**Lecture 6:** $\left\{ \begin{array}{l} \text{IBP. Dyson-Schwinger etc.} \\ \text{prepare for small scale limit} \end{array} \right.$ (depend on time)

First: Recall Wick (Isserlis) Theorem:

Given a mean-zero multivar Gaussian $\{\Phi(x)\}_{x\in\Lambda}$.

e.g.: $\Phi$ is massive GFF on $\Lambda_{\varepsilon,M}$ with cov $(m-\Delta_\varepsilon)^{-1}$ as in previous lectures.

Then.
$$\mathbb{E}\left[\Phi(x_1)\cdots\Phi(x_n)\right] = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sum_P \prod_{\{i,j\}\in P} \underbrace{\mathbb{E}\left[\Phi(x_i)\Phi(x_j)\right]}_{=\mathrm{Cov}} & \text{if } n \text{ even} \end{cases}$$

Here $\Sigma_P$ is over all pairings $P$ of $\{1,\cdots,n\}$

i.e. partitions of $\{1,\cdots,n\}$ into pairs $\{i,j\}$

Example:
$$\begin{aligned}\mathbb{E}\left[\Phi(x_1)\cdots\Phi(x_4)\right] &= \mathbb{E}\left[\Phi(x_1)\Phi(x_2)\right]\mathbb{E}\left[\Phi(x_3)\Phi(x_4)\right] \\ &+ \mathbb{E}\left[\Phi(x_1)\Phi(x_3)\right]\mathbb{E}\left[\Phi(x_2)\Phi(x_4)\right] \\ &+ \mathbb{E}\left[\Phi(x_1)\Phi(x_4)\right]\mathbb{E}\left[\Phi(x_2)\Phi(x_3)\right]\end{aligned}$$

Wick theorem says:

For Gaussian, n-point correlations $\rightsquigarrow$ 2-point correlations (and perhaps mean)

Not true for Non-Gaussian eg. $\Phi^4$ but there are "Dyson-Schwinger" equations which relate correlations of different orders.

Fix finite lattice.

$$\nu = \nu_{\varepsilon, M} = e^{-\varepsilon^d \sum_x \left\{ \frac{(\nabla_i \varphi)^2}{2} + \frac{m^2 \varphi^2}{2} + \frac{\lambda \varphi^4}{4} \right\}} \prod_x d\varphi_x / Z$$

$$= e^{-V(\varphi)} \prod_x d\varphi_x / Z$$

For function $F(\varphi)$ write gradient: $DF(\varphi) = \left( \frac{\partial F}{\partial \varphi_x} \right)_{x \in \Lambda}$

Fix $x$

$$\int_{\mathbb{R}^{|\Lambda|}} DF(\varphi)(x) \, \nu(d\varphi) = \int \frac{\partial F(\varphi)}{\partial \varphi_x} \nu(d\varphi)$$

IBP for Leb measure

$$= -\int F(\varphi) \cdot \left( \frac{\partial}{\partial \varphi_x} e^{-V} \right) \cdot \prod_x d\varphi_x / Z$$

$$= \int F(\varphi) \frac{\partial V}{\partial \varphi_x} \cdot \nu(d\varphi)$$

$$= \int F(\varphi) \left( (m^2 - \Delta) \varphi_x + \lambda \varphi_x^3 \right) \nu(d\varphi)$$

This is IBP for $\nu$.

"under $E_\nu$, derivative is effectively same as multiply $(\cdots)$".

Dyson-Schwinger:

Take $F(\varphi) = \varphi(x_1) \cdots \varphi(x_n)$

$$DF(\varphi)(x) = \frac{\partial F(\varphi)}{\partial \varphi_x} = \sum_{i=1}^n \delta(x - x_i) \, \varphi(x_1) \cdots \varphi(x_{i-1}) \varphi(x_{i+1}) \cdots \varphi(x_n)$$

IBP $\Rightarrow$

$$\sum_{i=1}^n \delta(x - x_i) \, \mathbb{E}\left( \varphi(x_1) \cdots \varphi(x_{i-1}) \varphi(x_{i+1}) \cdots \varphi(x_n) \right)$$

n-1 point correlation

$$= \mathbb{E}\left( (m^2-\Delta_x)\varphi(x)\,\varphi(x_1)\cdots\varphi(x_n) \right)$$ n+1 point correlation

$$+ \lambda\, \mathbb{E}\left( \varphi^3(x)\,\varphi(x_1)\cdots\varphi(x_n) \right)$$ n+2 point correlation

e.g.: $n=1$

$$\delta(x-y) = (m^2-\Delta_x)\,\mathbb{E}(\varphi_x\varphi_y) + \lambda\,\mathbb{E}(\varphi_x^3\varphi_y)$$

In particular If $\lambda=0$. we recover:

for GFF, $\mathbb{E}(\varphi_x\varphi_y)$ is Green's function of $m^2-\Delta$.

$n=3$:

$$\delta(x-x_1)\,\mathbb{E}(\varphi(x_2)\varphi(x_3)) + \delta(x-x_2)\,\mathbb{E}(\varphi(x_1)\varphi(x_3))$$
$$+ \delta(x-x_3)\,\mathbb{E}(\varphi(x_1)\varphi(x_2))$$
$$= \mathbb{E}\left((m^2-\Delta_x)\,\varphi(x)\,\varphi(x_1)\cdots\varphi(x_3)\right) + \lambda\,\mathbb{E}\left(\varphi(x)^3\varphi(x_1)\cdots\varphi(x_3)\right)$$

Convolve with $C = (m-\Delta)^{-1}$ at $x$:

$$C(x-x_1)\,\mathbb{E}(\varphi(x_2)\varphi(x_3)) + C(x-x_2)\,\mathbb{E}(\varphi(x_1)\varphi(x_3))$$
$$+ C(x-x_3)\,\mathbb{E}(\varphi(x_1)\varphi(x_2))$$
$$= \mathbb{E}\left(\varphi(x)\varphi(x_1)\varphi(x_2)\varphi(x_3)\right] + \lambda\sum_z C(x-z)\,\mathbb{E}\left(\varphi(z)^3\varphi(x_1)\cdots\varphi(x_3)\right)$$

In particular If $\lambda=0$. we recover Wick theorem (4 pt) for GFF.
higher moments Wick follow similarly.

Note: for non-Gaussian. DS equ's don't close.

Next time, we'll discuss small scale limit $\varepsilon \to 0$.

$$\nu_{\varepsilon,M} = e^{-\varepsilon^d \sum_x \left\{ \frac{(\nabla_\varepsilon \varphi)^2}{2} + \frac{m^2 \varphi^2}{2} + \frac{\lambda \varphi^4}{4} \right\}} \prod_x d\varphi_x / Z$$

Fix $M = 1$

To prepare for it, Let's clarify scaling first:

Recall $\nabla_\varepsilon \varphi(x) = \frac{\varphi(x+\varepsilon) - \varphi(x)}{\varepsilon}$

Dynamics:

$$\frac{d\phi}{dt} = \varepsilon^d(\Delta_\varepsilon \phi - m^2\phi - \lambda\phi^4) + \frac{dB}{dt}$$

where $B = (B(x))_{x \in (\varepsilon \mathbb{Z})^d}$ indep standard BM's

Let $t = \varepsilon^{-d} \tilde{t}$

$$\frac{d\phi}{d\tilde{t}} = \Delta_\varepsilon \phi - m^2\phi - \lambda\phi^4 + \frac{dB(\varepsilon^{-d}\tilde{t})}{d\tilde{t}}$$

Recall: $\varepsilon^{d/2} B(\varepsilon^{-d}\tilde{t}) \overset{law}{=} B(\tilde{t})$

$$\frac{d\phi}{dt} = \Delta_\varepsilon \phi - m^2\phi - \lambda\phi^4 + \underbrace{\frac{dB}{dt} \cdot \varepsilon^{-d/2}}$$

Will converge in law to "space time white noise $\xi$"

Def: $\xi$ centered Gaussian. $\mathbb{E}[\xi(f)\xi(g)] = \iint fg \, dt\, dx$

$$\left(\frac{dB}{dt}\cdot\varepsilon^{-d/2}\right)(f) = \varepsilon^d \sum_x \int \varepsilon^{-d/2} \frac{dB}{dt}(t,x) f(t,x)\, dt$$

So, $\mathbb{E}\left[\varepsilon^{2d} \sum_{x\,y} \cdot \varepsilon^{-d} \iint \frac{dB}{\cancel{dt}}(t,x) \frac{dB}{\cancel{ds}}(s,y) f(t,x) g(s,y) \cancel{dt\,ds}\right]$

$$\overset{\text{Itô}}{=} \varepsilon^d \sum_x \int f(t,x) g(t,x)\, dt \longrightarrow \iint fg\, dt\, dx$$

Below it will be more convenient to consider

$$\frac{d\phi}{dt} = \Delta_\varepsilon \phi - m^2\phi - \lambda\phi^4 + \xi_\varepsilon \quad \text{where } \xi_\varepsilon \to \xi \text{ a.s.}$$

for example $\xi_\varepsilon$ is directly defined from $\xi$ by "discretization":

e.g. $\xi_\varepsilon(t,x) = \varepsilon^{-d} \langle \xi(t,\cdot), 1_{|\cdot - x| \le \frac{\varepsilon}{2}} \rangle_{L^2(\mathbb{T}^d)}$

Exercise: $\xi_\varepsilon \overset{\text{law}}{=} \varepsilon^{-d/2} \frac{dB}{dt}$.

Two alternative views of $\xi$ (useful in calculations)

1) $\mathbb{E}[\xi(t,x)\,\xi(t',x')] = \delta(t-t')\,\delta(x-x')$

$$\left( \begin{aligned} \mathbb{E}[\xi(f)\,\xi(g)] &= \mathbb{E}\int \xi(t,x)\,\xi(t',x')\, f(t,x)\, g(t',x')\, dt\,dt'\,dx\,dx' \\ &= \int \delta(t-t')\,\delta(x-x')\, f(t,x)\, g(t',x')\, dt\,dt'\,dx\,dx' \\ &= \int f(t,x)\, g(t,x)\, dt\, dx \end{aligned} \right)$$

2) $\xi_{(t)} = \sum_k \hat{\xi}(t,k)\, e^{ixk}$

where $\hat{\xi}(t,k) = \frac{1}{\sqrt{2}}\left( \frac{dB_k}{dt} + i\, \frac{dB'_k}{dt} \right)$

$\hat{\xi}(t,-k) = \overline{\hat{\xi}(t,k)}$

$(B_k, B'_k)_{k \in \mathbb{Z}^d \cap [-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]}$ are indep standard BM

Exercise: $\mathbb{E}[\hat{\xi}(t,k)\,\overline{\hat{\xi}(t',k')}] = 2\pi\, 1_{k=k'}\, \delta(t-t')$

$$\left( \begin{aligned} \text{So, } \mathbb{E}[\xi(t,x)\,\xi(t',x')] &= \mathbb{E}\sum_{k,k'} \hat{\xi}(t,k)\,\hat{\xi}(t',k')\, e^{ixk} e^{ix'k'} \\ &= \delta(t-t')\,\delta(x-x') \quad \text{since } \sum_k e^{i(x-x')k} = \delta(x-x') \end{aligned} \right.$$

For Friday lectures,
the following fact is important:

☆ $Y(t)$ is function if $d=1$. $Y(t)$ is distribution if $d\geq 2$.

We prove exact regularity next time, but for now we put in simpler way:

$$\mathbb{E}(Y(t,x)^2) = \begin{cases} <\infty & d=1 \\ =\infty & d\geq 2 \end{cases}$$

Indeed, $Y(t,x) = \int_{-\infty}^{t}\int P_m(t-s,x-y)\,\xi(s,y)\,ds\,dy$

$$\mathbb{E}(Y(t,x)^2) \overset{\text{Itô}}{=} \int_{-\infty}^{t}\int P_m(t-s,x-y)^2\,ds\,dy$$

$$\overset{\text{Parseval}}{=} \int_{-\infty}^{t}\int_{\mathbb{R}^d} e^{-(t-s)(|k|^2+m^2)}\,dk\,ds = \int_{\mathbb{R}^d}\frac{1}{|k|^2+m^2}\,dk = \begin{cases} <\infty & d=1 \\ =\infty & d\geq 2 \end{cases}$$

understood as $Y_\varepsilon$ and $\varepsilon\to 0$

But $Y$ is distribution since $\forall\,\varphi\in S(\mathbb{R}^d)$,

$$\mathbb{E}\Big[\Big(\int Y(t,x)\,\varphi(x)\,dx\Big)^2\Big] = \iint \varphi(x)\varphi(y)\,\mathbb{E}(Y(t,x)Y(t,y))\,dx\,dy$$

$$= \iint_{dx\,dy}\varphi(x)\varphi(y)\int_{-\infty}^{t}\int P_m(t-s,x-z)\,P_m(t-s,y-z)\,dz\,ds < \infty$$

One way to see is $= \sum_k |\hat{\varphi}(k)|^2\frac{1}{|k|^2+m^2} < \infty$

Another way:

$$\left|\int_{-\infty}^{t}\int P_m(t-s,x-z)\,P_m(t-s,y-z)\,dz\,ds\right| \lesssim \frac{1}{|x-y|^{d-2-\varepsilon}}$$

for $|x-y|$ small.

exercise: "$-d-d+(d+2) = -d+2$" (Maybe TA)

always integrable around 0.

HW for Thursday holiday:

Use Wick theorem above to compute 2nd moments of $\langle \Psi^2, \varphi \rangle$, $\langle \Psi^3, \varphi \rangle$ in 2D, 3D.

Which pairings converge?

Which pairings diverge?