Lecture 8: Small scale limit of ϕ^4_2

Recall: $X = Y + Z$

$$\begin{cases} dY_t = (\Delta_\varepsilon - m^2) Y_t \, dt + \sqrt{2}\, dB_t \cdot \varepsilon^{-1} \\ \partial_t Z_t = (\Delta_\varepsilon - m^2) Z_t - \frac{1}{2} V'(Y_t + Z_t) \end{cases}$$

$$V'(\varphi) = \lambda \varphi^3 + \beta \varphi$$

$$V'(Y+Z) = \lambda(Y^3 + 3Y^2 Z + 3YZ^2 + Z^3) + \beta(Y+Z)$$

(pointwise)

- recall Littlewood–Paley theory and Besov space

and $Y^\varepsilon \in C(\mathbb{R}; C^{-\frac{d-2}{2}-\kappa})$ a.s. uniformly in ε

- recall $\mathbb{Y}^2 \in C(\mathbb{R}; C^{2\alpha})$ $\quad \alpha = -\frac{d-2}{2} - \kappa < 0$, $d \geq 2$

$$\mathbb{Y}^3 \in \begin{cases} C(\mathbb{R}; C^{-\kappa}) & d=2 \\ C^{-\kappa}(\mathbb{R}; C^{3\alpha}) & d=3 \end{cases}$$

Take: $\beta = \beta_\varepsilon = -3\lambda C_\varepsilon + \beta'$ where $\beta' \in \mathbb{R}$ fixed

$$\lambda(Y^3 + 3Y^2 Z + 3YZ^2 + Z^3) + \beta_\varepsilon(Y+Z)$$

$$= \lambda(Y^3 - 3C_\varepsilon Y + 3(Y^2 - C_\varepsilon)Z + 3YZ^2 + Z^3) + \beta'(Y+Z)$$

$$= \lambda(\mathbb{Y}^3 + 3\mathbb{Y}^2 Z + 3YZ^2 + Z^3) + \beta'(Y+Z)$$

"magic" is that one constant C_ε works for all terms.

Energy identity: $\int_{\mathbb{T}_\varepsilon^d} = \varepsilon^d \sum_{x\in \mathbb{T}_\varepsilon^d}$

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^d} Z_t^2\,dx + \int_{\mathbb{T}_\varepsilon^d} |\nabla_\varepsilon Z_t|^2 + m^2 |Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\,dx$$

$$= -\frac{1}{2}\int_{\mathbb{T}_\varepsilon^d} \lambda\left(Y^3 Z + 3 Y^2 Z^2 + 3 Y Z^3\right) + \beta'(YZ + Z^2)\,dx$$

goal: bound RHS by ① norms of Y, Y^2, Y^3

② small const $\times$ $\|Z\|_{L^2}^2$, $\|\nabla Z\|_{L^2}^2$, $\|Z\|_{L^4}^4$

and absorb them to LHS.

The following can be found in [GH] (TA session)

Lemma (Besov duality): $\alpha\in\mathbb{R}$, $p, p', q, q' \in [1,\infty]$.

$$\frac{1}{p}+\frac{1}{p'}=1 \qquad \frac{1}{q}+\frac{1}{q'}=1$$

Then $\langle f, g\rangle_\varepsilon \lesssim \|f\|_{B^{\alpha,\varepsilon}_{p,q}} \|g\|_{B^{-\alpha,\varepsilon}_{p',q'}}$

Lemma (Interpolate): $\|f\|_{B^\alpha_{p,q}} \le \|f\|^{\theta}_{B^{\alpha_0}_{p_0 q_0}} \|f\|^{1-\theta}_{B^{\alpha_1}_{p_1 q_1}}$

where $\alpha = \theta\alpha_0 + (1-\theta)\alpha_1$.
$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$$
$$\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{1-q_1}$$

Facts: $L^2 = B^0_{2,2}$, $C^\alpha = B^\alpha_{\infty\infty}$. $H^\alpha = B^\alpha_{2,2}$.

Besov duality

$\alpha = -\kappa$

$$\left|\int_{\mathbb{T}_\varepsilon^d} Y^3 Z \, dx\right| \leq \|Y^3\|_{B^{3\alpha}_{\infty\infty}} \underbrace{\|Z\|_{B^{4\kappa}_{1,1}}}$$

Want: $L^2 = B^0_{2,2}$ and $H^1 = B^1_{2,2}$.

Besov embedding

$$\|Z\|_{B^{4\kappa}_{1,1}} \leq \|Z\|_{B^{4\kappa}_{2,2}}$$

Interpolate

$$\leq \|Z\|^{\theta}_{B^0_{22}} \|Z\|^{1-\theta}_{B^1_{22}}$$

$$4\kappa = \theta \cdot 0 + (1-\theta)\cdot 1$$

$$\theta = 1 - 4\kappa$$

$$= \|Z\|^{1-4\kappa}_{L^2} \|Z\|^{4\kappa}_{H^1}$$

Then use Young to write $a \cdot b^{1-4\kappa} \cdot c^{4\kappa}$ into a sum.

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \qquad p_2 = \frac{2}{1-4\kappa} \qquad p_3 = \frac{2}{4\kappa}$$

So $\frac{1}{p_1} + \frac{1-4\kappa}{2} + \frac{4\kappa}{2} = 1 \Rightarrow p_1 = 2$

Therefore,

$$\left|\int_{\mathbb{T}_\varepsilon^d} Y^3 Z \, dx\right| \leq C_\delta \|Y^3\|^2_{C^{3\alpha}} + \delta \|\nabla_\varepsilon Z\|^2_{L^2} + \delta \|Z\|^2_{L^2}$$

Similarly:

$$\left|\int_{\mathbb{T}_\varepsilon^d} Y^2 Z^2 \, dx\right| \leq \|Y^2\|_{C^{2\alpha}} \|Z^2\|_{B^{3\kappa}_{1,1}}$$

$$\leq C_\delta \|Y^2\|^{\cdots}_{C^{2\alpha}} + \delta \|\nabla Z\|^2_{L^2} + \delta \|Z\|^4_{L^4}$$

If time permits we do one more. or, TA session

$$\left|\int_{\mathbb{T}_\varepsilon^d} Y^1 Z^3 \, dx\right| \leq \|Y^1\|_{C^\alpha} \|Z^3\|_{B^{2\kappa}_{1,1}}$$

$$\leq C_\delta \|Y^2\|^{\cdots}_{C^\alpha} + \delta \|\nabla Z\|^2_{L^2} + \delta \|Z\|^4_{L^4}$$

Therefore,

$$\frac{1}{2}\partial_t \int_{\mathbb{T}_\varepsilon^d} Z_t^2\,dx + (1-\delta)\int_{\mathbb{T}_\varepsilon^d} |\nabla_\varepsilon Z_t|^2 + m^2|Z_t|^2 + \frac{\lambda}{2}|Z_t|^4\,dx \le Q_t$$

$$Q_t = 1 + C\left(\|Y_t\|_{C^\alpha}^K + \|Y_t^2\|_{C^{2\alpha}}^K + \|Y_t^3\|_{C^{3\alpha}}^K\right)$$

for some large power K.

Now, we can study what happens when $\varepsilon \to 0$:

recall stationary coupling $\mathbb{P}^\varepsilon$.

under $\mathbb{P}^\varepsilon$, Y, Z are stationary, $X = Y + Z$ is stationary

$X_t \sim \nu^\varepsilon$

Take $\mathbb{E}$ ∂_t term vanishes by stationarity.

$$\mathbb{E}\int_{\mathbb{T}_\varepsilon^d} |\nabla_\varepsilon Z_0|^2 + m^2|Z_0|^2 + \frac{\lambda}{2}|Z_0|^4\,dx \le \mathbb{E}Q_t = \mathbb{E}Q_0 < \infty$$

uniformly in ε

Claim: above bound $\Longrightarrow$ tightness of $(\nu^\varepsilon)_{\varepsilon>0}$

Rmk: ν^ε lives on lattices with different ε. To compare them. We should extend fields ϕ on Λ_ε to continuum.

e.g. $(\mathcal{E}^\varepsilon \phi)(x) = \varepsilon^d \sum_{y\in\Lambda_\varepsilon} \delta^\varepsilon(x-y) f(y)$ $\quad \delta^\varepsilon \to \delta$

But we ignore this technical detail.

Indeed, write $\gamma^\varepsilon = \text{Law of }(Y_0, Z_0)$

def of γ

$$\sup_\varepsilon \int \left(\|\psi\|^2_{C^{-\alpha}} + \|\nabla\zeta\|^2_{L^2} + \|\zeta\|^2_{L^2} + \|\zeta\|^4_{L^4} \right) \gamma^\varepsilon(d\psi \times d\zeta)$$

$$= \sup_\varepsilon \mathbb{E}_{P^\varepsilon} \left[\|Y_0\|^2_{C^{-\alpha}} + \|\nabla Z_0\|^2_{L^2} + \|Z_0\|^2_{L^2} + \|Z_0\|^4_{L^4} \right]$$

$$\lesssim \sup_\varepsilon \mathbb{E}\, Q_0 < \infty$$

above bound and trivial bound on Y_0 by Q_0

why L^4 doesn't lose regularity?

This gives tightness of $(\gamma^\varepsilon)_{\varepsilon>0}$ on $C^{-2\alpha} \times (H^{1-\kappa} \cap L^4)$

since $C^{-\alpha} \hookrightarrow C^{-2\alpha}$, $H^1 \hookrightarrow H^{1-\kappa}$ compact embedding.

But what we care about is ν^ε

which is projection of γ^ε (by summing two factors)

$$\int \|\varphi\|^2_{B^{-\alpha}_{22}} \nu^\varepsilon(d\varphi) = \int \|\psi + \zeta\|^2_{B^{-\alpha}_{2,2}} \gamma^\varepsilon(d\psi \times d\zeta)$$

$$\leq 2 \int \left(\|\psi\|^2_{B^{-\alpha}_{22}} + \|\zeta\|^2_{B^{-\alpha}_{22}} \right) \gamma^\varepsilon(d\psi \times d\zeta)$$

$$\lesssim 2 \int \left(\|\psi\|^2_{C^{-\alpha}} + \|\zeta\|^2_{H^1} \right) \gamma^\varepsilon(d\psi \times d\zeta) \lesssim 1 \quad \text{unif in } \varepsilon$$

$$C^{-\alpha} = B^{-\alpha}_{\infty\infty} \hookrightarrow B^{-\alpha}_{2,2}$$

$$H^1 = B^1_{22} \hookrightarrow B^{-\alpha}_{22}$$

This gives tightness of ν^ε on $B^{-2\alpha}_{22}(\mathbb{T}^2)$

Since $B^{-\alpha}_{22} \hookrightarrow B^{-2\alpha}_{22}$ is compact embedding.

So $\exists$ subseq $\longrightarrow$ limit ν on $B^{-2\alpha}_{22}(\mathbb{T}^2) = H^{-2\alpha}(\mathbb{T}^2)$

Remarks:

① Combined with the infinite volume limit argument <u>with weights</u>
One can construct Φ^4 on whole $\mathbb{R}^2$.

② This limiting measure is <u>not</u> Gaussian
It follows by an interesting argument, but we postpone it to 3D.

③ IBP formula/Dyson-Schwinger extends to continuum:
assume F is cylinder functional on $S'(\mathbb{R}^2)$

i.e. $F(\varphi) = \mathrm{poly}(\varphi(f_1), \dots, \varphi(f_n))$ $\quad f_i \in S(\mathbb{R}^2)$

Compute like Wednesday but now with renormalization C_ε

$$\int DF(\varphi)(x)\,\nu(d\varphi) = \int F(\varphi)\,(m^2-\Delta)\,\varphi(x)\,\nu(d\varphi) + \lambda \int F(\varphi)\underbrace{\left(\varphi(x)^3 - 3C\varphi(x)\right)}_{=:\,[\varphi^3]}\nu(d\varphi)$$

meaningful as distributions in x
(f_i smooth)

$\varphi = Y_0 + Z_0$

$$(Y_0^3 - 3cY_0) + 3(Y_0^2 - c)Z_0 + 3Y_0 Z_0^2 + Z_0^3$$

Difficulty in 3D:

$Y^2 \in C^{-1-\kappa}$.

① For duality, $\langle Y^2, Z^2 \rangle \leq \|Y^2\|_{B^{-1-\kappa}_{\dots}} \|Z^2\|_{B^{1+\kappa}_{\dots}}$

But LHS only has $\|\nabla Z\|_{L^2} \approx \|Z\|_{H^1}$

There's no way to control $B^{1+\kappa}$ norm of Z.

TA session

If time permits:

② The problem is even "before energy estimate".

Classical result for product (Young theorem)

$$\|fg\|_{C^{\min(\alpha,\beta)}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \quad \text{if } \alpha+\beta>0$$

So for $Ψ^2 \cdot Z$ in the Z equation, Z must be $C^{1+\kappa}$.

There's no way to control $C^{1+\kappa}$ norm of Z

(could mention Schauder)

③ One more renormalization beyond Wick is necessary

$$Z \approx (\partial_t - \Delta)^{-1}(Ψ^3 + 3Ψ^2)$$

$$\mathbb{E}[Ψ^2 Z] \approx \mathbb{E}[Ψ^2 (\partial_t - \Delta)^{-1}(Ψ^3 + 3Ψ^2)]$$

both terms diverge

"$-1+2-1 \approx \log$"

Comments about Φ^4_3 (if time permits):

- local solution: Hairer 2013/3 $\xi_\varepsilon = \xi * \psi_\varepsilon$ "smooth regulariz."
  - Using Reg. Stru.
  - Catellier – Chouk 2013/10.
    using Paracontrol distribution (Gubinelli Imkeller Perkowski)
- Global solution: Mourrat – Weber 2016. on $\mathbb{T}^3$. → meas on $\mathbb{T}^3$.

  Gubinelli – Hofmanova 2018.4
  Moinat – Weber 2018.11 global bound over space-time

  Gubinelli – Hofmanova 2018: PDE construction of Φ^4 on $\mathbb{R}^3$ (lattice approx) & axioms.