Distributions

Ref: Wikipedia, Max's notes

Formalize the notion of an object which is pointwise ill-defined, yet "averages" are well-defined.

Notation. Given multi-index
$$\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d,$$
let

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} .$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

Natural space of test functions: Schwartz space

$$\mathcal{S} = \mathcal{S}(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d, \mathbb{C}) : \forall \alpha, \beta \in \mathbb{N}^d, \ \|f\|_{\alpha,\beta} < \infty \},$$

where

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| .$$

In words, f and all of its derivatives decreases faster than any polynomial.

Examples: $C_c^\infty(\mathbb{R}^d)$, $x^\alpha e^{-|x|^2}$.

Topology on $\mathcal{S}$ characterized by:

$$f_k \to f \quad \text{if}$$

$$\|f_k - f\|_{\alpha,\beta} \to 0 \quad \forall\, \alpha, \beta.$$

This makes $\mathcal{S}$ into a Frechét space (a complete metrizable TVS whose topology comes from countable family of seminorms).

Def. A tempered distribution is a continuous linear functional

$$\phi : \mathcal{S} \to \mathbb{C}.$$

The TVS of tempered distributions is denoted $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^d)$.

Ex. Dirac delta $\delta_0(f) = f(0)$. Continuous because if

$$f_k \to f \quad \text{in} \quad \mathcal{S},$$

then

$$\delta_0(f_k) = f_k(0) \to f(0) = \delta_0(f).$$

Derivative of BM on $\mathbb{R}$. Let B be a standard BM,

$$\xi \text{ "=" } \frac{dB}{dt},$$

which means

$$\xi(f) = \int f \, dB$$

One can then show that a.s., this is continuous, and thus is a tempered distribution.

Fourier transform

For $f \in L^1(\mathbb{R}^d, \mathbb{C})$, define

$$\mathcal{F}f(p) := \int_{\mathbb{R}^d} dx\, f(x)\, e^{-2\pi i p\cdot x},$$

$p \in \mathbb{R}^d$.

Define also inverse Fourier transform

$$\mathcal{F}^{-1}f(x) := \int_{\mathbb{R}^n} dp\, f(p)\, e^{2\pi i p\cdot x}$$

Lemma. If $f \in \mathcal{S}$, then $\mathcal{F}f, \mathcal{F}^{-1}f \in \mathcal{S}$, and

$$\mathcal{F}\mathcal{F}^{-1}f = \mathcal{F}^{-1}\mathcal{F}f = f.$$

Lemma. $\mathcal{F}\partial^\alpha f(p) = (2\pi i)^{|\alpha|} p^\alpha\, \mathcal{F}f(p)$.

Corollary. $\mathcal{F}\Delta f(p) = -4\pi^2 |p|^2 \mathcal{F}f(p)$.

Plancherel: for $f, g \in \mathcal{S}$,

$$(f, g) = (\mathcal{F}f, \mathcal{F}g),$$

where

$$(f, g) = \int_{\mathbb{R}^d} f(x)\,\overline{g(x)}\,dx.$$

(Since $\mathcal{S} \subseteq L^2$ is dense, this allows to extend $\mathcal{F}$ to L^2.)

Lemma. $\widehat{f * g} = \hat{f}\,\hat{g}$.

For $f \in L^1(\mathbb{T}^d, \mathbb{C})$, define

$$\mathcal{F}f(n) := \hat{f}(n) := \int_{\mathbb{T}^d} dx \, f(x) e^{-2\pi i n \cdot x}, \quad n \in \mathbb{Z}^d.$$

Then (Fourier inversion)

$$f = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e_n,$$

where $e_n(x) = e^{2\pi i n \cdot x}$.

If $f \in C^\infty(\mathbb{T}^d, \mathbb{C})$, then series converges ptwise unif., if $f \in L^2(\mathbb{T}^d, \mathbb{C})$, then series converges in L^2.

Plancherel:

$$\int_{\mathbb{T}^d} dx\, f(x)\, \overline{g(x)} = \sum_{n \in \mathbb{Z}^d} \hat{f}(n)\, \overline{\hat{g}(n)} .$$

Fourier transform on the lattice

As in Max's notes,

$$\varepsilon = 2^{-N}, \quad M = 2^{N'},$$

$$\Lambda_\varepsilon = (\varepsilon \mathbb{Z})^d$$

$$\Lambda_{\varepsilon, M} = \Lambda_\varepsilon \cap \mathbb{T}_M^d$$

$$= (\varepsilon \mathbb{Z})^d \cap \left[-\frac{M}{2}, \frac{M}{2}\right)^d$$

periodic boundary

Let $\hat{\Lambda}_\varepsilon = (\varepsilon^{-1}[-\frac{1}{2},\frac{1}{2}))^d$ be the dual of Λ_ε.

For $f : \Lambda_\varepsilon \to \mathbb{C}$, $g : \hat{\Lambda}_\varepsilon \to \mathbb{C}$,

$$\mathcal{F}_\varepsilon f(k) = \varepsilon^d \sum_{x \in \Lambda_\varepsilon} f(x) e^{-2\pi i k \cdot x}$$

$$\mathcal{F}_\varepsilon^{-1} g(x) = \int_{\hat{\Lambda}_\varepsilon} g(k) e^{2\pi i k \cdot x} \, dk,$$

Remark. The ε^{-1} factor in $\hat{\Lambda}_\varepsilon$ arises because if

$$k, k' \in \mathbb{R}^d, \quad k - k' \in \varepsilon^{-1}\mathbb{Z}^d,$$

then

$$x \mapsto e^{2\pi i k \cdot x}, \quad x \mapsto e^{2\pi i k' \cdot x}$$

define the same function.

Let $\hat{\Delta}_{\varepsilon,M} = \left((M^{-1}\mathbb{Z}) \cap \varepsilon^{-1}[-\tfrac{1}{2}, \tfrac{1}{2}) \right)^d.$

Remark. Note that if $k \notin M^{-1}\mathbb{Z}$, then

$$x \mapsto e^{2\pi i k \cdot x}$$

is not periodic in x.

For $f : \Delta_{\varepsilon,M} \to \mathbb{C}$, $g : \hat{\Delta}_{\varepsilon,M} \to \mathbb{C}$, define

$$\mathcal{F}_{\varepsilon,M} f(k) = \varepsilon^d \sum_{x \in \Delta_{\varepsilon,M}} f(x) e^{-2\pi i k \cdot x}$$

$$\mathcal{F}^{-1}_{\varepsilon,M} g(x) = \frac{1}{M^d} \sum_{k \in \hat{\Delta}_{\varepsilon,M}} g(k) e^{2\pi i k \cdot x}$$

Now, for $f : \Lambda_\varepsilon \to \mathbb{C}$, define

$$\nabla_\varepsilon^i f(x) = \frac{f(x+\varepsilon e_i) - f(x)}{\varepsilon},$$

$$\nabla_\varepsilon^{-,i} f(x) = \frac{f(x) - f(x-\varepsilon e_i)}{\varepsilon},$$

$$\Delta_\varepsilon f(x) = \sum_{i=1}^{d} \nabla_\varepsilon^{-,i} \nabla_\varepsilon^i f(x)$$

$$= \varepsilon^{-2} \sum_{i=1}^{d} \left(f(x+\varepsilon e_i) + f(x-\varepsilon e_i) - 2f(x) \right).$$

Have that

$$\nabla_\varepsilon^j f(x) = \int_{\Lambda_\varepsilon^*} dk\, \mathcal{F}_\varepsilon f(k)\, \frac{e^{2\pi i \varepsilon k_j} - 1}{\varepsilon} e^{2\pi i k \cdot x},$$

$$\Delta_\varepsilon f(x) = -\int_{\Lambda_\varepsilon^1} dk \; \mathcal{F}f(k) \sum_{j=1}^{d} (2\varepsilon^{-1}\sin(\pi\varepsilon k_j))^2 \; e^{2\pi i k\cdot x}$$

For $m^2 > 0$, have that

$$(m^2 - \Delta_\varepsilon)^{-1}(x,y)$$

$$= \frac{1}{M^d} \sum_{k \in (M^{-1}\mathbb{Z} \cap \varepsilon^{-1}[-\frac{1}{2},\frac{1}{2}))^d} e^{ik\cdot(x-y)} \times \frac{1}{m^2 + \sum_{i=1}^{d} (2\varepsilon^{-1}\sin(\pi\varepsilon k_i))^2}$$

Appendix.

How to remember the formal identity

$$\int_{\mathbb{R}^d} dp \; e^{i2\pi p\cdot x} = \delta_0(x) \;?$$

Or, equivalently,

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dp \; e^{ip\cdot x} = \delta_0(x).$$

Let K_t be the heat kernel on $\mathbb{R}^d$, i.e.

$$K_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}}\exp\left(-\frac{|x|^2}{2t}\right).$$

We have that

$$K_t \to \delta_0 \quad \text{as} \quad t \to 0.$$

Want to find $g_t(p)$ st

$$K_t(x) = \int_{\mathbb{R}^d} dp \, g_t(p) e^{ip \cdot x}.$$

Then take $\lim_{t \downarrow 0} g_t$ to obtain formal identity for δ_0. We compute

$$\left(\frac{t}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} dp \, \exp\left(-\frac{t}{2}|p|^2\right) e^{ip \cdot x}$$

$$= \exp\left(\frac{-|x|^2}{2t}\right).$$

(Characteristic fn of $N(0, \frac{1}{t})$.)

Thus,

$$K_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{2t}\right)$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} dp \, \exp\left(-\frac{t}{2}|p|^2\right) e^{ip\cdot x}$$

Taking $t \to 0$, obtain the desired formal identity.