SL Math Stochastic Quantization Summer School TA Session: Wiener Chaos and practice with Wick's theorem

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1 Wiener Chaos

In this note, we shall follow the second and third chapters of the book of Janson [Jan97]. There are also some nice, accessible notes by Alberts and Khoshnevisan available in Section 3 here `https://www.math.utah.edu/~davar/math7880/F18/GaussianAnalysis.pdf`. Some additional books to look at are [NN18, Nua06].

We recall here a theorem stated in the main lectures, known as Isserlis' Theorem. It is also sometimes called the Wick Theorem, but in this notes, we shall refer to a more general result, which we will call the Wick Theorem.

Theorem 1.1 (Isserlis' Theorem). *Let $X_1, \dots, X_n$ be jointly Gaussian random varaibles, each with mean 0. Then,*

$$\mathbb{E}[X_1 \cdots X_n] = \begin{cases} 0 & n \text{ odd} \\ \sum_{P \in \mathcal{M}_n} \prod_{(i,j) \in P} \mathbb{E}[X_i X_j] & n \text{ even} \end{cases},$$

where $\mathcal{M}_n$ is the set of perfect matchings/pairings P of the set $\{1, \dots, n\}$.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, A **Gaussian Hilbert space** H is a complete real vector space consisting of mean-zero Gaussian random variables. Such a space must be a subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ since Gaussian random variables have finite second moment.

For $n \geq 0$, define $\overline{\mathcal{P}}_n(H)$ to be the closure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the vector space

$$\mathcal{P}_n(H) := \{p(\xi_1, \dots, \xi_m) : p \text{ is a polynomial of degree } \leq n; \xi_1, \dots, \xi_m \in H; m < \infty\}.$$

For $n \geq 0$, define the subspace $H^{:n:}$ inductively as follows: first $H^{:0:} = \overline{\mathcal{P}}_0(H) = \mathbb{R}$ (the space of constants), and for $n \geq 1$,

$$H^{:n:} := \overline{\mathcal{P}}_n(H) \cap \overline{\mathcal{P}}_{n-1}(H)^{\perp}.$$

Here, $\mathcal{P}_{n-1}(H)^{\perp}$ is the orthogonal complement of $\overline{\mathcal{P}}_{n-1}(H)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We note here that taking the closure is only necessary when H is infinite-dimensional. By construction, the spaces $(H^{:n:})_{n\in\mathbb{N}}$ are mutually orthogonal and closed. The following theorem is known as the **Wiener chaos decomposition**. It is fairly straightforward to prove; one just needs to show that the orthogonal sum on the left is the entire L^2 space. The details may be found, for example, in [Jan97, Theorem 2.6].

Theorem 1.2. *We have the orthogonal decomposition*

$$\bigoplus_{n=0}^{\infty} H^{:n:} = L^2(\Omega, \mathcal{F}(H), \mathbb{P}),$$

where $\mathcal{F}(H)$ *is the* σ*-algebra generated by the random variables in* H*.*

Assume that the random variables in H generate $\mathcal{F}$. Then, for $n \geq 0$ and $\xi_1, \ldots, \xi_n \in H$, we let $\pi_n : L^2(\Omega, \mathcal{F}(H), \mathbb{P}) \to H^{:n:}$ denote orthogonal projection (Recall that this means $\pi_n(x)$ is the unique vector in $H^{:n:}$ such $x - \pi_n(x) \in (H^{:n:})^{\perp}$. For $\xi_1, \ldots \xi_n \in H$, define the **Wick product**

$$:\xi_1 \cdots \xi_n := \pi_n(\xi_1 \ldots, \xi_n).$$

To state the next theorem, we define a Feynman diagram. A **Feynman diagram** is a graph with n vertices and $r \leq \frac{n}{2}$ edges, such that each vertex is incident to at most one edge. There are r pairs of edges, and $n - 2r$ unpaired vertices. Let F_n be the set of Feynman diagrams on n vertices. For $\gamma \in F_n$, let $r(\gamma)$ be the number of edges, let $E(\gamma) = \{(i, j) : i, j \text{ connected}\}$ be the set of edges, and let $U(\gamma)$ be the set of unmatched vertices. For such a Feynman diagram and $\xi_1, \ldots, \xi_n \in H$, define

$$v(\gamma; \xi_1, \ldots, \xi_n) = \prod_{(i,j)\in E(\gamma)} \mathbb{E}[\xi_i \xi_j] \prod_{i\in U(\gamma)} \xi_i.$$

We now state the following consequence of Isserlis' Theorem. Here, we will refer to it as Wick's Theorem. This is in fact, a more general version of Isserlis' Theorem. Indeed, for $n \geq 1$, we must have $\mathbb{E}[:\xi_1 \cdots \xi_n:] = 0$ by orthogonality. It is an exercise to verify that the appropriate terms cancel, and we are left the equality in Isserlis' Theorem.

Theorem 1.3 (Wick's Theorem)**.** *For* $\xi_1, \ldots, \xi_n \in H$*,*

$$:\xi_1 \cdots \xi_n := \sum_{\gamma\in F_n} (-1)^{r(\gamma)} v(\gamma; \xi_1, \ldots, \xi_n). \tag{1.1}$$

Proof. Denote the right-hand side of (1.1) as ψ. We need to show that $\psi \in H^{:n:} = \overline{\mathcal{P}}_n(H) \cap \overline{\mathcal{P}}_{n-1}(H)^{\perp}$ and that $\psi - (\xi_1 \cdots \xi_n) \in (H^{:n:})^{\perp}$. Observe first that $\psi \in \overline{\mathcal{P}}_n(H)$, and the only term of degree n in the sum (1.1) is $\xi_1 \cdots \xi_n$. All other terms in the sum (1.1) involve matching at least two terms and hence are of order $\leq n - 2$. Thus, $\psi - (\xi_1 \cdots \xi_n) \in \mathcal{P}_{n-2}(H) \subseteq \overline{\mathcal{P}}_{n-1}(H) \subseteq (H^{:n})^{\perp}$. It remains to show that $\psi \in \overline{\mathcal{P}}_{n-1}(H)$. Let $m \geq 1$ and choose $\xi_{n+1}, \dots \xi_{n+m} \in H$. We show that, if $m \leq n-1$, then $\mathbb{E}[\psi \xi_{n+1} \cdots \xi_{n+m}] = 0$. We write things out as follows:

$$\begin{aligned}\mathbb{E}[\psi \xi_{n+1} \cdots \xi_{n+m}] &= \sum_{\gamma \in F_n} (-1)^{r(\gamma)} \mathbb{E}[v(\gamma; \xi_1, \dots, \xi_n) \xi_{n+1} \cdots \xi_{n+m}] \\ &= \sum_{\gamma \in F_n} (-1)^{r(\gamma)} \prod_{(i,j) \in E(\gamma)} \mathbb{E}[\xi_{i_k} \xi_{j_k}] \mathbb{E}\Big[\prod_{i \in U(\gamma)} \xi_i \prod_{j=1}^{m} \xi_{n+j}\Big].\end{aligned}$$

Using Isserlis' Theorem, we see this is 0 if $n+m$ is odd, and if $n+m$ is even, we obtain

$$\begin{aligned}\mathbb{E}[\psi \xi_{n+1} \cdots \xi_{n+m}] &= \sum_{\gamma \in F_n} (-1)^{r(\gamma)} \sum_{\substack{P \in \mathcal{M}_{n+m} \\ P \text{ extends } \gamma}} \prod_{(i,j) \in P} \mathbb{E}[\xi_i \xi_j] \\ &= \sum_{P \in \mathcal{M}_{n+m}} \prod_{(i,j) \in P} \mathbb{E}[\xi_i \xi_j] \sum_{\substack{\gamma \in F_n \\ P \text{ extends } \gamma}} (-1)^{r(\gamma)}.\end{aligned} \tag{1.2}$$

Above, we say that P **extends** γ if every pair of vertices in P is also a pair in γ.

Given $P \in \mathcal{M}_{n+m}$, we seek to compute the sum

$$\sum_{\substack{\gamma \in F_n \\ P \text{ extends } \gamma}} (-1)^{r(\gamma)}.$$

If P has ℓ pairs of vertices (i,j) for $i, j \leq n$, then there are 2^ℓ elements of $\gamma \in F_n$ such that P_n extends γ. This is because, for each of the ℓ pairs $(i,j) \in P$ with $i, j \leq n$, we have 2 choices: we may keep or remove that pairing for γ. For each $1 \leq r' \leq \ell$, there are $\binom{\ell}{r'}$ choices of $\gamma \in F_n$ such that P extends γ and $r(\gamma) = r'$. Therefore,

$$\sum_{\substack{\gamma \in F_n \\ P \text{ extends } \gamma}} (-1)^{r(\gamma)} = \sum_{r=0}^{\ell} \binom{\ell}{r} (-1)^r = \begin{cases} 1 & \ell = 0 \\ (1 + (-1))^\ell = 0 & \ell > 0. \end{cases}$$

Hence, by (1.2), we have

$$\mathbb{E}[\psi \xi_{n+1} \cdots \xi_{n+m}] = \sum_{\substack{P \in \mathcal{M}_{n+m} \\ (i,j) \notin P \text{ if } i,j \leq n}} \prod_{(i,j) \in P} \mathbb{E}[\xi_i \xi_j].$$

In words, the sum is over perfect matchings $P \in \mathcal{M}_{n+m}$ so that no two vertices less than n are matched. If $m \leq n-1$, there are no such perfect matchings, and so the expectation is 0. Hence, we have shown $\psi \perp \overline{\mathcal{P}}_{n-1}$, so $\psi \in H^{:n:}$ as desired. □

In the proof of Theorem 1.3, we have also shown the following:

Lemma 1.4. *For $n, m \geq 1$ and $\xi_1, \ldots, \xi_{n+m} \in H$,*

$$\mathbb{E}[(:\xi_1 \cdots \xi_n:)\xi_{n+1} \cdots \xi_{n+m}] = \sum_{\substack{P \in \mathcal{M}_{n+m} \\ (i,j) \notin P \text{ if } i,j \leq n}} \prod_{(i,j) \in P} \mathbb{E}[\xi_i \xi_j].$$

This has the following consequence.

Corollary 1.5. *For $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m \in H$, we have*

$$\mathbb{E}[(:\xi_1 \cdots \xi_n:)(:\eta_1 \cdots \eta_m:)] = \begin{cases} 0 & n \neq m \\ \sum_{\sigma \in \mathfrak{S}(n)} \prod_{i=1}^n \mathbb{E}[\xi_i \eta_{\sigma(i)}] & n = m, \end{cases}$$

where $\mathfrak{S}_n$ is the symmetric group.

Proof. Note that $:\xi_1 \cdots \xi_n: \in H^{:n:}$ and $:\eta_1 \cdots \eta_m: \in H^{:m:}$, and if $n \neq m$, $H^{:n:} \perp H^{:m:}$, so the result follows. For $n = m$, we note that

$$:\xi_1, \cdots \xi_n: \perp \eta_1 \cdots \eta_m - :\eta_1 \cdots \eta_m:,$$

and so

$$\mathbb{E}[(:\xi_1, \cdots \xi_n:)(:\eta_1 \cdots \eta_n:)] = \mathbb{E}[(:\xi_1, \cdots \xi_n:)\eta_1 \cdots \eta_n].$$

The result now follows from Lemma 1.4, noting that the only perfect matching of $2n$ vertices so that ξ_i is not paired with to $\xi_j j$ for $i, j \leq n$ are those for which each ξ_i is paired with one of the η_j. □

Using Theorem 1.3, we explicitly write $:\xi_1 \cdots \xi_n:$ for $n \leq 4$:

$$\begin{aligned} :\xi_1: &= \xi_1, \\ :\xi_1\xi_2: &= \xi_1\xi_2 - \mathbb{E}[\xi_1\xi_2], \\ :\xi_1\xi_2\xi_3: &= \xi_1\xi_2\xi_3 - \mathbb{E}[\xi_1\xi_2]\xi_3 - \mathbb{E}[\xi_1\xi_3]\xi_2 - \mathbb{E}[\xi_2\xi_3]\xi_1 \\ :\xi_1\xi_2\xi_3\xi_4: &= \xi_1\xi_2\xi_3\xi_4 - \xi_1\xi_2\mathbb{E}[\xi_2\xi_3] - \xi_1\xi_3\mathbb{E}[\xi_2\xi_4] - \xi_1\xi_4\mathbb{E}[\xi_2\xi_3] \\ &\quad - \xi_2\xi_3\mathbb{E}[\xi_1\xi_4] - \xi_2\xi_4\mathbb{E}[\xi_1\xi_3] - \xi_3\xi_4\mathbb{E}[\xi_1\xi_2] \\ &\quad + \mathbb{E}[\xi_1\xi_2]\mathbb{E}[\xi_2\xi_3] + \mathbb{E}[\xi_1\xi_3]\mathbb{E}[\xi_2\xi_4] + \mathbb{E}[\xi_1\xi_4]\mathbb{E}[\xi_2\xi_3] \end{aligned}$$

We can see directly from here that if $\xi \sim \mathcal{N}(0,1)$, then $:\xi^2: = \xi^2 - 1 = H_2(\xi)$, $:\xi^3: = \xi^3 - 3\xi = H_3(\xi)$, and $:\xi^4: = \xi^4 - 6\xi^2 + 3 = H_4(\xi)$, where H_n is the nth Hermite polynomial. This is a consequence of the fact that $(:\xi^n:)_{n \geq 0}$ are orthogonal. In general, we have the following:

Lemma 1.6. *If $\xi \sim \mathcal{N}(0, \sigma^2)$, then, for $n \geq 0$, $:\xi^n: = \sigma^n H_n\left(\frac{\xi}{\sigma}\right)$.*

This observation along with Wick's Theorem gives us a combinatorial way of calculating the Hermite polynomials:

Corollary 1.7. *For $n \geq 0$,*

$$H_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m n!}{m!(n-2m)!2^m} x^{n-2m}.$$

Proof. By Wick's Theorem, the coefficient of x^m in $H_n(x)$ is the number of Feynman diagrams with on n vertices that leave m vertices unmatched. The details are left as an exercise. □

2 Equivalence of moments/hypercontractivity

We know from standard analysis (see, for example, [Fol99, Proposition 6.12]) that, for any random variable X, $(\mathbb{E}[|X|^p])^{1/p} \leq (\mathbb{E}[|X|^q)^{1/q}$ whenever $0 \leq p \leq q \leq \infty$. That is, higher moments can be used to bound the lower moments. In Gaussian spaces, we can also do the reverse. That is, lower moments can be used to bound the higher moments. Let's see this with a simple example: if $\xi \sim \mathcal{N}(0, \sigma^2)$, then $\xi \stackrel{d}{=} \sigma \xi_0$, where $\xi_0 \sim \mathcal{N}(0,1)$. Then, for $p > 0$,

$$\mathbb{E}[|\xi|^p] = \sigma^p \mathbb{E}[|\xi_0|^p],$$

and we see that for $q \neq p$, the ratio $\frac{\|\xi\|_q}{\|\xi\|_p}$ does not depend on σ. In particular, for $q, p > 0$, there exists a constant $C > 0$ so that $\|\xi\|_q \leq C(p,q)\|\xi\|_p$ for all centered Gaussian random variables ξ. Using the results developed in the previous section, one can actually say something much stronger: For $n \in \mathbb{N}$, such a bound holds for all polynomials of centered Gaussian random variables of degree at most n. The full proof is somewhat lengthy; we omit it here. However, Wick's Theorem and Corollary 1.5 are the key building blocks. See [Jan97, Theorem 3.50] for a proof.

Theorem 2.1. *Let H be a Gaussian Hilbert space. For $n \in \mathbb{N}$ and $p, q > 0$, there exists a constant $C(n,p,q)$ so that, for all $X \in \overline{\mathcal{P}}_n(H)$,*

$$\|X\|_q \leq C(n,p,q)\|X\|_p.$$

The typical application of Theorem 2.1 will be for $p = 2$ and $q \geq 2$. This is because the Itô's isometry often allows for nice computations of the second moment.

The property in Theorem 2.1 is sometimes called **hypercontractivity**.

3 Practice with Wick's/Isserlis' Theorem

Let $Y = Y(t,x)$, $t > 0, x \in \mathbb{T}^d$ be the solution to the linear equation

$$dY = (\Delta - m^2)Y + \xi,$$

from the main lecture. We have seen that Gaussian free field on $\mathbb{T}^d$ for $d \geq 2$ is invariant for this equation. Recall from the lectures that this is a centred Gaussian process indexed by test functions φ. First, we will consider the fixed-time stationary solution to the linear equation and write $Y(x) = Y(0,x)$. We formally write

$$\langle Y, \varphi \rangle = \int_{\mathbb{T}^d} Y(x)\varphi(x)\,dx,$$

noting that the pointwise values $Y(x)$ are not well-defined. For mass $m > 0$, we have the formal covariance function

$$\mathbb{E}[Y(x)Y(y)] = \sum_{k \in \mathbb{Z}^d} \frac{1}{|k|^2 + m^2} e^{i\langle k, x-y\rangle} \quad \begin{cases} = \infty & x = y, d \geq 2 \\ < \infty & \text{otherwise} \end{cases} \tag{3.1}$$

Furthermore, it was made in a remark that

$$\mathbb{E}[Y(x)Y(y)] \lesssim \frac{1}{|x-y|^{d-2+\delta}} \tag{3.2}$$

for small $\delta > 0$.

We have seen that we may construct Y by first taking a discrete approximation Y_ε, and then send $\varepsilon \searrow 0$. Let $\varphi \in C^\infty(\mathbb{T}^2)$ be a smooth test function. We now wish to study the moments of Y_ε^2 and Y_ε^3. In light of Theorem 2.1, we wish to find the second moment of $\langle Y_\varepsilon^2, \varphi\rangle$ and $\langle Y_\varepsilon^3, \varphi\rangle$.

Observe that

$$\mathbb{E}\big[\langle Y_\varepsilon^2, \varphi\rangle^2\big] \approx \int_{\mathbb{T}^d \times \mathbb{T}^d} \mathbb{E}[Y_\varepsilon(x)^2 Y_\varepsilon(y)^2]\varphi(x)\varphi(y)\, dx\, dy.$$

The $\approx$ just comes because we are really taking a sum instead of an integral. By Isserlis' Theorem, we may compute the expectation inside the integral:

$$\mathbb{E}[Y_\varepsilon(x)^2 Y_\varepsilon(y)^2] = \mathbb{E}[Y_\varepsilon(x)^2]\mathbb{E}[Y_\varepsilon(y)^2] + 2\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)]^2$$

In light of (3.1), we see that, for $d \geq 2$, $\mathbb{E}[Y_\varepsilon(x)^2]\mathbb{E}[Y_\varepsilon(y)^2]$ will diverge as $\varepsilon \searrow 0$, while $\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)]^2$ will converge. To remedy this, replace $Y_\varepsilon(x)^2$ with $: Y_\varepsilon(x)^2 :$. By Lemma 1.6, this is equal to $Y_\varepsilon(x)^2 - C_\varepsilon$, where C_ε is the variance of $Y_\varepsilon(x)$. Then, by Corollary 1.5,

$$\mathbb{E}[(: Y_\varepsilon(x)^2 :)(: Y_\varepsilon(y)^2 :)] = 2\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)]^2,$$

And then, by (3.2),

$$\mathbb{E}\big[\langle Y_\varepsilon^2, \varphi\rangle^2\big] \lesssim \int_{\mathbb{T}^d \times \mathbb{T}^d} \frac{\varphi(x)\varphi(y)}{|x-y|^{2d-4+2\delta}}\, dx\, dy.$$

For this to be finite, we have the condition $2d - 4 < d$, which means $d < 4$. Hence, this renormalization works in dimensions 2 and 3.

We can do a similar thing with Y_ε^3:

$$\mathbb{E}[\langle Y_\varepsilon^3, \varphi\rangle^2] \approx \int_{\mathbb{T}^d \times \mathbb{T}^d} \mathbb{E}[Y_\varepsilon(x)^3 Y_\varepsilon(y)^3]\varphi(x)\varphi(y)\, dx\, dy.$$

Then, Isserlis' Theorem tells us

$$\mathbb{E}[Y_\varepsilon(x)^3 Y_\varepsilon(y)^3] = 9\mathbb{E}[Y_\varepsilon(x)^2]\mathbb{E}[Y_\varepsilon(y)^2]\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)] + 6\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)]^3,$$

and the term $\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)]^2$ diverges. To remedy this, we may again replace $Y_\varepsilon(x)$ with $: Y_\varepsilon(x)^3 := Y_\varepsilon(x)^3 - 3C_\varepsilon Y_\varepsilon(x)$, and then by Lemma 1.4, we have

$$\mathbb{E}[(: Y_\varepsilon(x)^3 :)(: Y_\varepsilon(y)^3 :)] = 6\mathbb{E}[Y_\varepsilon(x)Y_\varepsilon(y)]^3.$$

Then,

$$\mathbb{E}[\langle Y_\varepsilon^3, \varphi\rangle^2] \lesssim \int_{\mathbb{T}^d \times \mathbb{T}^d} \frac{\varphi(x)\varphi(y)}{|x-y|^{3d-6+3\delta}}\, dx\, dy,$$

and for this to be finite, we have the condition $3d - 6 < d$, which gives $d < 3$. Hence, this renormalization for fixed time makes sense in dimension 2, but not dimension 3.

To remedy this issue in $d = 3$, we must no longer treat the time t as fixed, and consider $Y = Y(t, x)$ formally as a distribution on $\mathbb{R} \times \mathbb{T}^d$. Then, we instead consider smooth functions $\varphi \in C^\infty(\mathbb{R} \times \mathbb{T}^n)$, so that, formally,

$$\langle Y, \varphi \rangle = \int_{\mathbb{R} \times \mathbb{T}^d} Y(t, x)\varphi(t, x)\, dt\, dx.$$

In this setting, the relevant metric is a parabolic distance on $\mathbb{R} \times \mathbb{T}^d$ so that $d((t, x), (t', x')) = \sqrt{|t - t'|} + |x - x'|_d$. Then, $\mathbb{R} \times \mathbb{T}^d$ essentially becomes a $d+2$ dimensional space. A similar calculation as above leads to

$$\mathbb{E}[\langle Y_\varepsilon^2, \varphi \rangle^2] \lesssim \int_{(\mathbb{R} \times \mathbb{T}^2) \times (\mathbb{R} \times \mathbb{T}^2)} \frac{\varphi(t, x)\varphi(s, y)}{d((t, x), (s, y))^{3d-6+3\delta}}\, dt\, dx\, ds\, dy,$$

and this is finite if $3d - 6 < d + 2$, which holds for $d < 4$.

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