclidean invariant (+ other technical regularity assumptions).

Let us look therefore at what is available. We can try to take a process X such that the family $(X_t(f))_{t\in\mathbb{R}, f\in\mathscr{D}(\mathbb{R}^n)}$ is jointly Gaussian and we know that to get a RP process we can look at OU processes.

Let us start assuming that we build $X_t(f)$ as a sum of finitely many independent OU (complex) processes (Y_t^i) as follows

$$X_t(f) = \sum_i \hat{f}(k_i) Y_t^i, \tag{6}$$

where $\hat{f}$ is the Fourier transform of f and $k_i \in \mathbb{R}^n$ are parameters. Precisely we take

$$X_t(f) = \sum_i \mathrm{Re}(\hat{f}(k_i)) Y_t^{1,i} + \mathrm{Im}(\hat{f}(k_i)) Y_t^{2,i}.$$

Then we have to arrange the covariances such that

$$\mathbb{E}[X_t(f) X_0(f)] = \sum_{i,j} \mathrm{Re}(\hat{f}(k_i))\mathrm{Re}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{1,i} Y_t^{1,j}]}_{=\delta_{i,j} c_i e^{-\lambda_i |t|}} + \sum_{i,j} \mathrm{Im}(\hat{f}(k_i))\mathrm{Im}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{2,i} Y_t^{2,j}]}_{=\delta_{i,j} c_i e^{-\lambda_i |t|}}$$

$$+ \sum_{i,j} \mathrm{Re}(\hat{f}(k_i))\mathrm{Im}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{1,i} Y_t^{2,j}]}_{=0} + \sum_{i,j} \mathrm{Im}(\hat{f}(k_i))\mathrm{Re}(\hat{f}(k_j)) \underbrace{\mathbb{E}[Y_t^{2,i} Y_t^{1,j}]}_{=0}$$

Then the covariance reads:

$$\mathbb{E}[X_t(f) X_0(f)] = \sum_i |\hat{f}(k_i)|^2 c_i e^{-\lambda_i |t|}.$$

Exercise. make this precise.

Note that

$$e^{-\lambda_i |t|} = \pi \int_{\mathbb{R}} \frac{e^{i\omega t}}{\omega^2 + \lambda_i^2} \mathrm{d}\omega.$$

(modulo the right coefficient). So

$$\mathbb{E}[X_t(f) X_0(f)] \propto \int_{\mathbb{R}} \sum_i |\hat{f}(k_i)|^2 c_i \frac{e^{i\omega t}}{\omega^2 + \lambda_i^2} \mathrm{d}\omega$$

Now taking more and more points and appropriate c_i we can enforce convergence to a integral over $k \in \mathbb{R}^n$:

$$\mathbb{E}[X_t(f)X_0(f)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}} |\hat{f}(k)|^2 \frac{e^{i\omega t}}{\omega^2 + \lambda(k)^2} \mathrm{d}k\mathrm{d}\omega$$

(we would need now infinitely many degrees of freedom, i.e. OU processes).

This expression implies that the spatial covariance has the form

$$\mathbb{E}[X_t(f(\cdot + x))X_0(f)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}} |\hat{f}(k)|^2 \frac{e^{i(\omega t + k\cdot x)}}{\omega^2 + \lambda(k)^2} \mathrm{d}k\mathrm{d}\omega$$

so it makes sense to take $\lambda(k)^2 = k^2 + m^2$ for a fixed constant $m > 0$. This give rise to an Euclidean invariant covariance. To see this take $f \to \delta$ so $\hat{f}(k) \to 1$ and observe that one obtains

$$\mathbb{E}[X_t(\delta_x)X_0(\delta_0)] = \int_{\mathbb{R}^d} \frac{e^{ik\cdot(t,x)}}{k^2 + m^2} \mathrm{d}k.$$

The associated process is therefore Euclidean invariant in $\mathbb{R}^d$, since the covariance is. This form of the covariance is constrained by RP and Euclidean invariance.

Definition. *This process is called the Gaussian Free field with mass $m > 0$ in (Euclidean) dimension d. This this an example of EQFT.*

For $d = 1$ is just the OU process we have seen.

Note that we have problems… :

$$\mathbb{E}[|X_0(\delta_0)|^2] = \int_{\mathbb{R}^d} \frac{1}{k^2 + m^2} \mathrm{d}k = +\infty,$$

if $d \geqslant 2$. Note that $d = 1$ is just QM… (finitely many degrees of freedom). I cannot evaluate X at any fixed space-time point. It is still true that for any $f \in \mathscr{D}(\mathbb{R}^n)$ the process $(X_t(f))_{t\in\mathbb{R}}$ is a nice and continuous Gaussian process.

From eq. (6) we have

$$\mathrm{d}X_t(f) = \sum_i \hat{f}(k_i)\mathrm{d}Y_t^i = \sum_i \hat{f}(k_i)\lambda_i Y_t^i \mathrm{d}t + \sum_i \hat{f}(k_i)\mathrm{d}B_t^i = X_t((m^2 - \Delta)^{1/2} f)\mathrm{d}t + \mathrm{d}B_t(f)$$

with

$$B_t(f) = \sum_i \hat{f}(k_i) B_t^i.$$

In the limit we obtain therefore the Euclidean-time dynamics

$$\mathrm{d}X_t(f) = X_t((m^2 - \Delta)^{1/2} f)\mathrm{d}t + \mathrm{d}B_t(f)$$

with $B_t(f)$ a Brownian motion such that $\mathbb{E}[B_t(f)^2] = t\|f\|_{L^2(\mathbb{R}^n)}^2$.

Note that our construction guarantees that it is reflection positive in the time direction. By rotation invariant we have also that it is reflection positive along any direction.

Full Euclidean invariance is important because, within the reconstruction of QM guarantees that the quantum theory is Poincaré invariant.

> **Exercise.** Try to work out some details of the reconstruction in this case, it would be nice to arrive at a point where one can check explicitly the Poincaré invariance of the quantum theory.

3.5 Interacting Euclidean Quantum Fields?

From now on we stop to treat the physical time as a special variable and consider it part of the Euclidean coordinates and we will work in $\mathbb{R}^d$ forgetting about the Minkowski geometry. This is certainly one advantage of the Euclidean approach. As we have shown in the one dimensional setting we can go back to a full quantum theory. Here again the theory is trivial: the dynamics do not mix the different OU processes. This also implies that one can factorize the quantum Hilbert space into a product of independent components, each of them behaving as we described in one dimension: the system is full of non-interacting particles.

To construct other examples out of the GFF one can use the perturbative approach and consider an interaction potential of the form $V(X_t)$. However doing so we break the Euclidean invariance of the theory: one one hand we need that the Gibbsian perturbation has the form

$$\exp\Big(\int_{-T}^{T} V(X_s)\mathrm{d}s\Big)$$

to maintain RP, on the other hand this special treatment of the time direction should be compensated by a specific form of V to reestablish some invariance under rotations. We are led to take the perturbation to be in the form

$$\exp\Big(\int_{[-T,T]^d} V(X(x))\mathrm{d}x\Big)$$

where we denote $X(x_1, x_2, \ldots, x_d) = X_{x_1}(x_2, \ldots, x_d)$ the OU process and consider the new measure

$$\mathrm{d}\mathbb{Q}^T = \frac{1}{Z_T}\exp\Big(\int_{[-T,T]^d} V(X(x))\mathrm{d}x\Big)\,\mathrm{d}\mathbb{P} \tag{7}$$

with suitable normalization constant.

Now we encounter more difficulties: first of all we had to cut of the integral in all the dimensions to have a well defined quantity: we call T an infrared cutoff: it limits the perturbation to a compact region of space, giving hope to gave a meaning to the perturbed measure and thus breaking even more the translation invariance of the model. As with the finite dimensional situation however we can hope that the $T \to \infty$ limit reestablish such invariance. This is also confirmed by statistical mechanical models which have very strong similarities with the measure (7) and for which one understand well what happens when $T \to \infty$.

A more serious problem is due to UV (ultraviolet) divergences. As soon as one stops a moment and think, one realises that $X(x)$ is not a well defined random variable, indeed we have

$$\mathbb{E}[X(f)^2] = \int_{\mathbb{R}^d} \frac{|f(k)|^2}{k^2+m^2}\mathrm{d}k.$$

so as $f \to \delta_x$ this variance explodes if $d \geqslant 2$ due to failure of the integrability at infinity caused by the slow decay of the integrand (whose specific form was motivated by Euclidean invariance). Recall that the case $d = 1$ corresponds to the one dimenional OU process, so of no interest of us at this point.

We are forced to admit the possibility that the samples of the random variable X are not regular enough to be evaluated at points $x \in \mathbb{R}^d$ and we have to resort to a more careful handling. In particular the expression $V(X(x))$ does not make sense. This is a problem due to the fluctuations at small scales so it is called an UV divergence (because UV light rays oscillates at smaller scales than IR light rays).

To have a sensible starting point we therefore add another layer of approximation and avoid very large fluctuations. This can be achieved in many ways. An easy approach is to fix a smooth function $\rho: \mathbb{R}^d \to \mathbb{R}_+$ so that $\int \rho(x) \mathrm{d}x = 1$ and with compact support, let $\rho_\varepsilon(x) = \varepsilon^{-d} \rho(x / \varepsilon)$ to rescale it while preserving the L^1 norm and then let $X_\varepsilon = \rho_\varepsilon * X_\varepsilon$ which can be checked to be a nice and smooth gaussian process, that is we can prove that $\mathbb{P}$-almost surely $X_\varepsilon \in C^\infty(\mathbb{R}^d)$ (it is not bounded however and the control of the smoothness degrades with the distance from the origin). We will come later on to details on these considerations, for the moment let's assume this fact. Keeping $\varepsilon > 0$ we can now consider

$$\mathrm{d}\mathbb{Q}^{\varepsilon,T} = \frac{1}{Z_T} \exp\left(\int_{[-T,T]^d} V(X_\varepsilon(x)) \mathrm{d}x \right) \mathrm{d}\mathbb{P} \tag{8}$$

which is well defined, e.g. when V is bounded below and satisfying some additional mild assumptions. For example we can take $V(\varphi) = \varphi^4$ or any other polynomial bounded from below and things would be fine.

We have arrived finally to a meaningful and nontrivial mathematical problem: prove that the family of measures $\mathbb{Q}^{\varepsilon,T}$ converges weakly to a measure $\mathbb{Q}$ on random distributions on $\mathbb{R}^d$.

Of course this is not enough to carry on the reconstruction program: one needs that the limit satisfies RP, stationarity and an additional analytic condition.

If moreover the measure $\mathbb{Q}$ is also rotation invariant, then the reconstructed QM is Poincaré invariant.

The above choice of UV regularisation destroys completely RP and it is not clear how to recover this property in the limit. Another possible regularisation is to apply the smoothening with ρ_ε in all directions but one, for which then the proof of RP given above works.

In these notes we will follow a technically easier approach to construct first a model on a lattice instead of continuous space, i.e. we replace $[-T, T]^d$ by $\mathbb{T}^d_{L,\varepsilon} := [-L, L]^d \cap (\varepsilon\mathbb{Z})^d$ a discrete set of $(L / \varepsilon)^d$ points which we endow with periodic boundary conditions. The boundary conditions are useful to have the translation invariance of the model, since the GFF is not periodic of period L we need also to periodicize it. The result is that we will consider a family of Gaussian r.v. $(X^{L,\varepsilon}(x))_{x \in \mathbb{T}^d_{L,\varepsilon}}$ with covariance

$$\mathbb{E}[X^{L,\varepsilon}(x) X^{L,\varepsilon}(y)] = (m^2 - \Delta_\varepsilon)^{-1}(x, y),$$

where Δ_ε is the discrete Laplacian with periodic boundary conditions. A Fourier transform formula for this function reads

$$\mathbb{E}[X^{L,\varepsilon}(x) X^{L,\varepsilon}(y)] = \frac{1}{(2\pi L)^d} \sum_{k \in ((\mathbb{Z}/L) \cap [-\varepsilon^{-1}, \varepsilon])^d} \frac{e^{ik \cdot (x-y)}}{(\sin(\varepsilon k) \varepsilon^{-1})^2 + m^2}, \qquad x, y \in [-L, L]^d \cap (\varepsilon\mathbb{Z})^d.$$

The exact formula is not very important.

Then we take $\mu^{L,\varepsilon}$ to be the law of $X^{L,\varepsilon}$ and define $\nu^{L,\varepsilon}$ as

$$\mathrm{d}\nu^{L,\varepsilon} = \frac{1}{Z_{L,\varepsilon}} \exp\left(-\varepsilon^d \sum_{x \in \mathbb{T}^d_{L,\varepsilon}} V(X^{L,\varepsilon}(x))\right) \mathrm{d}\mu^{L,\varepsilon} \tag{9}$$

which is now a very well defined (even elementary) expression.

The advantage of this discretisation is that it is invariant under lattice translations and that satisfy a discrete version of reflection positivity [see. e.g. Sacha Friedli and Yvan Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction* (Cambridge, United Kingdom; New York, NY: Cambridge University Press, 2017).].

This will be our starting point to discuss stochastic quantisation.

4 Stochastic quantisation

Let's recall the setting of the definition of the two measures $\mu^{L,\varepsilon}, \nu^{L,\varepsilon}$.

Let $\varepsilon = 2^{-N}$ and $M = 2^{N'}$. Let $\Lambda_\varepsilon = (\varepsilon \mathbb{Z})^d \subseteq \mathbb{R}^d$ the square lattice in dimension d of side length ε, $\Lambda_{\varepsilon,M} = \Lambda_\varepsilon \cap \mathbb{T}^d_M = (\varepsilon \mathbb{Z})^d \cap [-M/2, M/2)^d$ a finite box of $(M/\varepsilon + 1)^d$ points which we think with periodic boundary conditions in every directions.

Fourier transform on Λ_ε is defined as

$$\mathscr{F}_\varepsilon f(x) = \varepsilon^d \sum_{x \in \Lambda_\varepsilon} f(x) e^{-2\pi i k \cdot x}, \qquad \mathscr{F}_\varepsilon^{-1} g(x) = \int_{\hat{\Lambda}_\varepsilon} g(k) e^{2\pi i k \cdot x} \mathrm{d}k,$$

with $\hat{\Lambda}_\varepsilon = (\varepsilon^{-1}[-1,1))^d$ the dual of Λ_ε. These definitions can be extended to the finite lattice in a natural way, with $\hat{\Lambda}_{\varepsilon,M} = ((\mathbb{Z}/M) \cap [-\varepsilon^{-1}/2, \varepsilon^{-1}/2))^d$ and

$$\mathscr{F}_{\varepsilon,M} f(x) = \varepsilon^d \sum_{x \in \Lambda_{\varepsilon,M}} f(x) e^{-2\pi i k \cdot x}, \qquad \mathscr{F}_{\varepsilon,M}^{-1} g(x) = \frac{1}{M^d} \sum_{k \in \hat{\Lambda}_{\varepsilon,M}} g(k) e^{2\pi i k \cdot x}.$$

The measure μ is the law of a family of Gaussian r.v. $(X^{\varepsilon,M}(x))_{x \in \Lambda_{\varepsilon,M}}$ with covariance

$$\mathbb{E}_\mu[X^{\varepsilon,M}(x) X^{\varepsilon,M}(y)] = (m^2 - \Delta_\varepsilon)^{-1}(x,y), \qquad x, y \in \Lambda_{\varepsilon,M}$$

where Δ_ε is the discrete Laplacian with periodic boundary conditions, i.e.

$$\Delta_\varepsilon f(x) = \varepsilon^{-2} \sum_{i=1,\ldots,d} (f(x + \varepsilon e_i) - 2f(x) + f(x - \varepsilon e_i)), \qquad x \in \Lambda_\varepsilon$$

where $(e_i)_{i=1,\ldots,d}$ is the canonical basis of $\mathbb{R}^d$. We introduce also discrete derivatives

$$\nabla_\varepsilon^i f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}, \qquad \nabla_\varepsilon^{-,i} f(x) = \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon}$$

and note that $(\nabla_\varepsilon^i)^* = -\nabla_\varepsilon^{-,i}$ and $\Delta_\varepsilon = \sum_{i=1}^d \nabla_\varepsilon^{-,i} \nabla_\varepsilon^i$. Moreover

$$(\nabla_\varepsilon^i \mathscr{F}^{-1} g)(x) = \int_{\hat{\Lambda}_\varepsilon} g(k) \frac{e^{2\pi i \varepsilon k_i} - 1}{\varepsilon} e^{2\pi i k \cdot x} \mathrm{d}k$$

$$(\nabla_\varepsilon^{-,i}\nabla_\varepsilon^i\mathscr{F}^{-1}g)(x) = \int_{\hat{\Lambda}_\varepsilon} g(k)\frac{e^{2\pi i\varepsilon k_i}-1}{\varepsilon}\frac{1-e^{-2\pi i\varepsilon k_i}}{\varepsilon}e^{2\pi ik\cdot x}\mathrm{d}k$$

$$=-\int_{\hat{\Lambda}_\varepsilon} g(k)(2\,\varepsilon^{-1}\sin(\pi\,\varepsilon\,k_i))^2 e^{2\pi ik\cdot x}\mathrm{d}k$$

A Fourier transform formula for the correlation function reads

$$(m^2-\Delta_\varepsilon)^{-1}(x,y) = \frac{1}{M^d}\sum_{k\in((\mathbb{Z}/M)\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d}\frac{e^{ik\cdot(x-y)}}{(m^2+\sum_i(2\,\varepsilon^{-1}\sin(\pi\,\varepsilon\,k_i))^2)},\qquad x,y\in\Lambda_{\varepsilon,M}$$

So $\varphi^{\varepsilon,M}$ is an approximation of the GFF φ. We denote $\mu^{\varepsilon,M}$ its law, note that it is a law on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ which is a finite dimensional space. By abuse of notation I will also consider it as a measure on $\mathbb{R}^{\Lambda_\varepsilon}$ by periodic extension.

Both our discrete versions of translation invariance and RP will converge nicely to their continuum counterpart. Finally

► Define the measure $\nu^{\varepsilon,M}$ on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ (or by extension on $\mathbb{R}^{\Lambda_\varepsilon}$)

$$\nu^{\varepsilon,M}(\varphi) = \frac{1}{Z_{\varepsilon,M}}\exp\left(-\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}}V(\varphi(x))\right)\mu^{\varepsilon,M}(\mathrm{d}\varphi) \tag{10}$$

for some $V:\mathbb{R}\to\mathbb{R}$ bounded below.

Exercise. Prove that if $V(\varphi)=\beta\,\varphi^2$ and $\beta>-m^2$ then we get another GFF with a different mass.

This approximation now is elementary and it has the advantage that it preserves discrete translation invariance wrt. the lattice Λ_ε and moreover a discrete and periodic version of RP.

Reference for discrete RP: S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction* (Cambridge, United Kingdom; New York, NY: Cambridge University Press, 2017).

► Both our discrete versions of translation invariance and RP will converge nicely to their continuum counterpart as $\varepsilon\to 0$ (to get rid of discreteness) and $M\to\infty$ (to get rid of periodicity).

The rest of the lectures will concern the analysis of these measures in order to prove the existence of the limits above.

What is a proper choice of V? Any V (non-quadratic) is ok, as soon as it works. The problem is that not so many choices are available. In $d=1$ one could take any $V\in C(\mathbb{R},\mathbb{R}_+)$ (or even unbounded with some conditions). In $d=2$ one can take polynomial functions, exponential, trigonometric functions. In $d=3$ we know only how to take V a fourth order polynomial bounded below, in this case we say we are looking at Φ_3^4.

Definition 7. *A Φ_3^4 measure is any non-Gaussian, Euclidean invariant and RP accumulation point of the family $(\nu^{\varepsilon,M})_{\varepsilon,M}$ as $\varepsilon\to 0$ and $M\to\infty$ where one can take as V any 4-th order polynomial, bounded below and with ε,M dependent coefficient.*

One of the big successes of constructive EQFT in '70,'80 is the proof that this limits exists and has many nice properties. It was proven by Glimm, Jaffe, Feldman, Osterwalder, Seneor, …

For ε fixed and $M \to \infty$ this is a problem of statistical mechanics: the infinite volume limit of a system of unbounded spins with nearest neighbor interaction.

As I said at the beginning a stochastic quantisation (in these lectures) of a given measure ρ is a map F_ρ which sends a Gaussian r.v. to a r.v. with law ρ.

Even in the case $\nu^{\varepsilon,M}$ there are many interesting ways to do this.

1. ***Langevin dynamics / parabolic SQ*** : the Gaussian process W is a family of Brownian motions and the map $\nu \sim F_\nu(W) = \phi(0)$ is given by the stationary solution

$$\phi \colon \mathbb{R} \times \Lambda_{\varepsilon,M} \to \mathbb{R},$$

   of the SDE

$$\mathrm{d}\phi(t,x) = \{[(-m^2 + \Delta_\varepsilon)\,\phi(t)](x) - V'(\phi(t,x))\}\mathrm{d}t + \mathrm{d}W(t,x), \qquad x \in \Lambda_{\varepsilon,M}, t \in \mathbb{R}$$

   with V' the derivative of V. Here $t \in \mathbb{R}$ is a fictious time (it is not the Euclidean time!!!)

2. ***Elliptic SQ***: $\nu \sim F_\nu(\xi) = (\phi(0,x))_{x \in \Lambda_{\varepsilon,M}}$ but now $\phi \colon \mathbb{R}^2 \times \Lambda_{\varepsilon,M} \to \mathbb{R}$ is the solution to the elliptic PDE

$$(m^2 - \Delta_{\mathbb{R}^2} - \Delta_{\Lambda_{\varepsilon,M}})\,\phi(z,x) + V'(\phi(z,x)) = \xi(z,x), \qquad x \in \Lambda_{\varepsilon,M}, z \in \mathbb{R}^2$$

   where ξ is a space-time white noise.

3. ***Canonical SQ***: the Gaussian process W is a family of Brownian motions and the map $\nu \sim F_\nu(W) = \phi(0)$ is given by the stationary solution

$$\phi \colon \mathbb{R} \times \Lambda_{\varepsilon,M} \to \mathbb{R}$$

   of the SDE (discrete wave equation)

$$\partial_t^2 \phi(t,x) = -\gamma\,\partial_t \phi(t,x) + [(-m^2 + \Delta_\varepsilon)\,\phi(t)](x) - V'(\phi(t,x)) + \partial_t W(t,x)$$

   (approximatively). Without noise this is an Hamiltonian equation.

4. ***Variational representation*** (see Barashkov/G.)

5. There is even another possible approach which require to consider a stochastic evolution in the Euclidean time and it looks like

$$\partial_{x_0} \phi = \{-(m^2 - \Delta_\varepsilon)^{1/2}\,\phi - V'(\phi)\}\mathrm{d}t + \partial_{x_0} W, \qquad x \in \mathbb{R} \times \Lambda_{\varepsilon,M}^{d-1}.$$

   In this case we cannot discretize the Euclidean time and also the measure $\nu^{\varepsilon,M}$ has to be taken slightly differently. This is essentially the Markovian point of view wrt. the EQFT we introduced in the first part of the course, where we perturb the OU process ϕ with a drift $-V'(\phi)$.

Remark. While the measure $\nu^{\varepsilon,M}$ is defined via a density wrt. to a Gaussian the goal of SQ is to define it as the push-forward of a Gaussian measure. In infinite dimensions it seems that push-forwards are more robust.

Example. Let $(B_t)_{t\geqslant 0}$ a one dim BM and let $X_t = B_t + t$. Then while there is no problem to see the law of X as push-forward of that of B, they are not absolutely continuous as measures on $C(\mathbb{R}_{\geqslant 0}; \mathbb{R})$.

Exercise. Prove it.

Historical note. Stochastic quantisation was introduced Nelson, Parisi & Wu. Rigorous construction of EQFT with stochastic quantisation was done in $d=1$ by Jona–Lasinio and Faris ('80), Jona–Lasinio and Mitter (~'84) in $d=2$ bounded volume and then Mitter et al. in infinite volume, this was done using probabilistic tools (martingale problems and Girsanov's formula). For $P(\Phi)_2$ (polynomial interaction in $d=2$) another approach was introduced by Da Prato and Debussche. Only in 2013 Hairer managed to prove a local existence and uniqueness result for the parabolic SQ of Φ_3^4 using regularity structures. And the we had many more results… still many problems remain open.

References

For more details on the history of EQFT and SQ and a broad list of reference s look at the introductions of these papers:

- M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.
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- S. Albeverio et al., `Grassmannian Stochastic Analysis and the Stochastic Quantization of Euclidean Fermions', *ArXiv:2004.09637 [Math-Ph]*, 25 May 2020, http://arxiv.org/abs/2004.09637.

Hyperbolic SQ and the variational method are discussed here:

- M. Gubinelli, H. Koch, and T. Oh, `Renormalization of the Two-Dimensional Stochastic Nonlinear Wave Equations', *Transactions of the American Mathematical Society*, 2018, 1, https://doi.org/10.1090/tran/7452.
- N. Barashkov and M. Gubinelli, `A Variational Method for Φ_3^4 ', *Duke Mathematical Journal* 169, no. 17 (November 2020): 3339–3415, https://doi.org/10.1215/00127094-2020-0029.

A broad literature on stochastic quantisation from the physicist point of view:

- Poul Henrik Damgaard and Helmuth Hüffel, *Stochastic Quantization* (World Scientific, 1988).

For an alternative approach to the infinite volume and infinite time limit of the dynamics:

- J. Dimock, `A Cluster Expansion for Stochastic Lattice Fields', *Journal of Statistical Physics* 58, no. 5 (1 March 1990): 1181–1207, https://doi.org/10.1007/BF01026571.

For a broad discussion of early applications of stochastic calculus in stochastic quantisation:

- Lorenzo Bertini, Giovanni Jona-Lasinio, and Claudio Parrinello, `Stochastic Quantization, Stochastic Calculus and Path Integrals: Selected Topics', *Progress of Theoretical Physics Supplement* 111 (1 January 1993): 83–113, https://doi.org/10.1143/PTPS.111.83.

Remark. An interesting recent talk of A. Jaffe "Is relativity compatible with quantum theory?"(December 2020)

https://www.youtube.com/watch?v=RgQixyA2Gcs

It discusses the history and challenges in a mathematical theory of quantum fields.

5 Langevin dynamics

We start by constructing the parabolic stochastic quantization of the measure $\nu^{\varepsilon,M}$ for fixed ε,M. Since in this section these parameters do not play any role we will avoid to write the whenever it does not lead to ambiguities. In particular here Λ will denote the finite set $\Lambda_{\varepsilon,M}$ and Δ the discrete Laplacian and take $\varepsilon=1$ sometimes.

The law $\mu^{\varepsilon,M}$ is Gaussian, we can therefore introduce a fictious time $t\in\mathbb{R}$ (this is !not! the physical time) and a stationary OU process $(X_t^{\varepsilon,M})_{t\geqslant 0}$ such that $X_t^{\varepsilon,M}\sim\mu^{\varepsilon,M}$. There is not a unique choice, however it is not difficult to guess that a suitable dynamics is given by

$$\mathrm{d}X_t^{\varepsilon,M}=(\Delta_\varepsilon-m^2)\,X_t^{\varepsilon,M}\mathrm{d}t+2^{1/2}\mathrm{d}B_t^{\varepsilon,M}, \tag{11}$$

where $(B_t^{\varepsilon,M}(x))_{x\in\Lambda_{\varepsilon,M}}$ is a family of independent standard Brownian motions.

Exercise. Check the invariance of $\mu^{\varepsilon,M}$ under this dynamics, in particular pay attention to the normalization.

I want to construct now a dynamics which leave invariant the measure $\nu^{\varepsilon,M}$ instead. Let us guess what this dynamics should be: we write something similar as what we had before but with an unknown vector-field $F(t)$

$$\mathrm{d}X_t=AX_t\mathrm{d}t+F(t)\mathrm{d}t+2^{1/2}\mathrm{d}B_t.$$

with $A=(\Delta-m^2)$. Then if we denote $\mathbb{P}$ the law of the solution X of this equation with $X_0\sim\mu^{\varepsilon,M}$ and independent B, we want to have

$$\int_{\mathbb{R}^\Lambda}f(\varphi)\,\nu(\mathrm{d}\varphi)=\int_{\mathbb{R}^\Lambda}f(\varphi)\frac{e^{-U(\varphi)}}{Z}\,\mu(\mathrm{d}\varphi)=\frac{1}{Z}\mathbb{E}[f(X_0)\,e^{-U(X_0)}]=\frac{1}{Z}\mathbb{E}[f(X_t)\,e^{-U(X_0)}],$$

for all test functions f and all $t\geqslant 0$ with

$$U(\varphi)=\varepsilon^d\sum_{x\in\Lambda_{\varepsilon,M}}V(\varphi(x)),\qquad U\colon\mathbb{R}^{\Lambda_{\varepsilon,M}}\to\mathbb{R}.$$

Note that under the measure $\mathbb{P}^U$ defined as

$$\mathbb{P}^U(\mathrm{d}X)=\frac{e^{-U(X_0)}}{Z}\mathbb{P}(\mathrm{d}X),$$

the process X is still solution to the equation and $X_0\sim\nu^{\varepsilon,M}$.

► By Girsanov's formula, we have, for two test functions f,g

$$\mathbb{E}[f(X_t)\,e^{-U(X_0)}\,g(X_0)]=\mathbb{E}_{\mathbb{Q}}\Big[f(X_t)\,e^{\int_0^t F(s)2^{1/2}\mathrm{d}W_s-\int_0^t|F(s)|^2\mathrm{d}s-U(X_0)}\,g(X_0)\Big] \tag{12}$$

where under $\mathbb{Q}$ the process X satisfy the linear SDE

$$\mathrm{d}X_t = AX_t\mathrm{d}t + 2^{1/2}\mathrm{d}W_t$$

where W is a BM under $\mathbb{Q}$. Note that $X_0 \sim \mu$. So under $\mathbb{Q}$ X is an stationary OU process.

► Note that Ito formula gives

$$U(X_t) = U(X_0) + \int_0^t \mathrm{D}U(X_s)\mathrm{d}X_s + \int_0^t D^2 U(X_s)\mathrm{d}s$$

$$= U(X_0) + \int_0^t \mathrm{D}U(X_s)\, 2^{1/2}\mathrm{d}W_s + \int_0^t \underbrace{(\mathrm{D}^2 U(X_s) + \mathrm{DU}(X_s) A X_s)}_{=:Q(X_s)}\mathrm{d}s$$

so we can rewrite (12) as

$$= \mathbb{E}_{\mathbb{Q}}\left[f(X_t)\, g(X_0)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t (Q(X_s)\mathrm{d}s - |F(s)|^2)\mathrm{d}s} e^{\int_0^t F(s) 2^{1/2}\mathrm{d}W_s + \frac{1}{2}\int_0^t \mathrm{D}U(X_s) 2^{1/2}\mathrm{d}W_s}\right]$$

and then take $F(s) = -\frac{1}{2}\mathrm{D}U(X_s)$ to cancel the stochastic integral in the exponent to get

$$\mathbb{E}[f(X_t)\, e^{-U(X_0)}\, g(X_0)] = \mathbb{E}_{\mathbb{Q}}\left[f(X_t)\, g(X_0)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right]$$

and since $\mathbb{Q}$ is time reflection invariant (because under $\mathbb{Q}$ the process X is just a stationary OU process) we can rewrite this as

$$= \mathbb{E}_{\mathbb{Q}}\left[f(X_0)\, g(X_t)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right]$$

where we exchanged the two functions. Taking $f = 1$ we have

$$\mathbb{E}_{\mathbb{Q}}\left[g(X_t)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right] = \mathbb{E}[e^{-U(X_0)}\, g(X_0)]$$

and on the other hand, taking $g = 1$ we have (taking $g = f$ in the previous formula)

$$\mathbb{E}[f(X_t)\, e^{-U(X_0)}] = \mathbb{E}_{\mathbb{Q}}\left[f(X_t)\, e^{-\frac{1}{2}U(X_t) - \frac{1}{2}U(X_0) + \frac{1}{2}\int_0^t \left(Q(X_s)\mathrm{d}s - \frac{1}{4}|\mathrm{D}U(X_s)|^2\right)\mathrm{d}s}\right] = \mathbb{E}[e^{-U(X_0)}\, f(X_0)]$$

that is what we were looking for.

Remark. All this is ok provided we can perform all these computations. The only problems are related to the integrability of the exponential function involving the time integral. For example if we require that U is bounded below and moreover that

$$H(\varphi) = \frac{1}{2}Q(\varphi) - \frac{1}{8}|\nabla U(\varphi)|^2 = \mathrm{D}^2 U(\varphi) + \mathrm{DU}(\varphi) \cdot A\varphi - |\mathrm{D}U(\varphi)|^2$$

satisfies

$$\mathbb{E}_{\mathbb{Q}}\left[e^{\int_0^t H(X_s)\mathrm{d}s}\right] < \infty,$$

for some $t > 0$. Indeed it is enough to establish invariance for small time and the for all times.

▶ We learned that the solution to

$$\mathrm{d}X_t = AX_t\mathrm{d}t - \frac{1}{2}\mathrm{D}U(X_t)\mathrm{d}t + 2^{1/2}\mathrm{d}B_t \tag{13}$$

leaves the measure $\nu(\mathrm{d}\varphi) = Z^{-1} e^{-U(\varphi)} \mu(\mathrm{d}\varphi)$ invariant provided U is nice enough.

We will take this equation as stochastic quantization.

Exercise. Note that the process X is time-reversal invariant (we essentially gave a proof of this above, you can fill in the details).

We would actually like to have U which are unbounded, but bounded below, the relevant example in these lectures being

$$U(\varphi) = \varepsilon^d \sum_x \left(\frac{\lambda}{4}\varphi(x)^4 + \frac{\beta}{2}\varphi(x)^2\right)$$

for some $\lambda > 0$ and $\beta \in \mathbb{R}$.

We have two order of problems with such potentials. First

$$\mathrm{D}_x U(\varphi) = \frac{\partial}{\partial\varphi(x)} U(\varphi) = \lambda\,\varphi(x)^3 + \beta\,\varphi(x)$$

is not globally Lipschitz and the solutions to the SDE (13) could explode in finite time.

Then we still have to worry about invariance (i.e. fixing the details of the argument above) of the measure ν on this dynamics.

The second problem is merely technical and could be handled via a careful control of approximations with nice U and the above invariance argument. The first problem seems more worrisome but the key is to exploit the coercivity of the dynamics.

First method: one could use the invariance of the measure ν to conclude that solutions of the SDE do not explode, we will not do it here.

Second method: A direct approach is to test the equation with X_t, i.e. write

$$\frac{1}{2}\mathrm{d}\sum_x |X_t(x)|^2 = \sum_x X_t(x)\mathrm{d}X_t(x) + \sum_x \mathrm{d}t$$

$$= \sum_x [X_t(x)(AX_t)(x) - X_t(x)\mathrm{D}_x U(X_t)]\mathrm{d}t + 2^{1/2}\sum_x X_t(x)\mathrm{d}B_t(x) + \sum_x \mathrm{d}t$$

$$= -G(X_t)\mathrm{d}t + \beta\sum_x X_t(x)^2\mathrm{d}t + 2^{1/2}\sum_x X_t(x)\mathrm{d}B_t(x) + \sum_x \mathrm{d}t$$

with in the polynomial case (summing by parts the Laplacian)

$$G(\varphi) = \sum_x (|\nabla_\varepsilon \varphi(x)|^2 + m^2 \varphi(x)^2 + \lambda \varphi(x)^4) \geqslant 0.$$

By taking averages we could get some interesting estimates, for example

$$\mathbb{E} \sum_x |X_t(x)|^2 + \int_0^t G(X_s) \mathrm{d}s = \mathbb{E} \beta \int_0^t \sum_x X_s(x)^2 \mathrm{d}s + \sum_x \mathrm{d}t,$$

where now the r.h.s. can be controlled via the l.h.s. or via Gronwall lemma. But this is not robust enough for what is going on next week.

Third and last method: a more elementary and useful in the following approach which do not rely on Ito's formula goes as follows (this essentially what is called the Da Prato–Debussche trick).

First one write $X = Y + Z$ where Y is the solution to the linear equation

$$\mathrm{d}Y_t = A Y_t \mathrm{d}t + 2^{1/2} \mathrm{d}B_t,$$

that is an OU process, and Z is what remains. Then Z must solve

$$\frac{\mathrm{d}Z_t}{\mathrm{d}t} = \left(A Z_t - \frac{1}{2} \nabla U(Y_t + Z_t) \right) \tag{14}$$

which is an ODE with random coefficients, not a stochastic differential equation anymore since the effect of the Brownian perturbation is completely taken into account by Y.

We can now test this equation with Z (without the need of Ito's formula) and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_x |Z_t(x)|^2 + G(Z_t) = \sum_x \lambda (Y_t(x)^3 Z_t(x) + 3 Y_t(x)^2 Z_t(x)^2 + 3 Y_t(x) Z_t(x)^3)$$

$$+ \beta \sum_x (Z_t(x) Y_t(x) + Z_t(x)^2)$$

where

$$G(\varphi) = \|\nabla \varphi\|_{L^2}^2 + m^2 \|\varphi\|_{L^2}^2 + \lambda \|\varphi\|_{L^4}^4,$$

with the natural Lebesgue spaces on $\Lambda = \Lambda_{\varepsilon,M}$ (with counting measure).

The key property being that in the r.h.s. we have all terms which we can bound via Hölder inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_x |Z_t(x)|^2 + G(Z_t) \leqslant C_\delta \|Y_t\|_{L^4}^4 + \delta G(Z_t),$$

for $\delta > 0$ small as we wish, e.g. $\delta = 1/2$. We conclude that

$$\|Z_t\|_{L^2}^2 + \frac{1}{2} \int_0^t G(Z_s) \mathrm{d}s \leqslant \|Z_0\|_{L^2}^2 + C \int_0^t \|Y_s\|_{L^4}^4 \mathrm{d}s. \tag{15}$$

This bound implies that solutions cannot explode and we have an explicit bound on its growth in term of Y and Z_0.

Of the two we know very well Y (it is the OU process, it is Gaussian, I know everything I want on it). On the other hand I do not know so well

$$Z_0 = X_0 - Y_0 \sim \text{“}\nu - \mu\text{”}$$

because we do not really know very well ν (which is actually the object we want to study). For example we do not know estimates uniform in ε, M.

Note that even if X is stationary and Y is stationary (because we take $X_0 \sim \nu$ and $Y_0 \sim \mu$ and independent). But they are not independent and more importantly Z is not stationary.

One would like to prove that there exists a coupling of X_0 and Y_0 (i.e. find a joint law with marginals ν and μ respectively) so that the process (X, Y) is stationary (as a pair) from which would follow that Z is stationary.

In any case what we have so far is that for any f and any t we have

$$\int f(\varphi)\,\nu(\mathrm{d}\phi) = \mathbb{E}[f(X_t)] = \frac{1}{t}\int_0^t \mathbb{E}[f(X_s)]\mathrm{d}s = = \frac{1}{t}\int_0^t \mathbb{E}[f(Y_s + Z_s)]\mathrm{d}s$$

using stationarity. This is the stochastic quantization equation. Estimates on X are given via Y and Z.

Let's construct a stationary coupling of Y and Z. One uses the Krylov-Bogoliubov argument. We can construct a measure γ_T on a pair of fields $(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ by the formula

$$\int f(\varphi, \psi)\mathrm{d}\gamma_T(\varphi, \psi) := \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s, Z_s)]\mathrm{d}s,$$

for any bounded function f of the pair $(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ where Y, Z are started as above.

We have that

$$\int [G(\psi) + \|\varphi\|_{L^4}^4]\mathrm{d}\gamma_T(\varphi, \psi) = \frac{1}{T}\int_0^T \mathbb{E}[G(Z_s) + \|Y_s\|_{L^4}^4]\mathrm{d}s \leqslant \frac{2}{T}\Big(\mathbb{E}\|Z_0\|_{L^2}^2 + C'\int_0^T \mathbb{E}\|Y_s\|_{L^4}^4\mathrm{d}s\Big),$$

$$\leqslant \left(\frac{2}{T}\mathbb{E}\|Z_0\|_{L^2}^2\right) + 2C'\,\mathbb{E}\|Y_0\|_{L^4}^4,$$

which is uniformly bounded in T. This implies that the family $(\gamma_T)_T$ is tight on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ and one can extract a weakly convergent subsequence to a limit γ. Note also that

$$\int f(\varphi + \psi)\mathrm{d}\gamma_T(\varphi, \psi) = \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s + Z_s)]\mathrm{d}s = \frac{1}{T}\int_0^T \mathbb{E}[f(X_s)]\mathrm{d}s = \mathbb{E}[f(X_0)].$$

Therefore the law of $\varphi + \psi$ under γ_T is always given by ν for any T. As a consequence the law of $\varphi + \psi$ under γ is ν.

The measure γ is stationary under the joint dynamics of (Z, Y), i.e. if $(Z_0, Y_0) \sim \gamma$ then $(Z_t, Y_t) \sim \gamma$.

Exercise. Prove it. Also try to understand if the dynamics of the pair is time-symmetric.

In this way one can construct a stationary coupling of (Z, Y) which gives a useful representation of the stationary process X.

6 Infinite volume limit

What happens when we want to take the limit $M \to \infty$? The estimate (15) is not good enough because both $\|Z_0\|_{L^2(\Lambda_{\varepsilon,M})}$ and $\|Y_s\|_{L^4(\Lambda_{\varepsilon,M})}$ cannot remain finite since both random field are stationary and one expects that

$$\|Z_0\|_{L^2(\Lambda_{\varepsilon,M})} \sim M^d, \qquad \|Y_s\|_{L^4(\Lambda_{\varepsilon,M})} \sim M^d.$$

In this section we explicit the dependence on M and use Λ_ε for the full lattice. Moreover we extend any periodic field to the full lattice periodically. (we fix an origin)

However we can modify our apriori estimate introducing a polynomial weight $\rho\colon \Lambda = (\varepsilon\mathbb{Z})^d \to \mathbb{R}$

$$\rho(x) = (1 + \ell|x|)^{-\sigma}$$

with $\ell, \sigma > 0$ large enough, where $|x|$ is the distance from the origin of $\Lambda = \Lambda_\varepsilon$.

Now we test the equation for Z with $\rho^2 Z$ summing over the full lattice Λ and we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{x\in\Lambda_\varepsilon} |\rho(x) Z_t(x)|^2 + G(Z_t) \leqslant -\lambda \sum_{x\in\Lambda_\varepsilon} \rho(x)(Y_t(x)^3 Z_t(x) + 3 Y_t(x)^2 Z_t(x)^2 + 3 Y_t(x) Z_t(x)^3)$$

$$+\beta \sum_{x\in\Lambda_\varepsilon} \rho(x)(Z_t(x) Y_t(x) + Z_t(x)^2) + C_\rho \sum_{x\in\Lambda_\varepsilon} \rho(x) Z_t(x)^2$$

where C_ρ (and the inequality) is term coming from the integration by parts which can be made small by choosing ℓ small and where

$$G(\varphi) = \|\rho\nabla\varphi\|_{L^2(\Lambda_\varepsilon)}^2 + m^2\|\rho\varphi\|_{L^2(\Lambda_\varepsilon)}^2 + \lambda\|\rho^{1/2}\varphi\|_{L^4(\Lambda_\varepsilon)}^4.$$

And using similar estimates as above we obtain the apriori weighted estimates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho Z_t\|_{L^2(\Lambda_\varepsilon)}^2 + G(Z_t) \leqslant C_\delta\|\rho^{1/2} Y_t\|_{L^4(\Lambda_\varepsilon)}^4 + \delta G(Z_t)$$

indeed

$$\lambda\left|\sum_{x\in\Lambda_\varepsilon}\rho(x)\,Y_t(x)^3 Z_t(x)\right|\leqslant\lambda\left|\sum_{x\in\Lambda_\varepsilon}(\rho(x)^{3/2}Y_t(x)^3)(\rho(x)^{1/2}Z_t(x))\right|$$

$$\leqslant\lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4+\delta\,\lambda\|\rho^{1/2}Z_t\|_{L^4}^4\leqslant\lambda\frac{C}{\delta}\|\rho^{1/2}Y_t\|_{L^4}^4+\delta G(Z_t)$$

for any small $\delta>0$.

As a consequence one get the estimate

$$\|\rho Z_t\|_{L^2(\Lambda)}^2+\frac{1}{2}\int_0^t G(Z_s)\mathrm{d}s\leqslant\|\rho Z_0\|_{L^2(\Lambda)}^2+C\int_0^t\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s. \tag{16}$$

We have seen that we can construct a stationary coupling of (Y,Z), so we can use there this stationary coupling and take the average of this inequality to get

$$\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2+\frac{1}{2}\int_0^t\mathbb{E}\,G(Z_s)\mathrm{d}s\leqslant\mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2+C\int_0^t\mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s$$

but by stationarity we also have $\mathbb{E}\|\rho Z_t\|_{L^2(\Lambda)}^2=\mathbb{E}\|\rho Z_0\|_{L^2(\Lambda)}^2$ so the initial condition disappear!!! So

$$\frac{1}{2}\int_0^t\mathbb{E}\,G(Z_s)\mathrm{d}s\leqslant C\int_0^t\mathbb{E}\|\rho^{1/2}Y_s\|_{L^4(\Lambda)}^4\mathrm{d}s$$

and again by stationarity one get

$$\mathbb{E}\,G(Z_0)\leqslant 2C\,\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4.$$

Which give us very good apriori estimates on the law of Z_0 which are independent of M, indeed

$$\mathbb{E}\|\rho^{1/2}Y_0\|_{L^4(\Lambda)}^4=\mathbb{E}\sum_{x\in\Lambda_\varepsilon}\rho(x)^2|Y_0(x)|^4=\sum_{x\in\Lambda_\varepsilon}\rho(x)^2\,\mathbb{E}|Y_0(x)|^4=C\sum_{x\in\Lambda_\varepsilon}\rho(x)^2<\infty$$

uniformly in M provided $\sigma>d$ and the law of $Y_0(x)$ is translation invariant so does not depend on x and actually one can easily show that

$$(\mathbb{E}|Y_0(x)|^4)^{1/2}\leqslant C\,\mathbb{E}|Y_0(x)|^2$$

$$\lesssim(m^2-\Delta)^{-1}(x,x)\lesssim\frac{1}{M^d}\sum_{k\in((\mathbb{Z}/M)\cap[-\varepsilon^{-1},\varepsilon^{-1}))^d}\frac{1}{(m^2+\sum_i(2\,\varepsilon^{-1}\sin(\pi\,\varepsilon\,k_i))^2)}$$

$$\to\int_{[-\varepsilon^{-1},\varepsilon^{-1})^d}\frac{1}{(m^2+\sum_i(2\,\varepsilon^{-1}\sin(\pi\,\varepsilon\,k_i))^2)}<+\infty$$

uniformly in M.

Lemma 8. *For any $M>0$ we have that $\nu^M \sim X_0^M \sim Y_0^M + Z_0^M$ where $Y_0^M \sim \mu^M$ and Z_0^M is a r.v. such that*

$$\sup_M \int_{\mathbb{R}^\Lambda} G(\varphi)\, \nu^M(\mathrm{d}\varphi) = \sup_M \mathbb{E}\, G(Z_0^M) < \infty.$$

This is a key estimate to take the infinite volume limit since it allows to use tightness on the family $(\nu^M)_M$ on $\mathbb{R}^\Lambda$ in the topology of local convergence.

Remark. Integration by parts with a weight was treated a bit sloppily above. Let us make these considerations more precise: we have

$$\varepsilon \nabla_\varepsilon^i (fg)(x) = f(x+\varepsilon e_i)\, g(x+\varepsilon e_i) - f(x)\, g(x) = (\nabla_\varepsilon^i f(x) + f(x))\, g(x+\varepsilon e_i) - f(x)\, g(x)$$

$$= \nabla_\varepsilon^i f(x)\, g(x+\varepsilon e_i) + f(x) \nabla_\varepsilon^i g(x)$$

so

$$\begin{aligned} -\sum_x \rho^2 Z\, \Delta Z &= \sum_x \sum_i \nabla^i(\rho^2 Z) \nabla^i Z = \sum_x \sum_i \rho^2 |\nabla^i Z|^2 + \sum_x \sum_i \nabla^i(\rho^2)\, Z(\cdot + \varepsilon e_i) \nabla^i Z \\ &= \sum_x \sum_i \rho^2 |\nabla^i Z|^2 + \sum_x \sum_i \nabla^i(\rho^2)\, Z(\cdot + \varepsilon e_i) \nabla^i Z \end{aligned} \tag{17}$$

and the second term can be bounded by

$$\left| \sum_x \sum_i \nabla^i(\rho^2)\, Z(\cdot + \varepsilon e_i) \nabla^i Z \right| \leqslant C \| \nabla^i(\rho^2)\, \rho^{-2} \|_{L^\infty} \| \nabla Z \|_{L^2} \| Z \|_{L^2} \leqslant C_\rho \| \nabla Z \|_{L^2}^2 + C_\rho \| Z \|_{L^2}^2$$

with $C_\rho = C' \| \nabla^i(\rho^2)\, \rho^{-2} \|_{L^\infty}$. Note that this constant can be made as small as we want (uniformly in ε, M) by taking ℓ small, indeed this is essentially a discrete version of the fact that in the continuum ($\varepsilon \to 0$) one has

$$|\nabla(\rho^2)\, \rho^{-2}| = |\rho^{-1} \nabla \rho| = (1 + \ell |x|)^{-1}\, \ell \lesssim \ell.$$

This argument shows however that one has to be careful with weights which are compactly supported since the term $\rho^{-1} \nabla \rho$ is then more tricky to estimate. For example, in one dimension if one has a function $\rho(t) \approx t^\alpha$ with goes to zero as $t \to 0$ then

$$\rho^{-1} \nabla \rho \propto t^{-\alpha}\, t^{\alpha - 1} \propto t^{-1}$$

so the above L^∞ estimate is not true anymore. For our purpose here polynomial weights are enough.

More estimates can be obtained by testing with other functions. For example, testing (14) with $\rho(\rho Z)^{p-1}$ one gets

$$\frac{1}{p} \partial_t \sum_\Lambda (\rho Z_t)^p + \sum_\Lambda m^2 (\rho Z_t)^p + \rho(\rho Z_t)^{p-1} \Delta Z_t + \lambda (\rho^{p/(p+2)} Z_t)^{p+2}$$

$$= -\lambda \sum_\Lambda \rho(\rho Z_t)^{p-1} [(Y_t + Z_t)^3 - Z_t^3]$$

and by proceeding as above one obtains uniform weighted $L^p(\Lambda)$ estimates for ν^M.

Remark 9. It is also possible to obtain weighted L^∞ estimates.

These weighted estimates are uniform in M and allow to prove tightness of the family $(\nu^{\varepsilon,M})_M$ for fixed $\varepsilon>0$ and $M\to\infty$, in the topology of local convergence (i.e. convergence by testing with continuous functions on $\mathbb{R}^{\Lambda_\varepsilon}$ which depends only of finitely many points of Λ_ε). In particular we understood that the local (or weighted) $L^p(\Lambda_\varepsilon)$ norms of $\varphi:\mathbb{R}^{\Lambda_\varepsilon}\to\mathbb{R}$ under the measure $\nu^{\varepsilon,M}$ have finite moments:

$$\sup_M \int \|\rho\,\varphi\|_{L^p}^p \nu^{\varepsilon,M}(\mathrm{d}\varphi)<\infty$$

for any $p>1$. Actually by working a bit harder one can prove uniform integrability of functions like $\exp(\|\rho\,\varphi\|_{L^2})$.

7 Convergence to equilibrium

At this point we want also to probe the behavior of the dynamics for long times and also large distances.

The appropriate way to look at this problem is to imagine that we have two solutions Z^1 and Z^2 both driven by two OU processes Y^1, Y^2 and with arbitrary initial conditions Z_0^1, Z_0^2.

We will fix later the specific ways we choose these data, according to the property of the measures we would like to establish.

For the moment the only thing we care about is that their difference $H=Z^1-Z^2$ solves the equation

$$\partial_t H - AH = -\frac{1}{2}[U'(X^1)-U'(X^1+H+K)] =: Q$$

with $K:=Y^1-Y^2$ and $X^1=Y^1+Z^1$ as usual.

The r.h.s. can be Taylor-expanded as

$$Q=-\frac{1}{2}\int_0^1 \mathrm{d}\tau\, U''(X^1+\tau(H+K))(H+K).$$

Note that for our usual expression of U we have

$$U_x''(\varphi)=(4\cdot 3)\,\lambda\,\varphi^2+2\,\beta.$$

However we will not need here this specific expression. We can deal with a more general situation.

We need however to concentrate on estimates which are good when $H,K\ll 1$, since we want to show that the two solutions are near when the difference in the data, i.e. in K and H_0 is small.

We will make there the key hypothesis that

$$U''(\varphi) \geqslant -2\chi$$

for some constant $\chi \in \mathbb{R}$.

This a convexity assumption since it implies that $\varphi \mapsto U(\varphi) + 2\chi \sum_\Lambda \varphi^2$ is convex. (I'm a bit sloppy here about the precise meaning of the derivatives, but we are dealing with local functionals, so I leave to you to fill out the details)

We also let

$$G := \frac{1}{2}\int_0^1 \mathrm{d}\tau\, U''(X^1 + \tau(H+K)) \geqslant -\chi.$$

We test the equation with $\rho^2 H$ for some weight ρ. The r.h.s. can be bounded by (δ is small)

$$\sum_\Lambda \rho^2 HQ = \sum_\Lambda \rho^2 GKH - \sum_\Lambda \rho^2 GH^2$$

$$\leqslant C_\delta \|\rho GK\|_{L^2}^2 + \delta \|\rho H\|_{L^2}^2 - \underbrace{\sum_\Lambda \rho^2 GH^2}_{\geqslant \chi \|\rho H\|_{L^2}^2}$$

$$\leqslant C_\delta \|\rho GK\|_{L^2}^2 + (\chi + \delta)\|\rho H\|_{L^2}^2$$

So we have now, taking any $\delta \leqslant \chi$,

$$\frac{1}{2}\partial_t \|\rho H\|_{L^2}^2 + \sum_\Lambda \rho^2 H(m^2 - \Delta) H \leqslant C\|\rho GK\|_{L^2}^2 + 2\chi \|\rho H\|_{L^2}^2$$

Consider furthermore $F(t) = e^{ct}\|\rho H\|_{L^2}^2$, then

$$\frac{1}{2}\partial_t (e^{ct}\|\rho H\|_{L^2}^2) = \frac{c}{2}(e^{ct}\|\rho H\|_{L^2}^2) + \frac{e^{ct}}{2}\partial_t \|\rho H\|_{L^2}^2$$

$$\leqslant -e^{ct}\sum_\Lambda \rho^2 H\left(m^2 - 2\chi - \frac{c}{2} - \Delta\right) H + e^{ct} C\|\rho GK\|_{L^2}^2.$$

Let

$$\tilde{Q}(t) := C\|\rho G(t) K(t)\|_{L^2}^2.$$

Choose a different kind of weight here, exponential, in the form

$$\rho(x) = e^{-\theta|x|},$$

then for $\varepsilon\theta \leqslant 1$ we have

$$|\nabla^i(\rho^2)| = \left|\frac{e^{-\theta|x+\varepsilon e_i|} - e^{-\theta|x|}}{\varepsilon}\right| \leqslant \frac{e^{\varepsilon\theta} - 1}{\varepsilon} e^{-\theta|x|} \leqslant 2\theta e^{-\theta|x|}$$

and using (17) we get

$$\sum_{\Lambda} \rho^2 H(-\Delta)H \geqslant \sum_{\Lambda} \sum_{i} \rho^2 |\nabla_i H|^2 - \sum_{\Lambda} \sum_{i} 4\theta \rho^2 |H(\cdot + \varepsilon e_i)||\nabla^i H|$$

$$\geqslant \sum_{\Lambda} \sum_{i} \rho^2 |\nabla_i H|^2 - C\theta^2 \sum_{\Lambda} \rho^2 |H|^2 - \frac{1}{2} \sum_{\Lambda} \sum_{i} \rho^2 |\nabla_i H|^2$$

$$\geqslant \frac{1}{2} \sum_{\Lambda} \sum_{i} \rho^2 |\nabla_i H|^2 - C\theta^2 \sum_{\Lambda} \rho^2 |H|^2.$$

Putting all together this gives

$$\frac{1}{2}\partial_t (e^{ct} \|\rho H\|_{L^2}^2) + e^{ct}\Big(m^2 - 2\chi - \frac{c}{2} - C\theta^2\Big) \sum_{\Lambda} \rho^2 H^2 + e^{ct} \frac{1}{2} \sum_{\Lambda} \sum_{i} \rho^2 |\nabla_i H|^2$$

$$\leqslant \frac{1}{2}\partial_t (e^{ct} \|\rho H\|_{L^2}^2) + e^{ct} \sum_{\Lambda} \rho^2 H \Big(m^2 - \frac{c}{2} - 2\chi\Big) H + e^{ct} \sum_{\Lambda} \rho^2 H(-\Delta)H \leqslant e^{ct} \tilde{Q}(t).$$

Assuming that

$$m^2 - 2\chi - \frac{c}{2} - C\theta^2 \geqslant 0$$

we have

$$\frac{1}{2}\partial_t (e^{ct} \|\rho H\|_{L^2}^2) \leqslant e^{ct} \tilde{Q}(t).$$

Integrating this we conclude

$$\frac{e^{ct}}{2} \|\rho H_t\|_{L^2}^2 \leqslant \left[\frac{1}{2} \|\rho H_0\|_{L^2}^2 + \int_0^t e^{cs} \tilde{Q}(s) \mathrm{d}s\right]$$

that is

$$\|\rho H_t\|_{L^2}^2 \leqslant e^{-ct} \|\rho H_0\|_{L^2}^2 + 2\int_0^t e^{-c(t-s)} \tilde{Q}(s) \mathrm{d}s$$

where

$$\tilde{Q}(s) = C\|\rho G_s K_s\|_{L^2}^2 = C\int_0^1 \mathrm{d}\tau \|\rho^2 U''(X_s^1 + \tau(H_s + K_s))^2 K_s^2\|_{L^1(\Lambda)}$$

For example, taking averages of this source term we can estimate it as

$$\mathbb{E}[\tilde{Q}(s)] \leqslant \sum_{x \in \Lambda} \int_0^1 \mathrm{d}\tau \, \rho^2(x) \Big\{\mathbb{E}\Big[U_x''(X_s^1 + \tau(H_s + K_s))^4\Big]\Big\}^{1/2} \Big(\mathbb{E} K_s^4(x)\Big)^{1/2}$$

From our estimates above for Z^1, Z^2, Y^1, Y^2 is not difficult to deduce that

$$\mathbb{E}\Big[U_x''(X_s^1 + \tau(H_s + K_s))^4\Big] \leqslant C$$

uniformly in M for any polynomial U''. When K is stationary in time we have the simpler expression

$$\mathbb{E}[\tilde{Q}(s)] \lesssim \sum_{x \in \Lambda} \rho^2(x) (\mathbb{E} K_s^4(x))^{1/2} \lesssim \sum_{x \in \Lambda} \rho^2(x) (\mathbb{E} K_0^4(x))^{1/2}.$$

We summarize these computation as

Lemma. *Uniformly in M and provided $m^2-2\chi-\frac{c}{2}-C\theta^2\geqslant 0$ and K is stationary in time we have*

$$\|\rho H_t\|_{L^2}^2\leqslant e^{-ct}\|\rho H_0\|_{L^2}^2+C\sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}K_0^4(x))^{1/2}.$$

Remark. This estimate shows that as $t\to\infty$ and provided

$$\sum_{x\in\Lambda}\rho^2(x)(\mathbb{E}K_0^4(x))^{1/2}\to 0 \tag{18}$$

we have

$$\|\rho H_t\|_{L^2}^2\to 0.$$

Lemma 7 can be used in two different ways: by coupling two different invariant measures via a common dynamics one can show that the two measures are equal. This gives uniqueness. On can use noises which coincide in a bounded region to drive two different dynamics, e.g. started from the same invariant measure. Then one obtains that the quantity (18) can be made small choosing a large region, which shows that one has a certain decay of correlations, i.e. what happens outside a given region does not influence much the dynamics.

8 The small scale limit and renormalization ($d=2$)

In this section we address the $\varepsilon\to 0$ limit at M fixed (let's say $M=1$). This is the ultraviolet limit (UV limit). Obtain uniform estimates in this limit is more difficult are requires new ideas. There are various possible approaches: regularity structures (Hairer), renormalization group ideas (Kupiainen), or paracontrolled distributions (GIP, Catellier & Chouk). I will follow this last strategy. The main reference for us here is the paper I mentioned by Hofmanova & myself:

> M. Gubinelli and M. Hofmanova, `A PDE Construction of the Euclidean Φ_3^4 Quantum Field Theory', *ArXiv:1810.01700 [Math-Ph]*, 3 October 2018, http://arxiv.org/abs/1810.01700.

The main problem is that as $\varepsilon\to 0$ the process Y becomes a distribution. Recall our context. We had a dynamics on X which can be decomposed on a linear part

$$\mathrm{d}Y_t=(\Delta_\varepsilon-m^2)Y_t\mathrm{d}t+2^{1/2}\mathrm{d}B_t,$$

and the non-linear part Z:

$$\frac{\partial}{\partial t}Z_t=(\Delta_\varepsilon-m^2)Z_t-\frac{1}{2}V'(Y_t+Z_t), \tag{19}$$

with $V'(\varphi)=\lambda\varphi^3+\beta\varphi$. The computation of $V'(Y_t+Z_t)$ is point-wise in space:

$$V'(Y_t+Z_t)(x)=V'(Y_t(x)+Z_t(x))=\lambda(Y_t(x)+Z_t(x))^3+\beta(Y_t(x)+Z_t(x))$$

$$=\lambda Y_t(x)^3+3\lambda Y_t(x)^2Z_t(x)+3\lambda Y_t(x)Z_t(x)^2+\lambda Z_t(x)^3+\beta Y_t(x)+\beta Z_t(x).$$

The main problem is the following: ($M=1$) as $\varepsilon \to 0$

$$\mathbb{E}[Y_t(x)^2] = (m^2 - \Delta_\varepsilon)(x,x) = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{1}{(m^2 + \sum_i (2\varepsilon^{-1} \sin(\pi \varepsilon k_i))^2)}$$

$$\approx \sum_{k \in \mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1})^d} \frac{1}{(m^2 + 2\pi |k|^2)} \propto \begin{cases} \varepsilon^{2-d} & d > 2 \\ \log(\varepsilon^{-1}) & d = 2 \end{cases}$$

Which tells us that the typical size of $Y_t(x)$ is $\varepsilon^{2-d} \to \infty$. The estimates from last week are useless in this limit, because they depend on $L^p(\Lambda_\varepsilon)$ norms of Y_t.

This is a problem of small scales. It hints to the fact that Y^ε is not converging to a function on $\mathbb{T}^d \approx [0,1]^d$, not even locally.

For convenience we define $\mathbb{T}_\varepsilon^d = \Lambda_{\varepsilon,1} = (\varepsilon \mathbb{Z} \cap [-1/2, 1/2))^d$.

8.1 Besov spaces

One way to deal with this problem and analyze what is going on in the equation (19) is to split all our functions in “blocks” which are nice.

This is accomplished via Littlewood–Paley decomposition, i.e. a nice partition of unity in Fourier space. We split every function $f: \mathbb{T}_\varepsilon^d \to \mathbb{R}$ in very nice pieces $(\Delta_i f)_{i \geqslant -1}$ as follows

$$f(x) = \sum_{i \geqslant -1} (\Delta_i f)(x),$$

where

$$\Delta_i f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \rho(2^{-i} k) \hat{f}(k) e^{2\pi i k \cdot x} \qquad i \geqslant 0,$$

and

$$\Delta_{-1} f(x) := \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \chi(k) \hat{f}(k) e^{2\pi i k \cdot x},$$

where $\rho: \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ and is such that

$$\chi(k) + \sum_{i \geqslant 0} \rho(2^{-i} k) = 1,$$

for a nice function $\chi: \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ with support in a ball $\mathcal{B}$ of radius ≈ 1 around $k = 0 \in \mathbb{R}^d$ and ρ is supported on an annulus $\mathcal{A}$ of radius ≈ 1. All these functions are smooth (and some other properties we don't care about right now).

Therefore the Fourier transform of the LP block $\Delta_i f$ is supported on an annulus of size 2^i and that of $\Delta_{-1} f$ in a ball of radius 1.

Remark. For $\varepsilon > 0$ we have $\Delta_i f = 0$ if $2^i \gtrsim \varepsilon^{-1}$, so we sum over i up to $\approx \log_2 \varepsilon^{-1}$. Let us define N_ε to be this bound. So

$$f = \sum_{i=-1}^{N_\varepsilon} (\Delta_i f),$$

there is a technical subtlety here on how one handles the last last block but we will ignore it.

Now:

$$\mathbb{E}[(\Delta_i Y_t(x))^2] = \sum_{k \in (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i}k)^2}{(m^2 + \sum_i (2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2)}$$

$$\approx \sum_{k \in 2^i \mathscr{A} \subseteq (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i}k)^2}{(m^2 + |k|^2)} \approx \sum_{k \in 2^i \mathscr{A} \subseteq (\mathbb{Z} \cap [-\varepsilon^{-1}, \varepsilon^{-1}))^d} \frac{\rho(2^{-i}k)^2}{\underbrace{|k|^2}_{\approx (2^i)^2}} \lesssim (2^i)^{d-2}$$

Which says that $\Delta_i Y_t \approx (2^i)^{(d-2)/2}$ which is uniform in ε! (but of course not in i, and i can be large)

One can prove actually that $\Delta_i Y^\varepsilon$ converges to a nice C^∞ random function on $\mathbb{T}^d$ as $\varepsilon \to 0$.

This decomposition shift the problem of dealing with distribution to a problem of dealing with large sums.

Definition 10. *Let $\alpha \in \mathbb{R}$ and $p, q \in [1, +\infty]$. We say that $f \in B^\alpha_{p,q}$ (a Besov space) iff*

$$\|f\|_{B^\alpha_{p,q}} := \|i \geqslant -1 \mapsto 2^{i\alpha} \|\Delta_i f\|_{L^p}\|_{\ell^q} = \left[\sum_i (2^{i\alpha} \|\Delta_i f\|_{L^p})^q\right]^{1/q} < \infty.$$

These are Banach spaces.

In particular we will use $\mathscr{C}^\alpha = B^\alpha_{\infty,\infty}$ with norm $\|f\|_{\mathscr{C}^\alpha}$ such that

$$\|\Delta_i f\|_{L^\infty} \leqslant 2^{-\alpha i} \|f\|_{\mathscr{C}^\alpha}.$$

When $\alpha > 0$ there are spaces of regular functions, when $\alpha < 0$ these are just distributions (of course when $\varepsilon = 0$).

Moreover note that

$$\|f\|^2_{B^\alpha_{2,2}} = \sum_{i \geqslant -1} 2^{2i\alpha} \|\Delta_i f\|^2_{L^2(\mathbb{T}^d_\varepsilon)} = \varepsilon^d \sum_{x \in \mathbb{T}^d_\varepsilon} \sum_{i \geqslant -1} 2^{2i\alpha} |\Delta_i f(x)|^2 \approx \varepsilon^d \sum_{x \in \mathbb{T}^d_\varepsilon} |(1 - \Delta_\varepsilon)^{\alpha/2} f|^2 =: \|f\|^2_{H^\alpha}.$$

From the estimate above on Y one can prove that almost surely for any $t \in \mathbb{R}$, $Y^\varepsilon_t \in \mathscr{C}^{(d-2)/2-\kappa}$ for any small $\kappa > 0$ uniformly in ε. One can also prove that as a function of t is continuous and actually uniformly in ε

$$Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{(d-2)/2-\kappa})$$

almost surely.

In the following κ will always denote an arbitrary small positive quantity. (this is a small loss of regularity due to the fact that we want almost sure statements).

Note that when $d = 2$ we have $Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{-\kappa})$ and when $d = 3$ $Y^\varepsilon \in C(\mathbb{R}; \mathscr{C}^{-1/2-\kappa})$.

Take $f \in \mathscr{C}^\alpha$, $g \in \mathscr{C}^\beta$ then

$$fg = \sum_i \Delta_i f \sum_j \Delta_j g = \sum_{i,j} \Delta_i f \Delta_j g$$

to give a sense to this product one has to control the two (large) sums. The good way to do it is to split it in three pieces:

$$fg = \sum_{i,j} \Delta_i f \Delta_j g = \sum_{i<j-K} \Delta_i f \Delta_j g + \sum_{i>j+K} \Delta_i f \Delta_j g + \sum_{|i-j|\leq K} \Delta_i f \Delta_j g$$

$$=: (f \prec g) + (f \succ g) + (f \circ g)$$

and call them the paraproducts $(f \prec g), (f \succ g) = (g \prec f)$ and the resonant term $(f \circ g)$.

Theorem 11. *The paraproducts are always well defined and*

$$\|f \prec g\|_{\mathscr{C}^\beta} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}, \qquad \alpha > 0,$$

$$\|f \prec g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}, \qquad \alpha < 0.$$

The resonant product is well-defined only if $\underline{\alpha + \beta > 0}$ *and in this case*

$$\|f \circ g\|_{\mathscr{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}.$$

Therefore fg *is well defined (and continuous) if* $\alpha + \beta > 0$ *and in this case*

$$\|fg\| \lesssim \|f\|_{\mathscr{C}^\alpha} \|g\|_{\mathscr{C}^\beta}.$$

We are allowed to multiply things only if regularity is ok, and the problem is in the resonant term.

8.2 Renormalization

Let's go back with these tools to our equation (19). In the r.h.s. we have

$$V'(Y+Z) = \lambda Y^3 + 3\lambda Y^2 Z + 3\lambda Y Z^2 + \lambda Z^3 + \beta Y + \beta Z.$$

By the product theorem we see that Y^3 is problematic since the regularity $\alpha = (2-d)/2 - \kappa$ of Y is negative. However Y is explicit, and we can do a probabilistic computation to prove that Y^3 converge as $\varepsilon \to 0$ to a well defined distribution provided it is *renormalized*.

Theorem 12. *There exists a constant* c_ε *such that the random field (renormalized square)*

$$\mathbb{Y}_t^{\varepsilon,2}(x) := (Y_t^\varepsilon(x))^2 - c_\varepsilon,$$

converges (in law) as $\varepsilon \to 0$ *to a random field* $\mathbb{Y}^2$ *in* $C(\mathbb{R}; \mathscr{C}^{2\alpha})$ *with* $\alpha = (2-d)/2 - \kappa < 0$ *(if* $d \geqslant 2$*).*

Similarly if $d = 2$ *the renormalized cube*

$$\mathbb{Y}_t^{\varepsilon,3}(x) := (Y_t^\varepsilon(x))^3 - 3 c_\varepsilon Y_t^\varepsilon(x),$$

converges as $\varepsilon \to 0$ *to a random field in* $C(\mathbb{R}; \mathscr{C}^{3\alpha})$ *while if* $d = 3$ *then convergence holds* $C^{-\kappa}(\mathbb{R}; \mathscr{C}^{3\alpha})$ *(where* $C^{-\kappa}$ *is a space of distributions in the time variable with negative regularity).*

Moreover one can take

$$c_\varepsilon := \mathbb{E}[(Y_t^\varepsilon(x))^2] \approx \varepsilon^{(2-d)}.$$

With this choice the renormalization corresponds to "Wick ordering".

Now we see that replacing

$$\beta = \beta_\varepsilon = \beta' - 3\,\lambda\, c_\varepsilon,$$

on has (with $\mathbb{Y}^1 = Y$)

$$V'(Y+Z) = \lambda \underbrace{(Y^3 - 3\,c_\varepsilon\, Y)}_{\mathbb{Y}^3} + 3\,\lambda \underbrace{(Y^2 - c_\varepsilon)}_{\mathbb{Y}^2} Z + 3\,\lambda\, Y Z^2 + \lambda Z^3 + \beta'\, Y + \beta' Z$$

$$= \lambda\, \mathbb{Y}^3 + 3\,\lambda\, \mathbb{Y}^2 Z + 3\,\lambda\, \mathbb{Y}^1 Z^2 + \lambda Z^3 + \beta'\, \mathbb{Y}^1 + \beta' Z.$$

Magic: one constant works for both problematic terms... (there are reasons for that, namely subcriticality of this model).

Next problems: the products

$$\underbrace{\mathbb{Y}^2}_{\mathscr{C}^{2\alpha}} Z, \qquad \underbrace{\mathbb{Y}^1}_{\mathscr{C}^{\alpha}} Z^2.$$

Let's try to get some estimates for Z: we test the equation (19) with Z and integrate in space $\mathbb{T}_\varepsilon^d$

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{T}_\varepsilon^d} Z_t^2 + \int_{\mathbb{T}_\varepsilon^d} \left[|\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2} |Z_t|^4 \right]$$

$$= -\frac{1}{2} \int_{\mathbb{T}_\varepsilon^d} [\lambda\, \mathbb{Y}^3 Z + 3\,\lambda\, \mathbb{Y}^2 Z^2 + 3\,\lambda\, \mathbb{Y}^1 Z^3 + \beta'\, \mathbb{Y}^1 Z + \beta' Z^2].$$

The l.h.s tells me that I have control of the L^2, L^4 norm of Z but also of the H^1 norm of Z, this means we have some regularity for Z.

Note that we define:

$$\int_{\mathbb{T}_\varepsilon^d} \mathrm{d}x := \varepsilon^d \sum_{x \in \mathbb{T}_\varepsilon^d}.$$

Also remark that $H^1 = B_{2,2}^1$. In the Besov scale we have Sobolev spaces. The theory of products and paraproducts extends naturally to Besov space with indexes p, q other than ∞, ∞.

When $d = 2$ we have that $\mathbb{Y}^k \in \mathscr{C}^{-k\alpha}$ with $k = 1, 2, 3$ and $\alpha = -\kappa$ a small negative quantity. Therefore all the products in the a-priori r.h.s. are well defined assuming $Z \in H^1$ (the sums of regularities is positive!). For example one has estimates like (for some small δ and some large K)

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^3 Z \right| \lesssim \| \mathbb{Y}^3 \|_{\mathscr{C}^{3\alpha}} \| Z \|_{B_{1,1}^{4\kappa}} \lesssim C_\delta \| \mathbb{Y}^3 \|_{\mathscr{C}^{3\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^2}^2,$$

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^2 Z^2 \right| \lesssim \| \mathbb{Y}^2 \|_{\mathscr{C}^{2\alpha}} \| Z^2 \|_{B_{1,1}^{3\kappa}} \lesssim C_\delta \| \mathbb{Y}^2 \|_{\mathscr{C}^{2\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^4}^4,$$

$$\left| \int_{\mathbb{T}_\varepsilon^d} \mathbb{Y}^1 Z^3 \right| \lesssim \| \mathbb{Y}^1 \|_{\mathscr{C}^{\alpha}} \| Z^3 \|_{B_{1,1}^{2\kappa}} \lesssim C_\delta \| \mathbb{Y}^1 \|_{\mathscr{C}^{\alpha}}^K + \delta \| \nabla Z \|_{L^2}^2 + \delta \| Z \|_{L^4}^4,$$

So overall we can obtain (via PDE methods only, no probability here)

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2 + (1-\delta)\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \leqslant Q(\mathbb{Y}_t^\varepsilon),$$

where

$$Q(\mathbb{Y}_t^\varepsilon) := 1 + C\sum_{k=1,2,3}\|\mathbb{Y}_t^k\|_{\mathscr{C}^{k\alpha}}^K,$$

for some power K. This estimate holds for all paths of Y since note that $Y^\varepsilon \in C\big(\mathbb{R}\times\mathbb{R}^{\mathbb{T}_\varepsilon^d};\mathbb{R}\big)$ so it is clear that $Q(\mathbb{Y}_t^\varepsilon) < \infty$.

The real problem is: what happens when $\varepsilon \to 0$?

Remember that we constructed a stationary coupling $\mathbb{P}^\varepsilon$ such that under $\mathbb{P}^\varepsilon$ the processes Y and Z are stationary and

$$X = Y + Z$$

is also stationary and such that $X_t \sim \nu^\varepsilon$ (recall $M = 1$ here).

Under this coupling this estimate implies that

$$\underbrace{\frac{1}{2}\frac{\partial}{\partial t}\mathbb{E}\int_{\mathbb{T}_\varepsilon^d} Z_t^2}_{=0 \text{ (by stationarity)}} + \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\right] \lesssim \mathbb{E}\,Q(\mathbb{Y}_t^\varepsilon) = \mathbb{E}\,Q(\mathbb{Y}_0^\varepsilon),$$

but again, using some probability theory one can prove that

$$\sup_\varepsilon \mathbb{E}\,Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

since again Y is well known and estimates are relatively easy. As a result one obtain uniform estimates of the form

$$\sup_\varepsilon \mathbb{E}\int_{\mathbb{T}_\varepsilon^d}\left[|\nabla_\varepsilon Z_0|^2 + m^2|Z_0|^2 + \frac{\lambda}{2}|Z_0|^4\right] < \infty.$$

This estimate is a key point because from that one can derive tightness of the family $(\nu^\varepsilon)_{\varepsilon>0}$. Indeed let γ^ε the law of (Y_0, Z_0) under $\mathbb{P}^\varepsilon$, we have

$$\sup_\varepsilon \int (\|\psi\|_{\mathscr{C}^{-\alpha}}^2 + \|\nabla\zeta\|_{L^2}^2 + \|\zeta\|_{L^2}^2 + \|\zeta\|_{L^4}^4)\,\gamma^\varepsilon(\mathrm{d}\psi\times\mathrm{d}\zeta)$$

$$= \sup_\varepsilon \mathbb{E}_{\mathbb{P}^\varepsilon}[\|Y_0\|_{\mathscr{C}^{-\alpha}}^2 + \|\nabla Z_0\|_{L^2}^2 + \|Z_0\|_{L^2}^2 + \|Z_0\|_{L^4}^4] \lesssim \sup_\varepsilon \mathbb{E}\,Q(\mathbb{Y}_0^\varepsilon) < \infty,$$

note that $\|Y_0\|_{\mathscr{C}^{-\alpha}}^2 \lesssim Q(\mathbb{Y}_0^\varepsilon)$ if K large enough.

This gives tightness of $(\gamma^{\varepsilon})_{\varepsilon\geqslant 0}$ in $\mathscr{C}^{-2\alpha}\times(H^{1-\kappa}\cap L^4)$ (some loss of regularity to guarantee the required compactness). Projecting down to $(\nu^{\varepsilon})_{\varepsilon}$ (taking the sum of the two factors) one get tightness of $(\nu^{\varepsilon})_{\varepsilon}$ in $H^{-2\alpha}=B^{-2\alpha}_{2,2}$:

$$\int \|\varphi\|^2_{B^{-\alpha}_{2,2}}\nu^{\varepsilon}(\mathrm{d}\varphi)=\int \|\psi+\zeta\|^2_{B^{-\alpha}_{2,2}}\gamma^{\varepsilon}(\mathrm{d}\psi\times\mathrm{d}\zeta)\leqslant 2\int (\|\psi\|^2_{B^{-\alpha}_{2,2}}+\|\zeta\|^2_{B^{-\alpha}_{2,2}})\,\gamma^{\varepsilon}(\mathrm{d}\psi\times\mathrm{d}\zeta)$$

$$\leqslant 2\int (\|\psi\|^2_{\mathscr{C}^{-\alpha}}+\|\zeta\|^2_{H^1})\,\gamma^{\varepsilon}(\mathrm{d}\psi\times\mathrm{d}\zeta),$$

which is uniformly bounded in ε. So we can extract an accumulation point ν (a measure on $H^{-2\alpha}(\mathbb{T}^2)$.

Theorem 13. *Provided $d=2$ and we take $\beta=-3\,\lambda\,c_{\varepsilon}+\beta'$ for some constant $\beta'\in\mathbb{R}$ and $c_{\varepsilon}=\mathbb{E}[Y_t^{\varepsilon}(x)^2]$ then the family $(\nu_{\varepsilon})_{\varepsilon}$ is tight in $H^{-2\alpha}(\mathbb{T}^2)$.*