Lecture 4 : Infinite volume limit ($M \to \infty$)

Stationary coupling:

Below is an important thing which we will use later:

$$X = Y + Z$$

We want (X, Y) to be stationary (as a pair)

from which we know Z is stationary

(so far we could only make X and Y separately stationary)

Let $X_0 \sim \nu$ $Y_0 \sim \mu$ indep

Stationary coupling of Y and Z ("Krylov-Bogoliubov")

Define a measure γ_T on $\{(\varphi, \psi) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda\}$ by

$$\int f(\varphi, \psi)\, d\gamma_T(\varphi, \psi) = \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s, Z_s)]\, ds$$

$Z_s = X_s - Y_s$ stationary

$Z_0 = X_0 - Y_0$

We have

$$\int G(\psi) + \|\varphi\|_{L^4}^4 \, d\gamma_T(\varphi, \psi)$$

$$= \frac{1}{T}\int_0^T \mathbb{E}\left[G(Z_s) + \|Y_s\|_{L^4}^4\right] ds$$

(using above bound)

$$\leq \frac{2}{T}\left(\mathbb{E}\|Z_0\|_{\ell^2}^2 + C'\int_0^T \mathbb{E}\|Y_s\|_{\ell^4}^4\, ds\right)$$

(using Y stationary)

$$= \frac{2}{T}\mathbb{E}\|Z_0\|_{\ell^2}^2 + 2C'\mathbb{E}\|Y_0\|_{\ell^4}^4$$

$\leq$ const uniformly in $T \to \infty$

$\Rightarrow (\gamma_T)_T$ is tight. $\Rightarrow \exists$ weakly convergent subseq $\to \gamma$

TA session: tightness,

Prokhorov theorem (tight $\Rightarrow$ weakly relative compact)

DaPrato §6.

Note that

$$\int f(\varphi+\psi)\, d\gamma_T(\varphi,\psi) = \frac{1}{T}\int_0^T \mathbb{E}[f(Y_s+Z_s)]\,ds$$

$$= \frac{1}{T}\int_0^T \mathbb{E}[f(X_s)]\,ds = \mathbb{E}[f(X_0)]$$ (using X stationary)

So Law$(\varphi+\psi)$ under γ_T is given by $\nu = $ law(X_0) $\forall T$

So Law$(\varphi+\psi)$ under γ is given by $\nu = $ law(X_0) (1)

Claim: If $(Y_0, Z_0) \sim \gamma$ then $(Y_t, Z_t) \sim \gamma$ $\forall t$

Namely the pair (Y_t, Z_t) has invariant meas γ

Combined with (1), $X_t = Y_t + Z_t \sim \nu$ (ν is inv meas of X)

The claim is general i.e. Krylov-Bogoliubov theorem

TA session if no time. (Da Prato "Intro to infinite dim ..." P94)

Pf of KB Thm: By tight, $\lim_{n\to\infty}\int f(x)\,\gamma_{T_n}(dx) = \int f(x)\gamma(dx)$ $\forall f\in C_b$

By def of γ_T, LHS $= \lim_{n\to\infty}\frac{1}{T_n}\int_0^{T_n} P_t f(x_0)\, dt$

Should average x_0 over $\mu\times\nu$ but this is not important

Write $f = P_s g \in C_b$ (Feller)

$$\text{LHS} = \lim_n \frac{1}{T_n}\int_0^{T_n} P_{t+s}\, g(x_0)\,dt = \lim_n \frac{1}{T_n}\int_s^{T_n+s} P_t = \cdots \to \int g(x)\,\gamma(dx)$$

Infinite volume limit:

In above estimates, $\|Z_0\|_{\ell^2(\Lambda_{\varepsilon M})} = \sum_{\Lambda_{\varepsilon M}} Z_0^2 \sim \underline{M^d}$

$\|Y_s\|_{\ell^4(\Lambda_{\varepsilon M})} \sim \underline{M^d}$ as $M \to \infty$

bad

weight $\rho : (\varepsilon\mathbb{Z})^d \to \mathbb{R}$ The larger σ is, it means the weaker control we'll be looking for.

$$\rho(x) = (1 + \ell |x|)^{-\sigma} \qquad \ell, \sigma > 0$$

$|x|$: distance from 0

Extend Z, Y etc on $\Lambda_{\varepsilon, M}$ to $\Lambda_\varepsilon = (\varepsilon\mathbb{Z})^d$ periodically

Multiply equ for Z by $\rho^2 Z$, then sum over Λ_ε:

$$\frac{1}{2}\frac{d}{dt}\sum_{x\in\Lambda_\varepsilon} |\rho(x) Z_t(x)|^2 + G(Z_t)$$

$$\le -\lambda \sum_{x\in\Lambda_\varepsilon} \rho(x)^2 (Y^3 Z + 3Y^2 Z^2 + 3YZ^3)$$

lost of exact sum by parts

$$+ \beta \sum_{x\in\Lambda_\varepsilon} \rho(x)^2 (ZY + Z^2) + \underbrace{C_\rho \sum_{x\in\Lambda_\varepsilon} \rho(x)^2 Z_t(x)^2}_{} \quad \star$$

$$G(\varphi) = \|\rho \nabla \varphi\|^2_{\ell^2(\Lambda_\varepsilon)} + m^2 \|\rho\varphi\|^2_{\ell^2(\Lambda_\varepsilon)} + \lambda \|\rho^{1/2}\varphi\|^4_{\ell^4(\Lambda_\varepsilon)}$$

$$-\sum_x \Delta_\varepsilon Z \cdot \rho^2 \cdot Z = \sum_x \nabla Z \cdot \nabla(\rho^2 Z)$$

$$= \underbrace{\sum_x (\nabla Z)^2 \rho^2}_{\downarrow \; G(Z)} + \sum_x \nabla Z \cdot Z \cdot \nabla(\rho^2)$$

$$\sum_x \nabla Z \cdot Z \cdot \nabla(\rho^2) = \sum_x \nabla(\rho^2)\cdot\rho^{-2}\cdot \rho\nabla Z \cdot \rho Z$$

$$\leq \underbrace{\|\nabla(\rho^2)\cdot\rho^{-2}\|_{L^\infty}}_{\leq C_\rho \text{ (see below)}} \|\rho Z\|_{\ell^2} \|\rho\nabla Z\|_{\ell^2}$$

$$\leq C_\rho \underbrace{\|\rho\nabla Z\|_{\ell^2}^2}_{\searrow G(Z)} + C_\rho \|\rho Z\|_{\ell^2}^2$$

$$\nabla(\rho^2)\cdot\rho^{-2} \approx 2\rho\nabla\rho\cdot\rho^{-2} \approx \rho^{-1}\nabla\rho$$

$$\rho = (1+\ell|x|)^{-\sigma}$$

$$\nabla\rho \approx (1+\ell|x|)^{-\sigma-1}\cdot\ell$$

$$\rho^{-1}\nabla\rho \approx (1+\ell|x|)^{-1}\cdot\ell \lesssim \ell$$

So we have the bound by $C_\rho \approx \ell$ choose ℓ small s.t. C_ρ is small.

So, similarly as before.

$$\frac{d}{dt}\|\rho Z_t\|_{\ell^2(\Lambda_\varepsilon)}^2 + G(Z_t) \leq C_\delta \|\rho^{\frac{1}{2}} Y_t\|_{L^4(\Lambda_\varepsilon)}^4 + \delta G(Z_t)$$

For instance:

$$\sum_{x\in\Lambda_\varepsilon} \rho(x)^2 Y(x)^3 Z(x) = \sum_{x\in\Lambda_\varepsilon} \rho(x)^{\frac{3}{2}} Y(x)^3 \cdot \rho(x)^{\frac{1}{2}} Z(x)$$

$$\leq C_\delta \|\rho^{\frac{1}{2}} Y\|_{\ell^4}^4 + \underbrace{\delta\|\rho^{\frac{1}{2}} Z\|_{\ell^4}^4}_{\leq G(Z)}$$

Here, one might want to try Gronwall ineq. e.g

$$\frac{d}{dt}\|\rho Z\|_{\ell^2}^2 \lesssim \|\rho Z\|_{\ell^2}^2$$

but we only have 2 powers of ρ and Y also needs some weight.

I think RHS should also have $C_\delta \|\rho Y\|_{\ell^2}^2$:

Luckily we can absorb it to LHS

$$\sum_{x\in\Lambda_\varepsilon} \rho(x)^2 Z(x) Y(x) = \sum_x \rho Y\cdot \rho Z \le C_\delta \|\rho Y\|_{\ell^2}^2 + \underbrace{\delta\|\rho Z\|_{\ell^2}^2}_{\le G(Z)}$$

but. the rest of argument still works obviously.

Therefore

$$\cancel{\mathbb{E}\|\rho Z_t\|^2_{\ell^2(\Lambda_\varepsilon)}} + \frac{1}{2}\,\mathbb{E}\int_0^t G(Z_s)\,ds \le \cancel{\mathbb{E}\|\rho Z_0\|^2_{\ell^2(\Lambda_\varepsilon)}} + C\int_0^t \mathbb{E}\|\rho^{1/2} Y_s\|^4_{\ell^4(\Lambda_\varepsilon)}\,ds$$

(recalling stationary coupling).

$$\Rightarrow \mathbb{E}\, G(Z_0) \le 2C\,\mathbb{E}\|\rho^{\frac12} Y_0\|^4_{\ell^4(\Lambda_\varepsilon)}$$

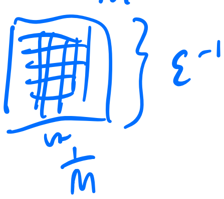
$$= \mathbb{E}\sum_{x\in\Lambda_\varepsilon}\rho(x)^2\,|Y_0(x)|^4 = C\sum_x \rho(x)^2 < \infty$$

(if $\sigma > d$)

i.e sufficient poly decay

$$\mathbb{E}\left[Y_0(x)^4\right]^{1/2} = 3\,\mathbb{E}\left[Y_0(x)^2\right]$$

$$\le \frac{1}{M^d}\sum_{k\in} \frac{1}{m^2+\sum_{i=1}^d\left(2\varepsilon^{-1}\sin(\pi\varepsilon k_i)\right)^2}$$



$$\xrightarrow{M\to\infty} \int_{[-\varepsilon^{-1},\varepsilon^{-1}]^d} \frac{1}{m^2+\sum_{i=1}^d\left(2\varepsilon^{-1}\sin(\pi\varepsilon k_i)\right)^2} < \infty$$

uniform in M

$(m > 0)$

Conclude:

$$\mathbb{E}\, G(Z_0) < \infty \quad \text{uniform in } M.$$

In particular. $\mathbb{E}\|\rho Z_0\|_{\ell^2}^2 < \infty$ uniform in M $\quad (\maltese)$

(Recall measure γ on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$. $\quad \nu^M \sim X_0 \sim \varphi + \psi$
$\varphi \quad \psi$ $\quad$ marginals of γ: Y_0. Z_0)

$$\sup_M \mathbb{E}\|\rho X_0\|_{\ell^2}^2 = \sup_M \mathbb{E}\int \|\rho(\varphi+\psi)\|_{\ell^2}^2 \gamma(d\varphi\, d\psi)$$

$$\leq \dots \leq \underbrace{\sup_M \mathbb{E}\|\rho Y_0\|_{\ell^2}^2}_{<\infty \text{ (same argument as last page)}} + \underbrace{\sup_M \mathbb{E}\|\rho Z_0\|_{\ell^2}^2}_{<\infty \text{ by } (\maltese)}$$

$\Rightarrow (\nu^M)_M$ is tight. $\quad \forall$ cylindrical F

$\int F\, d\nu^M \to \int F\, d\nu$

along subseq. to some ν

($\ell^p_\sigma = \{ f : \|\rho f\|_{\ell^p} < \infty.\ \rho = (1+\ell(x))^{-\sigma} \}$
$\sigma' > \sigma$. $\quad \ell^p_\sigma \hookrightarrow \ell^p_{\sigma'}$ compact embedding)

Uniqueness of ν not guaranteed.