LECTURE 2

$\alpha > 0$, stationary O.U. process:

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} \, dB_s \quad (B \text{ a BM})$$

Covariance $\quad \mathbb{E}(X_t X_{t'}) = \dfrac{e^{-\alpha|t-t'|}}{2\alpha}$

$t, t' \in \mathbb{R}$.

Has explained that the law of this process satisfies REFLECTION POSITIVITY:

for all $t_1, \dots, t_k > 0$ and $c_1, \dots, c_k \in \mathbb{C}$

$$\mathbb{E}\left(\left(\sum_k \overline{c_k} X_{-t_k} \right) \left(\sum_{k'} c_{k'} X_{t_{k'}} \right) \right) =$$

$$= \sum_{k,k'} \overline{c_k} \, c_{k'} \, \frac{e^{-\alpha(t_k' + t_k)}}{2\alpha}$$

$$= \frac{1}{2\alpha} \left| \sum_k c_k e^{-\alpha t_k} \right|^2 \geq 0.$$

The aim of this lecture is to construct a similar object in higher dimension (say $t \in \mathbb{R}^d$).

Let us note that

$i \in \mathbb{C}$
$i^2 = -1$

$$\frac{e^{-\alpha|t|}}{2\alpha} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\omega t}}{\omega^2 + \alpha^2} \, d\omega$$

(characteristic function of a Cauchy r.v.)

Let me remind some minimal facts about Fourier transform.

If $f \in L^1(\mathbb{R}^d; \mathbb{C})$ then we can define $\hat{f} : \mathbb{R}^d \to \mathbb{C}$

$$\hat{f}(k) := \int_{\mathbb{R}^d} e^{i\langle k, x\rangle} f(x)\, dx$$

probabilistic convention

Properties:

1) $\hat{f} \in C_0(\mathbb{R}^d; \mathbb{C})$ (continuous and vanishing at ∞)

2) $\widehat{f * g} = \hat{f}\,\hat{g}$ where $f * g \in L^1$; convolution

$$f * g\,(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$$

3) if $\hat{f} \in L^1$ then

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle k, x\rangle} \hat{f}(k)\, dk$$

4) if $f, g \in L^2(\mathbb{R}^d; \mathbb{C})$ then

$\hat{f}, \hat{g} \in L^2(\mathbb{R}^d; \mathbb{C})$ and

$$\langle f, g \rangle = \int_{\mathbb{R}^d} f(x) \overline{g(x)} \, dx = \frac{1}{(2\pi)^d} \langle \hat{f}, \hat{g} \rangle$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^n} \widehat{f(k)} \overline{\widehat{g(k)}} \, dk .$$

5) $\widehat{\partial_j f}(\omega) = -i \omega_j \hat{f}(\omega)$ (integration by parts)

Now: if $X \sim N(0, Q)$ on $\mathbb{R}^n$, (covariance matrix)

with $\det Q > 0$, then

$$N(0, Q)(dx) = \frac{1}{Z} \exp\left(-\frac{1}{2} \langle Q^{-1} x, x \rangle \right) dx$$

(Leb on $\mathbb{R}^n$)

For the O.U. process the covariance "matrix" is the integration operator

$$Qx(t) = \int_{\mathbb{R}} \frac{e^{-\alpha|t-t'|}}{2\alpha} x(t')\,dt'$$ (convolution)

Can we compute (even formally)

$\langle Q^{-1}x, x\rangle$? We have

$$\langle Q^{-1}x, x\rangle = \frac{1}{2\pi} \langle \widehat{Q^{-1}x}, \hat{x}\rangle$$

$$\widehat{Qx} = \widehat{\frac{e^{-\alpha|\cdot|}}{2\alpha}} \cdot \hat{x}$$

Since:

$$\frac{e^{-\alpha|t|}}{2\alpha} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-i\omega t}}{\omega^2+\alpha^2}\,d\omega$$

$$\Longrightarrow \widehat{\frac{e^{-\alpha|\cdot|}}{2\alpha}}(\omega) = \frac{1}{\omega^2+\alpha^2}$$

We obtain:

$$\widehat{Qx}(\omega) = \frac{\hat{x}(\omega)}{\omega^2+\alpha^2} \Rightarrow \widehat{Q^{-1}x}(\omega) = (\omega^2+\alpha^2)\,\hat{x}(\omega)$$

So that

$$\langle Q^{-1}x, x \rangle = \frac{1}{2\pi} \langle \widehat{Q^{-1}x}, \hat{x} \rangle$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} (\omega^2+\alpha^2) \, |\hat{x}(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left(|\widehat{\partial x}|^2 + \alpha^2 |\hat{x}|^2 \right) d\omega$$

$$= \int_{\mathbb{R}} \left(|\partial_t x|^2 + \alpha^2 |x_t|^2 \right) dt$$

Law of $(X_t)_{t\in\mathbb{R}}$: "$\frac{1}{Z} \exp\left(-\frac{1}{2} \int_{\mathbb{R}} \left(|\partial_t x|^2 + \alpha^2 |x_t|^2 \right) dt \right) dx$"

More precisely: $X(f) := \int_{\mathbb{R}} X_t f_t \, dt$

$$\Rightarrow \quad \mathbb{E}\left(X(f) X(g)\right) = \qquad f, g \in C_c^\infty(\mathbb{R}^d)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-\alpha |t-t'|}}{2\alpha} f(t) \, g(t') \, dt \, dt'$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i\omega(t-t')}}{\omega^2 + \alpha^2} \, d\omega \; f(t) g(t') \, dt \, dt'$$

$$= \int_{\mathbb{R}} \frac{1}{2\pi} \frac{1}{\omega^2 + \alpha^2} \hat{f}(\omega) \overline{\hat{g}(\omega)} \, d\omega$$

On $\mathbb{R}^d$: we want a process $\left(X(f)\right)_{f \in C_c(\mathbb{R}^d)}$

s.t. $\mathbb{E}\left(X(f) X(g)\right) =$

$$= \int_{\mathbb{R}^d} \frac{1}{2\pi} \frac{1}{|k|^2 + m^2} \hat{f}(k) \overline{\hat{g}(k)} \, dk .$$

GFF with $m > 0$

The Schwinger functions are

$$S_n(x_1, \dots x_n) = \mathbb{E}\left(X(\delta_{x_1}) \dots X(\delta_{x_n})\right)$$

$x_i \neq x_j,\ i \neq j$

Since the process is Gaussian, it is enough to study the 2-point function:

$$S_2(x, y) = \mathbb{E}\left(X(\delta_x)\, X(\delta_y)\right)$$

$$= \int_{\mathbb{R}^d} \frac{1}{2\pi} \frac{1}{|k|^2 + m^2} e^{i\langle k, x-y\rangle} dk$$

For $d \geq 2$, $S_2(x, x) = +\infty$, but

$$|S_2(x, y)| < +\infty \quad \forall x \neq y$$

Exercise: $\Delta_k e^{i\langle k, x\rangle} = -|x|^2 e^{i\langle k, x\rangle}$

$$\Rightarrow e^{i\langle k, x\rangle} = -\frac{1}{|x|^2} \Delta_k e^{i\langle k, x\rangle},$$

substitute in S_2 and integrate by parts.

An important remark is that this Schwinger function satisfies Euclidean covariance: $\forall R \in SO(d)$, $b \in \mathbb{R}^d$

$$S_2(Rx+b, Ry+b) = S_2(x,y).$$

Moreover we're going to see that a form of reflection positivity still holds.

Let us try to construct this process. We need the notion of white noise.

We consider $H = L^2(\mathbb{R}^d)$ and a c.o.s. $(e_n)_{n \in \mathbb{N}} \subseteq H$:

$\langle e_i, e_j \rangle_H = \delta_{i,j}$, $h = \sum_n \langle e_n, h \rangle e_n$

$\forall h \in H$.

given a i.i.d. sequence

$(Z_n)_{n \in \mathbb{N}}$ of $N(0,1)$-r.v.s , we set

$$W(h) := \sum_n Z_n \langle e_n, h \rangle.$$

Then $(W(h))_{h \in H}$ is a centered

Gaussian process s.t.

$$\mathbb{E}(W(h) W(h')) = \sum_n \langle e_n, h \rangle \langle e_n, h' \rangle$$

$$= \langle h, h' \rangle$$

$$= \int_{\mathbb{R}^d} h(x) h'(x) \, dx.$$

It has become common to use the notation $W(h) = \int_{\mathbb{R}^d} h(x) \underbrace{\xi(x)}_{} \, dx$

ξ = white noise on $\mathbb{R}^d$

We define for $d = n+1$, $f \in C_c^\infty(\mathbb{R}^n)$, $t \in \mathbb{R}$

$$X_t(f) :=$$

(two independent white noises: ξ^1, ξ^2)

$$= \mathrm{Re}\left[\int_{\mathbb{R}^n \times \mathbb{R}} \hat{f}(k) \, \mathbb{1}_{(s \le t)} \, e^{-(t-s)(|k|^2+m^2)} \left(\xi^1 + i \xi^2\right)(ds, dk) \right]$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}} \mathbb{1}_{(s \le t)} \, e^{-(t-s)(|k|^2+m^2)} \left[\mathrm{Re}\, \hat{f}(k) \, \xi^1 - \mathrm{Im}\, \hat{f}(k) \, \xi^2 \right](ds, dk)$$

$$\mathbb{E}\left(X_t(f) X_{t'}(g) \right) \overset{[t \ge t']}{=}$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}} \mathbb{1}_{(s \le t')} \, e^{-(t+t'-2s)(|k|^2+m^2)} \, \mathrm{Re}\left[\hat{f}(k) \overline{\hat{g}(k)} \right] ds \, dk$$

$$= \mathrm{Re} \int_{\mathbb{R}^n} \frac{1}{2} \frac{e^{-|t-t'|(|k|^2+m^2)}}{|k|^2+m^2} \hat{f}(k) \overline{\hat{g}(k)} \, dk .$$

If $F = F(t, x)$

$$\mathbb{E}\left(X(F)\,X(G)\right) = \iint dt\, dt'\ \mathbb{E}\left(X_t(F(t,\cdot))\, X_{t'}(F(t',\cdot))\right)$$

$$= \mathrm{Re} \iint dt\, dt' \int_{\mathbb{R}^n} \frac{1}{2} \frac{e^{-|t-t'|(|k|^2+m^2)}}{|k|^2+m^2} \widehat{F(t,\cdot)}\, \overline{\widehat{G(t,\cdot)}}(k)\, dk$$

$\bar{k} = (\omega, k)$, $z = (t, x)$

$$= \mathrm{Re} \frac{1}{2\pi} \int_{\mathbb{R}^{n+1}} \frac{e^{i\langle z, \bar{k}\rangle}}{\underbrace{|\bar{k}|^2}_{|k|^2+|\omega|^2} + m^2} \hat{F}(\bar{k})\, \overline{\hat{G}(\bar{k})}\, d\bar{k}$$

and we can remove the real part since we know that this quantity is real.