LECTURE 3

Wrap up messages:

1) Stochastic quantization means the following: given a probability measure μ (complicated, interesting), we want to find a Gaussian measure γ and a function F s.t. $F_* \gamma = \mu$, or in other words

$$\mu(A) = \gamma(F \in A) \quad \forall A$$

2) Yesterday we defined the GFF on $\mathbb{R}^d$ with mass $m > 0$ as the

centered Gaussian random field $(X(f))_{f\in C_c^\infty(\mathbb{R}^d)}$ with covariance

$$\mathbb{E}\left(X(f)X(g)\right) = \frac{1}{2\pi}\int_{\mathbb{R}^d} \frac{1}{|k|^2+m^2} \hat{f}(k)\overline{\hat{g}(k)}\,dk$$

The formal density is

$$(X(\delta_x))_{x\in\mathbb{R}^d} \sim \text{"}\frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}(m^2\phi^2+|\nabla\phi|^2)\,dx\right)d\phi\text{"}$$

This is the free model, meaning without interaction. If we want to add a potential $V:\mathbb{R}\to\mathbb{R}$ to the system, then we would like to define a measure with formal density

$$\frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{R}^d}(m^2\phi^2+|\nabla\phi|^2)\,dx - \int_{\mathbb{R}^d} V(\phi(x))\,dx\right)d\phi$$

but since a.s. $(X(\delta_x))_{x\in\mathbb{R}^d}$ is a distribution, then $V(X(\delta_x))$ "$V(\phi(x))$" is ill-defined as soon as V is not a polynomial of degree 2.

3) we need $m>0$ because otherwise

$$\int_{[-1,1]^d} \frac{1}{|k|^2}\,dk = +\infty \quad \text{if } d\in\{1,2\}$$

(infrared divergence)

The (more serious) divergence

$$\int_{\mathbb{R}^d} \frac{dk}{m^2+|k|^2} = +\infty \quad \text{if } d\geq 2$$

is called ultraviolet divergence.

In x-space, the former is related to large x behavior, the latter to small scale behavior.

The O.U. process satisfies REFLECTION POSITIVITY:

$\forall c_k \in \mathbb{C}, \ \lambda_k \in \mathbb{R}, \ t_k \geq 0$:

$$F(x) := \sum_k c_k e^{i\lambda_k x_{t_k}}$$

$$\Theta F(x) := F(X_{-\cdot}) = \sum_k c_k e^{i\lambda_k X_{-t_k}}$$

$$\Rightarrow \quad \mathbb{E}\left(\overline{\Theta F}\, F\right) \geq 0.$$

It can be seen (Glimm-Jaffe, Thm 6.2.2) that this is equivalent to the weaker property

$$\sum_{k,k'} \overline{c_{k'}}\, c_k\, \mathbb{E}(X_{t_k} X_{-t_{k'}}) \geq 0$$

$\forall c_k \in \mathbb{C}, \ t_k \geq 0.$

$$\downarrow = \sum_{k,k'} \overline{c_{k'}}\, c_k \frac{e^{-\alpha(t_k+t_{k'})}}{2\alpha} =$$

$$= \frac{1}{2\alpha}\left| \sum_k c_k e^{-\alpha t_k} \right|^2$$

<u>Thm</u> The GFF with mass $m>0$ satisfies a form of reflection positivity.

Moreover it satisfies euclidean invariance:

$$S_2(Rx+b, Ry+b) = S_2(x,y) \quad \forall R \in SO(d),\ b \in \mathbb{R}^d$$

Recall that $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}$

space time

Euclidean-invariance is related to Poincaré (relativistic) invariance in Minkowski space.

Let us go back to the problem of adding an interaction. Given $V : \mathbb{R} \to \mathbb{R}$, we would like to define

$$\frac{1}{Z} \exp\left(-\int_{\mathbb{R}^d} V(\phi_x)\,dx\right) \mathbb{P}(d\phi)$$

where $\mathbb{P}$ = law of the GFF $(m>0)$.

There are two problems:

1) ϕ is a distribution, $V(\phi_x)$ is ill-defined

2) even if we regularize ϕ, the integral $\int_{\mathbb{R}^d} V(\phi_x)\,dx$ may diverge.

For this, we introduce two "regularizations":

1) instead of $\mathbb{R}^d$, we consider a lattice $\varepsilon \mathbb{Z}^d$, $\varepsilon > 0$.

2) We eliminate large-scale problems by restricting further to

$$\left[-\frac{M}{2}, \frac{M}{2}\right[\cap \varepsilon \mathbb{Z}^d =: \Lambda_{\varepsilon, M}$$

with periodic boundary conditions

Now we can define a discrete GFF $(m > 0)$ on $\Lambda_{\varepsilon, M}$:

a Gaussian process $\left(X^{\varepsilon, M}(x)\right)_{x \in \Lambda_{\varepsilon, M}}$

which is centered and satisfies

$$\mathbb{E}\left(X^{\varepsilon,M}(x)\, X^{\varepsilon,M}(y)\right) = (m^2 - \Delta_\varepsilon)^{-1}(x,y)$$

$$x,y \in \Lambda_{\varepsilon,M}$$

(see Sky's or Max's lectures).

The law of $\left(X^{\varepsilon,M}(x)\right)_{x\in\Lambda_{\varepsilon,M}}$ is a measure on $\mathbb{R}^{\Lambda_{\varepsilon,M}}$ which finite-dimensional. The density is

$$\nu^{\varepsilon,M}(d\phi) := \frac{1}{Z_{\varepsilon,M}} \exp\left[-\frac{1}{2}\varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} \left[(m^2-\Delta_\varepsilon)\phi\right](x)\phi(x) - \varepsilon^d \sum_{x\in\Lambda_{\varepsilon,M}} V(\phi(x))\right] \prod_{x\in\Lambda_{\varepsilon,M}} d\phi_x$$

and this time it is perfectly well defined for $M,\varepsilon \in \,]0,\infty[$.

Then, one wants to find non-Gaussian limits as $\varepsilon_K \to 0$, $M_K \to +\infty$, and prove that such limits satisfy RP, Euclidean invariance, etc. so that it is possible to apply the Osterwalder Schrader Thm and reconstruct the associated QFT.

In principle, any V is fine, so long as it works!

So far:

1) if $d=1$, any $V: \mathbb{R} \to \mathbb{R}$ (continuous bounded below) works

2) if $d=2$, V can be a polynomial, exponential, trigonometric function

3) if $d=3$, V can only be a polynomial of degree 4 and bounded below, with coefficients depending on ε, M (the model is called ϕ^4_3)

4) if $d=4$, ϕ^4_4 has been proven recently (Aizenman - Duminil Copin) to have only Gaussian (i.e. physically trivial) limits.

NOW RECALL:

Stochastic quantization means the following: given a probability measure μ (complicated, interesting), we want to find a Gaussian measure γ and a function F

s.t. $F_* \gamma = \mu$, or in other words

$$\mu(A) = \gamma(F \in A) \quad \forall A.$$

Now, these lectures concentrate on the following choice: $\mu = \nu^{\varepsilon,M}$, γ the law of a white noise ξ on $\Lambda_{\varepsilon,M} \times \mathbb{R}$, and $F(\xi) := \phi_0$ where $(\phi_t)_{t\in\mathbb{R}}$ is the stationary solution to

$$d\phi_t(x) = (m^2 - \Delta_\varepsilon)\phi_t(x)dt - V'(\phi(x))dt$$
$$+ \xi(dt, x), \qquad t\in\mathbb{R}, x \in \Lambda_{\varepsilon,M}.$$

Here t is a fictitious time, it is NOT the physical time (which is already contained in $x \in \mathbb{R}^d$).

Before showing how one can handle the limits $\varepsilon \to 0$, $M \to \infty$, we need some preliminary results for fixed ε, M.

Thm. The SDE above has a unique solution $(\phi_t)_{t \geq 0}$ for any $\phi_0 \in \mathbb{R}^{\Lambda_{\varepsilon,M}}$ (existence for all times). Moreover $\nu^{\varepsilon,M}$ is the only invariant measure (namely s.t. if $\phi_0 \sim \nu^{\varepsilon,M}$, $\phi_t \sim \nu^{\varepsilon,M}$ $\forall t \geq 0$).

Finally, given $(\phi_t)_{t\geq s}$ a solution s.t. $\phi_s \in \Lambda^{\varepsilon,M}$ is fixed (say $\phi_s \equiv 0$)

as $s \to -\infty$ the process converges

to a stationary $(\phi_t)_{t\in\mathbb{R}}$ with

$\phi_t \sim \nu^{\varepsilon,M} \quad \forall t\in\mathbb{R}$.

We focus on ϕ^4

$$V(\phi) = \frac{\lambda}{4}\phi^4 + \frac{\beta}{2}\phi^2, \quad \lambda>0, \ \beta\in\mathbb{R}$$

$$d\phi_t = -\left[(m^2-\Delta_\varepsilon)\phi_t + \lambda\phi_t^3 + \beta\phi_t\right]dt + dW(t,x)$$

Problem: $\phi \mapsto \phi^3$ is not Lipschitz, the solution could explode in finite time. However $\lambda > 0$ helps:

by Itô's formula

$$d\,\phi_t^2(x) = -2\Big[(m^2 - \Delta_\varepsilon)\phi_t + \lambda\phi_t^3 + \beta\phi_t\Big](x)\,\phi_t(x)\,dt$$

$$+ (\cdots)\,dW_t(x) + dt$$

summation by parts:

$$\sum_x \Delta_\varepsilon \phi(x)\phi(x) = -\sum_{x\sim y} |\phi(x) - \phi(y)|^2$$

$$\Rightarrow \underbrace{\mathbb{E}\Big(\sum_{x\in\Lambda_{\varepsilon,M}} \phi_t^2(x)\Big)}_{u(t)} \leq \qquad t \leq T$$

$$\leq u(0) + |\Lambda_{\varepsilon,M}|\,T + |\beta| \int_0^t u(s)\,ds$$

By Gronwall:

$$u(t) \leq \Big(u(0) + |\Lambda_{\varepsilon,M}|\,T\Big)e^{|\beta|T} \quad \forall t \leq T$$

However this estimate is bad when

$\varepsilon \to 0$, $M \to +\infty$.

A second (more robust) argument:

the Da Prato - Debussche trick.

We write

$$dX = (AX - V'(X))dt + dW$$

$$dY = AY\,dt + dW \qquad A := m^2 - \Delta_\varepsilon$$

$$Z := X - Y$$

$$\Rightarrow \quad \dot{Z} = AZ - V'(Z+Y)$$

(a random ODE, not a SDE)

Then

$$\frac{1}{2}\frac{d}{dt}\sum_x Z_t^2(x) + \underbrace{\sum_x Z_t(m^2 - \Delta_\varepsilon)Z_t + \lambda \sum_x Z_t^4(x)}_{=: G(Z_t) \geq 0}$$

$$= \sum_x \lambda \left(Y^3 Z + 3 Y^2 Z^2 + 3 Y Z^3\right)$$

$$+ \beta \sum_x \left(Y Z + Z^2\right)$$

claim

$$\leq C_\delta \|Y_t\|_{L^4}^4 + \delta G(Z_t), \quad \text{e.g. } \delta = \frac{1}{2}$$

by the Young inequality:

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

$$p = \frac{4}{3}, \quad q = 4$$

so for example:

$$\sum_x Y^3(x) Z(x) \leq C_\delta \sum_x Y^4(x) + \delta \sum_x Z^4(x)$$

$$\Rightarrow \|Z_t\|_{L^2}^2 + \frac{1}{2} \int_0^t G(Z_s)\,ds$$

$$\leq \|Z_0\|_{L^2}^2 + \overbrace{C_{\frac{1}{2}} \int_0^t \|Y_s\|_{L^4}^4\,ds}^{\text{finite a.s.}}$$