Lecture 7: Probabilistic estimates

$$\partial_t Y = (\Delta - m^2) Y + \xi$$

$$\hat{u}(t,k) := \mathcal{F} u(t,\cdot)(k) = \int e^{-ikx} u(t,x)\, dx$$

$$\mathbb{E}\left[\hat{\xi}(t,k)\, \overline{\hat{\xi}(t',k')}\right] = 2\pi\, \mathbb{1}_{k=k'}\, \delta(t-t')$$

$$Y(t,x) = \int_{-\infty}^{t} (P_{t-s}\xi)(x)\, ds$$

$$\hat{Y}(t,k) = \int_{-\infty}^{t} e^{-k^2(t-s)} \hat{\xi}(s,k)\, ds$$

$$\mathbb{E}\left[\hat{Y}(t,k)\, \overline{\hat{Y}(t',k')}\right] \qquad (t \le t')$$

$$= \int_{-\infty}^{t} \int_{-\infty}^{t'} e^{-|k|^2(t-s) - |k'|^2(t'-s')} \mathbb{E}\left[\hat{\xi}(s,k)\, \overline{\hat{\xi}(s',k')}\right] ds\, ds'$$

$$= 2\pi\, \mathbb{1}_{k=k'} \int_{-\infty}^{t} e^{-|k|^2(t+t'-2s)}\, ds$$

$$= 2\pi\, \mathbb{1}_{k=k'}\, e^{-|k|^2(t+t')} \frac{1}{2k^2} e^{2|k|^2 t}$$

$$= \pi\, \mathbb{1}_{k=k'}\, e^{-|k|^2|t'-t|} / |k|^2 \qquad (\text{for } k \neq 0)$$

For $k=0$

$$\mathbb{E}\left[\hat{Y}(t,k)\, \overline{\hat{Y}(t',k')}\right] = 2\pi\, \mathbb{1}_{k=k'} (t \wedge t')$$

Could ignore. Since we have m. $|k|^2$ above should be k^2+m^2

Let $Y_{s,t} = Y(t) - Y(s)$

$$\mathbb{E}\left[\hat{Y}_{s,t}(k)\overline{\hat{Y}_{s,t}(k')}\right]$$

$$= \mathbb{E}\left[\left(\hat{Y}(t,k) - \hat{Y}(s,k)\right)\left(\overline{\hat{Y}(t,k') - \hat{Y}(s,k')}\right)\right]$$

$$= \frac{\pi 1_{k=k'}}{|k|^2}\left(2 - 2e^{-|k|^2|t-t'|}\right)$$

$$\lesssim |t-s|^{\delta} |k|^{2\delta-2} \quad \forall \delta \in [0,1], \ k \neq 0$$

$$1 - e^{-x} \leq |x|^{\delta}$$

Now we look at how analysis and probability interplay:

Lemma $\forall p \geq 1$ $\forall \alpha < -\frac{d-2}{2}$

$$\mathbb{E}\left[\|Y\|^p_{C(\mathbb{R};C^\alpha)}\right] < \infty \quad \text{i.e. } Y \in C(\mathbb{R}; C^\alpha) \text{ a.s.}$$

Pf: Recall: $\|u\|_{B^\alpha_{p,p}} = \left\{\sum_{j\geq -1}\left(2^{j\alpha}\|\Delta_j u\|_{L^p}\right)^p\right\}^{1/p}$

hypercontractivity:

$$\mathbb{E}\left[\|\Delta_\ell Y_{s,t}\|^{2p}_{L^{2p}}\right] = \mathbb{E}\int(\Delta_\ell Y_{s,t})^{2p}\,dx$$

$$\leq \int \mathbb{E}\left[(\Delta_\ell Y_{s,t})^2\right]^p \,dx$$

$$\mathbb{E}\left[(\Delta_\ell Y_{s,t})^2\right] = \frac{1}{2\pi}\sum_{k\,k'} \rho_\ell(k)\,\rho_\ell(k')\,e^{i(k-k')x}\ \mathbb{E}\left[Y_{s,t}(k)\,\overline{Y_{s,t}(k')}\right]$$

$$\lesssim \sum_k \rho_\ell(k)^2\,|t-s|^\delta\,|k|^{2\delta-2}$$

$$\lesssim |t-s|^\delta\, 2^{\ell(d+2\delta-2)}$$

$\text{supp}\,\rho_\ell \approx 2^{d\ell}$

$k \approx 2^\ell$

Hence,

$$\mathbb{E}\,\|Y(t)-Y(s)\|^{2p}_{B^\alpha_{2p,2p}}$$

$$= \sum_{\ell\geq -1} \mathbb{E}\left[\left(2^{\ell\alpha}\,\|\Delta_\ell Y_{s,t}\|_{2p}\right)^{2p}\right]$$

$$= \sum_{\ell\geq -1} 2^{2\ell\alpha p}\cdot\left(|t-s|^\delta\, 2^{\ell(d+2\delta-2)}\right)^p$$

For $\alpha < -\frac{d-2}{2}$. $2\ell\alpha p + (\ell d - 2\ell)p + 2\ell\delta p = p\ell(2\alpha + d - 2 + 2\delta) < 0$

sum converge. for δ small enough $\left(\delta < -\frac{d-2}{2} - \alpha\right)$

Kolmogorov: $E|X_t - X_s|^\alpha \leq |t-s|^{1+\beta}$ $\beta > 0$

$\Rightarrow X$ continuous version

here $|t-s|^{\delta p} = |t-s|^{1+(\delta p-1)}$

choose p large, s.t $\delta p - 1 > 0$ $p > \frac{1}{\delta}$

$\Rightarrow$ Y has cont version in $B^{\alpha}_{2p,2p}$ and $\mathbb{E}\left[\|Y\|^{2p}_{C(\mathbb{R};B^{\alpha}_{2p,2p})}\right]<\infty$

Besov embedding: $p_1 \le p_2$, $q_1 \le q_2$. $B^{\alpha}_{p_1,q_1} \hookrightarrow B^{\alpha-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,q_2}$

$$\Rightarrow \quad B^{\alpha}_{2p,2p} \hookrightarrow B^{\alpha-\frac{d}{2p}}_{\infty\infty}$$

Choose p large s.t given $\forall \alpha' < -\frac{d-2}{2}$

$$\alpha - \frac{d}{2p} = \alpha'$$

$\alpha < -\frac{d-2}{2}$ as above

i.e. p s.t $\alpha' + \frac{d}{2p} < -\frac{d-2}{2}$

$\mathbb{E}\left[\|Y\|^{2p}_{C(\mathbb{R};C^{\alpha})}\right]<\infty$, Then this holds $\forall p$ by Jensen. ∎

Rmk (1) moment bound in B^{α}_{pp} instead of C^{α}: $\mathbb{E}\sup \neq \sup \mathbb{E}$.

(2) p has to be large: Kolmogorov (like BM needs 4th moment)

Besov embedding doesn't lose too much regularity

(3) $\alpha = -\frac{d-2}{2}$ not work

To recap,

☆ $Y(t)$ is function if $d=1$. $Y(t)$ is distribution if $d\geq 2$.

Rmk: Actually $Y_{\varepsilon} \to Y$ in $C(\mathbb{R};C^{\alpha})$

(proof similar)

✱ If $d \geq 2$ $\lim_{\varepsilon \to 0} Y_\varepsilon(t,x)^2$ is not well-defined distribution.

The mean of $\langle Y_\varepsilon(t,x)^2, \varphi \rangle$ diverge. $\varphi \in S(\mathbb{R}^d)$

$$\mathbb{E} \int Y(t,x)^2 \varphi(x)\, dx = \mathbb{E}(Y(t,x)^2) \int \varphi(x)\, dx \to \infty$$

✱ If $d = 2, 3$. $\lim_{\varepsilon \to 0} Y_\varepsilon(t,x)^2 - C_\varepsilon$ is distribution on $\mathbb{R}^d$

$\forall \varphi \in S(\mathbb{R}^d)$ $\qquad C_\varepsilon = \mathbb{E}(Y_\varepsilon(t,x)^2)$

$$\mathbb{E}\left[\left(\int (Y(t,x)^2 - C)\, \varphi(x)\, dx\right)^2\right]$$

$$= \mathbb{E} \iint (Y(t,x)^2 - C)(Y(t,y)^2 - C)\, \varphi(x)\, \varphi(y)\, dx\, dy$$

Wick theorem:

$$\mathbb{E}[Y(x)^2 Y(y)^2] = \underbrace{\mathbb{E}[Y(x)^2]\, \mathbb{E}[Y(y)^2]}_{\text{Canceled by } C \text{ terms.}} + 2\mathbb{E}(Y(x)Y(y))^2$$

$$= 2 \iint \underbrace{\mathbb{E}[Y(t,x)\, Y(t,y)]^2}_{\lesssim \frac{1}{|x-y|^{2d-4-\varepsilon}}} \varphi(x)\, \varphi(y)\, dx\, dy$$

by above exercise (for $|x-y|$ small)

integrable. $2d - 4 < d \iff d < 4$

With extra work (like proof of above lemma)

One can show $\Psi^2 = \lim_{\varepsilon \to 0} Y_\varepsilon^2 - C_\varepsilon \in C(\mathbb{R}, C^{-(d-2)-\varepsilon})$

☆ If $d=2$, $\lim_{\varepsilon\to 0} Y_\varepsilon(t,x)^3 - 3C_\varepsilon Y_\varepsilon(t,x)$ is distribution on $\mathbb{R}^d$

If $d=3$ this is not well defined distribution on $\mathbb{R}^d$.

but is distribution on $\mathbb{R}\times\mathbb{R}^d$. (but can't fix t)

Test against $\varphi\in S(\mathbb{R}^d)$ and compute 2nd moment as above.

$$\mathbb{E}(Y(x)^3 Y(y)^3) = 6\mathbb{E}[Y(x)Y(y)]^3$$

+ terms which will be canceled by C terms

please check yourself!

$$\iint \frac{1}{|x-y|^{3d-6-\varepsilon}}\varphi(x)\varphi(y)\,dx\,dy$$

$=-\varepsilon$

$3d-6<d \iff d<3$ so integrable if $d=2$

If $d=3$. $\varphi\in S(\mathbb{R}\times\mathbb{R}^d)$

$$\iint \underbrace{\mathbb{E}(Y(t,x)Y(s,y))}_{\lesssim \frac{1}{(\sqrt{|t-s|}+|x-y|)^{d-2-\varepsilon}}}{}^3\,\varphi(t,x)\varphi(s,y)\,ds\,dt\,dx\,dy$$

$3d-6<d+2$

$\iff d<4$

So integrable if $d=3$

With extra work (like proof of above lemma)

One can show $\mathbb{Y}^3 = \lim_{\varepsilon\to 0} Y_\varepsilon^3 - 3C_\varepsilon Y_\varepsilon \in C(\mathbb{R}, C^{-\varepsilon})$ $d=2$

$\in C^{-\varepsilon}(\mathbb{R}, C^{-3/2}) \cap C^{-3/2-\varepsilon}(\mathbb{R}\times\mathbb{R}^d)$ $d=3$

Rmk: (1) Hermite polynomial

(2) Iterated integral