Lecture 5: Converge to equilibrium

Last time we showed
existence of infinite volume ϕ^4 measure
by tightness limit of μ_M as $M \to \infty$

To show uniqueness of infinite volume ϕ^4 measure.
One could argue by the following steps

Step 1: On the whole $\mathbb{Z}^d$

$$dX = (AX - X^3)\,dt + dB \qquad (A = \Delta - m^2)$$

has unique (strong) solution.

(Write its semigroup by P_t)

Step 2: Invariant measure of P_t is unique.
(call it μ)

Step 3: Every tight limit of μ_M must be μ.

Step 1 is "standard":

(e.g. tight limit in $C(\mathbb{R}_+, \mathbb{R}^{\mathbb{Z}^d})$
identify it as solution to the right martingale problem
Pathwise uniqueness by Gronwall type argument)

Step 3: just argue that every tight limit of u_n is an invariant meas of P_t

Step 2 is the "crucial step" and relies on some additional structure of P_t

we will show "convexity" is sufficient

To better understand the role of convexity consider general $dX = (AX - \frac{1}{2} U'(X))dt + dB$

Corresponding to $e^{-\frac{1}{2}\sum (\nabla\phi)^2 + m^2\phi^2 + U(\phi)} \prod_x d\phi_x$

Imagine two solutions Z_1 Z_2.

with initial conditions $Z_1(0)$. $Z_2(0)$

$$\partial_t Z_1 = AZ_1 - \tfrac{1}{2}\mathcal{U}'(X_1)$$
$$\partial_t Z_2 = AZ_2 - \tfrac{1}{2}\mathcal{U}'(X_2)$$

$X_j = Y_j + Z_j$ $j = 1, 2$.

$$\boxed{H = Z_2 - Z_1.}$$

$$\partial_t H - AH = \tfrac{1}{2}\left(\mathcal{U}'(X_1) - \mathcal{U}'(X_2)\right)$$
$$= \tfrac{1}{2}\left(\mathcal{U}'(X_1) - \mathcal{U}'(X_1 + H + K)\right) \overset{\text{def}}{=} Q$$

where $\boxed{K = Y_2 - Y_1}$

$$Q = -\frac{1}{2}\int_0^1 d\tau\ \mathcal{U}''(X_1 + \tau(H+K))(H+K)$$

$$\boxed{\text{Assume: } \quad \mathcal{U}''(\varphi) \geq -2\chi \quad \text{for some } \chi \in \mathbb{R}_+}$$

e.g. $\mathcal{U}(\varphi) = \beta\varphi^2 + \lambda\varphi^4$ $\lambda > 0$

This implies $\varphi \mapsto \mathcal{U}(\varphi) + 2\chi\sum\varphi^2$ is convex.

Let $\boxed{G \overset{\text{def}}{=} \frac{1}{2}\int_0^1 d\tau\ \mathcal{U}''(X_1 + \tau(H+K))} \geq -\chi$

Then $Q = -G(H+K)$

Test equ with $\rho^2 H$ for some ρ

RHS:

$$\sum \rho^2 H \cdot Q = -\sum \rho^2 H \cdot GH - \sum \rho^2 H \cdot GK$$

$$\le C_\delta \|\rho GK\|_{\ell^2}^2 + \delta \|\rho H\|_{\ell^2}^2 + \sum \rho^2 H^2 \cdot \chi$$

$$= C_\delta \|\rho GK\|_{\ell^2}^2 + (\delta + \chi) \|\rho H\|_{\ell^2}^2$$

Take $\delta \le \chi$

$$\frac{1}{2}\partial_t \|\rho H\|_{\ell^2}^2 + \sum \rho^2 H (m^2 - \Delta) H \le C \|\rho GK\|_{\ell^2}^2 + 2\chi \|\rho H\|_{\ell^2}^2$$

Consider $F(t) = e^{ct} \|\rho H\|_{\ell^2}^2$.

Then,

$$\frac{1}{2}\partial_t \left(e^{ct} \|\rho H\|_{\ell^2}^2\right) = \frac{c}{2} e^{ct} \|\rho H\|_{\ell^2}^2 + \frac{1}{2} e^{ct} \partial_t \|\rho H\|_{\ell^2}^2$$

(1) $$\le -e^{ct} \sum \rho^2 H \left(m^2 - 2\chi - \frac{c}{2} - \Delta\right) H + C e^{ct} \|\rho GK\|_{\ell^2}^2$$

Let $\tilde{Q}(t) = C \|\rho G(t) K(t)\|_{\ell^2}^2$

Choose $\rho(x) = e^{-\theta |x|}$

For $\boxed{\varepsilon\theta \le 1}$

$$|\nabla(\rho^2)| \le \cancel{2}\theta \rho^2$$

check!

$$\sum_{\Lambda} \rho^2 H(-\Delta)H = \sum_{\Lambda} \nabla H\, \nabla(\rho^2 H) = \sum_{\Lambda} \nabla H\,(\nabla\rho^2\cdot H + \rho^2\nabla H)$$

$$\geq \sum_{\Lambda} (\nabla H)^2\cdot\rho^2 - 4\theta \underbrace{\sum_{\Lambda} |\nabla H|\cdot|H|\cdot\rho^2}_{\leq C\theta^2\sum_{\Lambda}\rho^2H^2 + \frac{1}{2}\sum_{\Lambda}\rho^2(\nabla H)^2} \quad (2)$$

$$\geq \frac{1}{2}\sum \rho^2(\nabla H)^2 - C\theta^2\sum_{\Lambda}\rho^2H^2$$

Then

$$\frac{1}{2}\partial_t\left(e^{ct}\|\rho H\|_{\ell^2}^2\right) + e^{ct}\left(m^2 - 2\chi - \frac{c}{2} - C\theta^2\right)\sum_{\Lambda}\rho^2H^2$$

$$+ \frac{1}{2}e^{ct}\sum_{\Lambda}\rho^2(\nabla H)^2$$

$$\overset{(2)}{\leq} \frac{1}{2}\partial_t\left(e^{ct}\|\rho H\|_{\ell^2}^2\right) + e^{ct}\sum_{\Lambda}\rho^2 H\left(m^2 - \frac{c}{2} - 2\chi\right)H$$

$$+ e^{ct}\sum_{\Lambda}\rho^2 H(-\Delta)H$$

$$\overset{(1)}{\leq} e^{ct}\,\tilde{Q}(t)$$

Assume $\boxed{m^2 - 2\chi - \frac{c}{2} - C\theta^2 \geq 0}$?

$$\frac{1}{2}\partial_t\left(e^{ct}\|\rho H\|_{\ell^2}^2\right) \leq e^{ct}\,\tilde{Q}(t)$$

$$\Rightarrow \frac{1}{2}e^{ct}\|\rho H_{(t)}\|_{\ell^2}^2 \leq \frac{1}{2}\|\rho H(0)\|_{\ell^2}^2 + \int_0^t e^{cs}\tilde{Q}(s)\,ds$$

$$\Rightarrow \|\rho H(t)\|_{\ell^2}^2 \leq e^{-ct}\|\rho H(0)\|_{\ell^2}^2 + 2\int_0^t e^{-c(t-s)}\tilde{Q}(s)\,ds$$

$(\star)$

Where

$$\tilde{Q}(s) = C\|\rho G(s) K(s)\|_{\ell^2}^2$$

$$= C \sum_{\Lambda} \rho^2 \left(\int_0^1 d\tau \; u''(X_1^{(s)} + \tau(H^{(s)} + K^{(s)}))\right)^2 K(s)^2$$

$$\leq C \int_0^1 \sum_{\Lambda} \rho^2 \cdot u''(X_1(s) + \tau(H(s) + K(s)))^2 \cdot K(s)^2 \, d\tau$$

$$\mathbb{E}\,\tilde{Q}(s) \leq \sum_{\Lambda} \int_0^1 \rho^2 \cdot \mathbb{E}\left[u''(X_1(s) + \tau(H(s) + K(s)))^4\right]^{1/2} \mathbb{E}\left[K(s)^4\right]^{1/2} d\tau$$

need to explain how two processes are coupled...

$$\mathbb{E}\left[\text{poly}(X_1 + \tau(H + K))^4\right] \leq C \quad \text{uniform in } M \to \infty$$

$$\mathbb{E}\,\tilde{Q}(s) \leq \sum_{\Lambda} \rho^2 \cdot \mathbb{E}[K(0)^4]^{1/2} \quad (K \text{ in stationary})$$

By ($\star$) we have:

<u>Lem</u>: Under above assumptions.

$$\mathbb{E}\|\rho H_t\|_{\ell^2}^2 \leq e^{-ct}\,\mathbb{E}\|\rho H(0)\|_{\ell^2}^2 + C \sum_{\Lambda} \rho^2\, \mathbb{E}(K(0)^4)^{1/2}$$

<u>Remark</u>: As $t \to \infty$

If $\sum_{\Lambda} \rho^2\, \mathbb{E}(K(0)^4)^{1/2} \longrightarrow 0$.

We have $\mathbb{E}\|\rho H_t\|_{\ell^2}^2 \longrightarrow 0$

Write P_t : semigroup for dynamic X on $\mathbb{Z}^d$.

suppose, μ, ν are invariant measures of P_t.

One can find $\mu_M \to \mu$

$\nu_M \to \nu$

We have solution X_M to SDE starting from μ_M

Semigroup P_t^M

Similar as step 1. $(X_M)_M$ is tight. limit satisfy SDE on $\mathbb{Z}^d$ with i.c. μ

By Step 1. $X_M \to X$

So.
$$\int P_t^M F \, d\mu_M \longrightarrow \int P_t F \, d\mu \quad (M \to \infty)$$
$$\| \qquad\qquad\qquad \|$$
$$\mathbb{E} F(X_M(t)) \longrightarrow \mathbb{E} F(X(t)) \qquad (*)$$

Similarly:
$$\int P_t^M F \, d\nu_M \longrightarrow \int P_t F \, d\nu \quad (M \to \infty)$$

Therefore
$$\left| \int F \, d\mu - \int F \, d\nu \right|$$
$$= \left| \int P_t F \, d\mu - \int P_t F \, d\nu \right| \quad \text{(invariance of } \mu, \nu)$$
$$= \lim_{M \to \infty} \left| \int P_t^M F \, d\mu_M - \int P_t^M F \, d\nu_M \right| \quad (\text{by } (*))$$

$$= \lim_{M\to\infty} \inf_{\pi \in C(\mu_M P_t^M,\, \nu_M P_t^M)} \left| \int (F(x) - F(y))\, d\pi(x,y) \right|$$

$$\leq C \lim_{M\to\infty} W_2 (\mu_M P_t^M,\ \nu_M P_t^M)$$

Wasserstein distance

$$W_2(\mu,\nu) \overset{def}{=} \inf_{\pi\in C(\mu,\nu)} E_\pi($$

from the $t\to\infty$ decay result shown above

$$\leq C e^{-ct} \lim_{M\to\infty} W_2(\mu_M, \nu_M) \longrightarrow 0$$