SL Math Stochastic Quantization Summer School TA Session: Iterated Integrals and more on Wick's Theorem

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1 A stochastic integral identity for the Hermite polynomials

Proposition 1.1. *Let $B : [0,\infty) \to \mathbb{R}$ be a standard Brownian motion. We will denote this as $(B_s)_{s\ge 0}$. For $n \in \mathbb{N}$ and $t > 0$,*

$$n! \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} dB_{s_1} \cdots dB_{s_n} = t^{\frac{n}{2}} H_n\Big(\frac{B_t}{\sqrt{t}}\Big), \tag{1.1}$$

where H_n is the nth Hermite polynomial.

Proof. We prove this by induction on n. For $n = 1$,

$$\int_0^t dB_s = B_t = H_1(B_t) = t^{\frac{1}{2}} H_1\Big(\frac{B_t}{\sqrt{t}}\Big).$$

Now, assume that (1.1) holds for some $n \ge 1$. Then,

$$(n+1)! \int_0^t \int_0^{s_n} \cdots \int_0^{s_1} dB_{s_1} \cdots dB_{s_n} dB_{s_{n+1}} = (n+1) \int_0^t s^{\frac{n}{2}} H_n\Big(\frac{B_s}{\sqrt{s}}\Big)\, dB_s. \tag{1.2}$$

We show this equals $t^{\frac{n+1}{2}} H_{n+1}\Big(\frac{B_t}{\sqrt{t}}\Big)$ by an Itô formula calculation. Along the way, we need to recall the following two identities for the Hermite polynomials:

$$H_n'(x) = n H_{n-1}'(x), \qquad \text{and} \qquad H_{n+1}(x) = x H_n(x) - H_n'(x). \tag{1.3}$$

When differentiating $t^{\frac{n+1}{2}} H_{n+1}\Big(\frac{B_t}{\sqrt{t}}\Big)$, we get a dt term and a dB term. The dB term is

$$t^{\frac{n}{2}} H_{n+1}'\Big(\frac{B_t}{\sqrt{t}}\Big)\, dB_t = (n+1) t^{\frac{n}{2}} H_n\Big(\frac{B_t}{\sqrt{t}}\Big)\, dB_t,$$

and this agrees with (1.2). Thus, we just need to show that the dt term is 0. We compute this as follows:

$$\frac{n+1}{2}t^{\frac{n-1}{2}}H_{n+1}\Big(\frac{B_t}{\sqrt{t}}\Big) - \frac{1}{2}t^{\frac{n-2}{2}}B_tH'_{n+1}\Big(\frac{B_t}{\sqrt{t}}\Big) + \frac{1}{2}t^{\frac{n-1}{2}}H''_{n+1}\Big(\frac{B(t)}{\sqrt{t}}\Big),$$

where the last term is the Itô correction term. Using the relations in (1.3), this is

$$\frac{n+1}{2}t^{\frac{n-1}{2}}\Bigg(H_{n+1}\Big(\frac{B_t}{\sqrt{t}}\Big) - \frac{B_t}{\sqrt{t}}H_n\Big(\frac{B_t}{\sqrt{t}}\Big) + H'_n\Big(\frac{B_t}{\sqrt{t}}\Big)\Bigg) = 0.$$

□

2 A word of caution with multiple stochastic integrals

One may naively wish to write the left-hand side of (1.3) as

$$\int_0^t\int_0^t\cdots\int_0^t dB_{s_1}\cdots dB_{s_n} = B_t^n,$$

but this is not true, as seen from the right-hand side of (1.3). Indeed, one must be careful when dealing with multiple stochastic integrals. We define the left-hand side of (1.3) as an iterated integral. However, one may wish to make sense of an integral

$$\int_{\mathbb{R}_+^n} f(s_1,\dots,s_n)\,dB_{s_1}\cdots dB_{s_n}$$

for some function $f \in L^2(\mathbb{R}_+^n)$, without going through iterated integrals. To do so, we follow the conventions of the book [Tud23]. For a two-sided Brownian motion B, and a function $f \in L^2(\mathbb{R}_+)$, define $B(f) = \int_{\mathbb{R}_+} f(t)\,dB(t)$. For a Boret set $A \subseteq \mathbb{R}_+$ of finite Lebesgue measure, define $B(A) = B(\mathbb{1}_A)$.

Let $\mathcal{E}_n$ be the set of functions $f : \mathbb{R}_+^n \to \mathbb{R}$ that may be written as

$$f(t_1,\dots,t_n) = \sum_{i_1,\dots,i_n=1}^{N} a_{i_1,\dots,i_n}\mathbb{1}_{A_{i_1}\times\cdots\times A_{i_n}}(t_1,\dots,t_n),$$

where $N \geq 1$, the Borel sets $A_1,\dots,A_N \subseteq \mathbb{R}_+$ are disjoint, and $a_{i_1,\dots,i_n} = 0$ if any two indices are equal. For $f \in \mathcal{E}_n$, define

$$I_n(f) := \sum_{i_1,\dots,i_n=1}^{N} a_{i_1,\dots,i_n}B(A_{i_1})\cdots B(A_{i_n}).$$

The condition that $A_1,\dots,A_N$ are disjoint and $a_{i_1,\dots,i_n} = 0$ if any two indices are the same is essential; we will see that this explains why we don't obtain B_t^n

in (1.3). One can show that the set $\mathcal{E}_n$ is dense in $L^2(\mathbb{R}_+^n)$, and for $f \in L^2(\mathbb{R}_+^n)$, we define

$$I_n(f) = \lim_{k\to\infty} I_n(f_k),$$

where f_k is a sequence of functions in $\mathcal{E}_n$ converging to f in $L^2(\mathbb{R}_+^n)$. One can show, as in [Tud23, page 10] that this limit exists and is independent of the choice of approximating sequence.

To see a concrete example, let's look at the value of the integral

$$\int_{\mathbb{R}_+^2} \mathbb{1}_{[0,1]\times[0,1]}\, dB_{s_1}\, dB_{s_2}.$$

The function $\mathbb{1}_{[0,1]\times[0,1]}$ is a simple function which could be used for constructing the classical integral on $\mathbb{R}_+^2$, but it is **not** an element of $\mathcal{E}^2$. We instead approximate $\mathbb{1}_{[0,1]\times[0,1]}$ by the sum

$$\sum_{\substack{1\le j,k\le N\\ j\neq k}} \mathbb{1}_{(\frac{j}{N},\frac{j+1}{N}]\times(\frac{k}{N},\frac{k+1}{N}]},$$

and note each term $\mathbb{1}_{(\frac{j}{N},\frac{j+1}{N}]\times(\frac{k}{N},\frac{k+1}{N}]}$ for $j \neq k$ is in $\mathcal{E}_2$. Then,

$$I_2\Big(\sum_{\substack{1\le j,k\le N\\ j\neq k}} \mathbb{1}_{(\frac{j}{N},\frac{j+1}{N}]\times(\frac{k}{N},\frac{k+1}{N}]}\Big) = \sum_{\substack{1\le j,k\le N\\ j\neq k}} \Big(B_{\frac{j+1}{N}} - B_{\frac{j}{N}}\Big)\Big(B_{\frac{k+1}{N}} - B_{\frac{k}{N}}\Big)$$
$$= \sum_{j=1}^{N} \Big(B_{\frac{j+1}{N}} - B_{\frac{j}{N}}\Big)\Big(B_1 - B_{\frac{j+1}{N}} + B_{\frac{j}{N}}\Big) = B_1^2 - \sum_{j=1}^{N}\Big(B_{\frac{j+1}{N}} - B_{\frac{j}{N}}\Big)^2,$$

and this second term may be thought of as approaching $\int_0^1 (dB_t)^2 = \int_0^1 dt = 1$. Hence, we have

$$\int_{\mathbb{R}_+^2} \mathbb{1}_{[0,1]\times[0,1]}\, dB_{s_1}\, dB_{s_2} = B_1^2 - 1 = H_2(B_1).$$

Going back to (1.3), it can be shown that, for a symmetric function $f \in L^2(\mathbb{R}_+)$,

$$\begin{aligned} I_n(f) &= n! \int_0^\infty \int_0^{s_n} \cdots \int_0^{s_2} f(s_1,\dots,s_n) dB_{s_1}\cdots dB_{s_n} \\ &= n! \int_{0\le s_1\le\cdots\le s_n<\infty} f(s_1,\dots,s_n) dB_{s_1}\cdots dB_{s_n} \\ &= \int_{\mathbb{R}_+^n} f(s_1,\dots,s_n) dB_{s_1}\cdots dB_{s_n} \\ &\neq \int_0^\infty \cdots \int_0^\infty f(s_1,\dots,s_n) dB_{s_1}\cdots dB_{s_n}. \end{aligned} \tag{2.1}$$

To get a feel for why the second equality is true for $n = 2$, take $\mathbb{1}_{0 \le s_1 \le s_2 \le t}$ and approximate f by

$$\sum_{j<k} \mathbb{1}_{(\frac{j}{N}, \frac{j+1}{N}] \times (\frac{k}{N}, \frac{k+1}{N}]}.$$

Then,

$$I_2\Big(\sum_{j<k} \mathbb{1}_{(\frac{j}{N}, \frac{j+1}{N}] \times (\frac{k}{N}, \frac{k+1}{N}]}\Big) = \sum_{j<k}\Big(B_{\frac{j+1}{N}} - B_{\frac{j}{N}}\Big)\Big(B_{\frac{k+1}{N}} - B_{\frac{k}{N}}\Big)$$
$$= \sum_{k=1}^{N}\Big(B_{\frac{k+1}{N}} - B_{\frac{k}{N}}\Big) B_{\frac{k}{N}} \to \int_0^1 B_s \, dB_s = \int_0^1 \int_0^s dB_u \, dB_s.$$

In (2.1), we made the remark that f must be symmetric. Indeed, this certainly must be the case for the second equality of (2.1) to hold. However, we have also seen from the proof of (1.1) that it can be useful to use the iterative definition for doing proof by induction. For a function $f : \mathbb{R}_+^n \to \mathbb{R}_+$, we define the **symmetrization**

$$\widetilde{f}(s_1, \dots, s_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(s_{\sigma_1}, \dots, s_{\sigma_n}).$$

We now show the following:

Lemma 2.1. *For all $f \in L^2(\mathbb{R}_+^n)$, $I_n(f) = I_n(\widetilde{f})$.*

Proof. We show this for functions of the form $\mathbb{1}_{A_1 \times \cdots \times A_n}$. Then, the general result follows by linearity and passing to a limit. We have

$$\widetilde{f}(t_1, \dots, t_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbb{1}_{A_{\sigma(1)} \times \cdots \times A_{\sigma(m)}}(t_1, \dots, t_n).$$

Then,

$$I_n(\widetilde{f}) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} B(A_{\sigma(1)}) \cdots B(A_{\sigma(n)})$$
$$= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} B(A_1) \cdots B(A_n) = B(A_1) \cdots B(A_n) = I_n(f). \qquad \square$$

For $f \in L^2(\mathbb{R}_+^n)$ and a Borel subset $A \subseteq \mathbb{R}_+^n$, we define

$$\int_A f(s_1, \dots, s_n) \, dB_{s_1} \cdots dB_{s_n} := I_n(f \mathbb{1}_A).$$

Using Lemma 2.1, for a general function $f \in L^2(\mathbb{R}_+^n)$,

$$\begin{aligned}
I_n(f) &= \int_{\mathbb{R}_+^n} f(s_1, \dots, s_n) dB_{s_1} \cdots dB_{s_n} \\
&= \int_{\mathbb{R}_+^n} \widetilde{f}(s_1, \dots, s_n) dB_{s_1} \cdots dB_{s_n} \\
&= n! \int_{0 \le s_1 \le \cdots \le s_n < \infty} \widetilde{f}(s_1, \dots, s_n) dB_{s_1} \cdots dB_{s_n} \\
&= n! \int_0^\infty \int_0^{s_n} \cdots \int_0^{s_2} \widetilde{f}(s_1, \dots, s_n) dB_{s_1} \cdots dB_{s_n}.
\end{aligned}$$

We can also show the following starting from an argument of simple functions (see, for example [Tud23, Propositions 1.1-1.2])

Lemma 2.2. *For $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$,*

$$\mathbb{E}\Big(I_n(f) I_m(g)\Big) = \begin{cases} n! \langle \widetilde{f}, \widetilde{g} \rangle_{L^2(\mathbb{R}_+^n)} & n = m \\ 0 & n \neq m. \end{cases}$$

3 General Hermite polynomial identity and relation to Wiener Chaos:

The following generalizes (1.3). For a proof, see [NN18, Proposition 4.1.2] or [Tud23, Proposition 1.6]

Proposition 3.1. *For $f \in L^2(\mathbb{R}_+^2)$,*

$$n! \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f(s_1) \cdots f(s_n) dB_{s_1} \cdots dB_{s_n} = \|f\|_{L^2(\mathbb{R}_+)}^n H_n\left(\frac{B(f)}{\|f\|_{L^2(\mathbb{R}_+)}}\right).$$

To relate this to the Wiener Chaos discussed last time, consider the space $\Omega = C(\mathbb{R}_+)$ with the σ-algebra generated by the projection maps (the smallest σ-algebra so that B_t is a random variable for each t. We take the measure $\mathbb{P}$ on Ω to be that of a standard Brownain motion B. Let H be the Gaussian Hilbert space

$$\Big\{B(f) := \int_{\mathbb{R}_+} f(t)\, dB_t : f \in L^2(\mathbb{R}_+)\Big\}.$$

By the Itô isometry, for each $f \in L^2(\mathbb{R}_+)$, we have,

$$\mathbb{E}[B(f)^2] = \|f\|_{L^2(\mathbb{R}_+)}.$$

Hence, by our result proved last time combined with Proposition 3.1,

$$\begin{aligned} :B(f)^n: &= :\left(\int_{\mathbb{R}_+} f(t)\,dB_t\right)^n: \;= \|f\|^n_{L^2(\mathbb{R}_+)} H_n\left(\frac{B(f)}{\|f\|_{L^2(\mathbb{R}_+)}}\right) \\ &= n! \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f(s_1)\cdots f(s_n) dB_{s_1}\cdots dB_{s_n}. \end{aligned}$$

This representation of the iterated Chaos term can give us a nice tool for studying the regularity of the Wick powers.

Another way of saying this is that the Wick product turns the iterated integral

$$\int_0^\infty \cdots \int_0^\infty f(s_1)\cdots f(s_n)\,dB_{s_1}\cdots dB_{s_n}$$

into the multiple integral

$$\int_{\mathbb{R}_+^n} f(s_1)\cdots f(s_n)\,dB_{s_1}\cdots dB_{s_n},$$

and the two integrals are not the same for $n \geq 2$.

An analogue of this result also holds when we instead work with space-time white noise instead of one-dimensional Brownian motion. See [Nua06].

References

[NN18] David Nualart and Eulalia Nualart. *Introduction to Malliavin calculus*, volume 9 of *Institute of Mathematical Statistics Textbooks*. Cambridge University Press, Cambridge, 2018.

[Nua06] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[Tud23] Ciprian Tudor. *Non-Gaussian selfsimilar stochastic processes*. SpringerBriefs in Probability and Mathematical Statistics. Springer, Cham, [2023] ©2023.