Perturbation theory (2 lectures)

"$e^{-\frac{1}{2}\int |D\phi|^2 + m^2\phi^2 - \frac{\lambda}{4}\int :\Phi^4: dx} D\phi$" $\boxed{\mu : GFF_m}$

$\to e^{-\frac{\lambda}{4}\int :\phi^4: dx} \mu(d\phi) \Big/ \mathbb{E}\left[e^{-\frac{\lambda}{4}\int :\phi^4: dx}\right]$

Feynman diagram:

correlation:

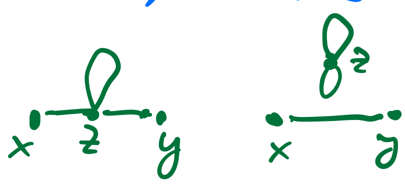
$$\mathbb{E}\left[\phi(x)\,\phi(y)\, e^{-\frac{\lambda}{4}\int :\phi^4: dz}\right] \Big/ \mathbb{E}\left[e^{-\frac{\lambda}{4}\int :\phi^4: dz}\right]$$

Numerator:

$$\mathbb{E}[\phi(x)\,\phi(y)] - \frac{\lambda}{4}\int \mathbb{E}[\phi(x)\,\phi(y) :\phi^4:(z)]\, dz$$
$$+ \frac{\lambda^2}{4^2\cdot 2}\iint \mathbb{E}[\phi(x)\,\phi(y) :\phi(z_1)^4: :\phi(z_2)^4:]\, dz_1 dz_2 + \cdots$$

Let $C(x-y) = (m^2-\Delta)^{-1} = \mathbb{E}[\phi(x)\,\phi(y)]$

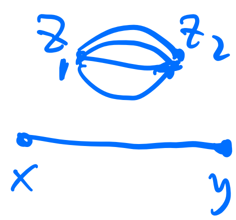
$O(\lambda)$: If without renormalization:



But Wick power $\Rightarrow$ no self-line (last week)

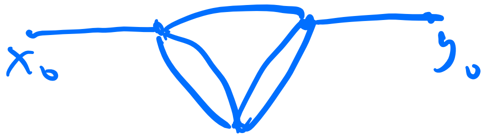
Thus $O(\lambda) = 0$

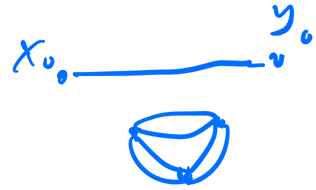
$O(\lambda^2)$: $\frac{\lambda^2}{2}\cdot(3\cdot 2\cdot 1)\cdot 2$ $\quad + \quad \frac{\lambda^2}{2\cdot 4^2}\cdot 4\,3\,2\,1 = \frac{\lambda^2}{2\cdot 4}\cdot 6$



Rmks ① "forget" wick theorem. just draw diagrams

e.g.: • $O(\lambda^3)$



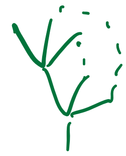


②

(sunset)

finite in 2D

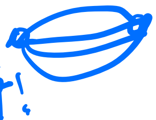
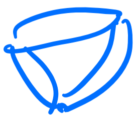
diverge in 3D



$-1-1-1+3 \sim \log$ div. "$\int \frac{1}{|x|^3} d^3x$"

Denominator:

$(1+a)^{-1} = 1 - a + a^2 \ldots$

$\left(1 + \frac{\lambda^2}{2\cdot 4^2} 4!\right.$  $- \frac{\lambda^3}{4^3\cdot 6}\cdot ?$  $\left. + \cdots \right)^{-1}$

$= 1 - \frac{\lambda^2}{4^2\cdot 2} 4!$  $+ O(\lambda^3)$.

"$\frac{3}{4}$"

Then: Numerator/Denominator =

$\Big($  $+ 6\lambda^2$  $+ \frac{3\lambda^2}{4}$  $+ O(\lambda^3) \Big)$

cancel

$\times \Big(1 - \frac{3}{4}\lambda^2$  $+ O(\lambda^3) \Big)$

It's a general fact that the "vacuum diagrams" will be canceled.

$=$  $+ \frac{\lambda^2}{2}\cdot 6$  $+ O(\lambda^3)$

$\Rightarrow$ inner pts connect to $x \cdot y$

Thm (S. Zhu. Zhu '21) Write S^2 the 2pt correlation of Φ^4_2.

$$\| S^2 - C \|_{C^{2-}} \le \lambda^2$$

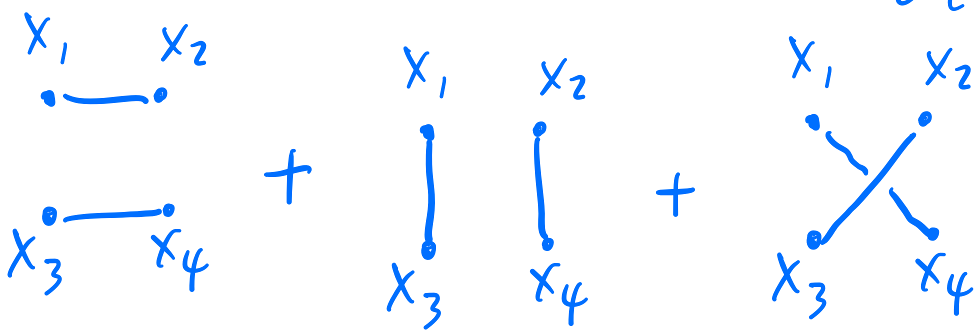
[small scale singularity $\approx$ Gaussian free field.

(Brydges-Frohlich-Sokal 82]

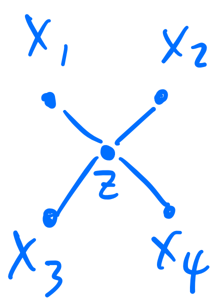
Higher order correlations are calculated similarly:

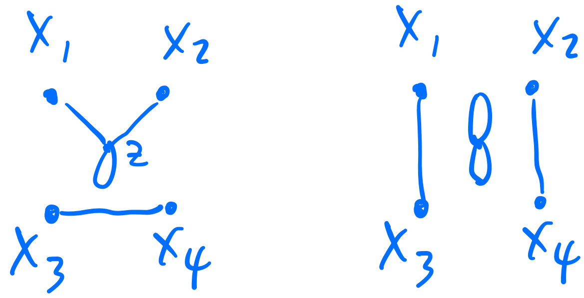
$$S^4 = \mathbb{E}\left(\phi(x_1) \cdots \phi(x_4)\, e^{-\frac{\lambda}{4}\int :\phi^4: dz}\right] \Big/ \mathbb{E}\left[e^{-\frac{\lambda}{4}\int :\phi^4: dz}\right]$$

$O(\lambda^0)$:



$O(\lambda)$:



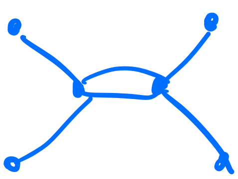


etc

$-\frac{\lambda}{4} \cdot 4!$

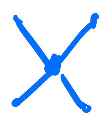
canceled by renormalization

$O(\lambda^2)$



etc.

here: $\sim = S^2$.

Thm: $\Big\| S^4 - ($  $) + 6\lambda$  $\Big\|_{C^{2-}} \le \lambda^2$

(Non-Gaussian)

problem: not a convergent series [Jaffe 1965]

$$Z(\lambda) = \int_{\mathbb{R}} e^{-x^2 - \lambda x^4} dx$$

$$Z(0) = \sqrt{\pi}$$

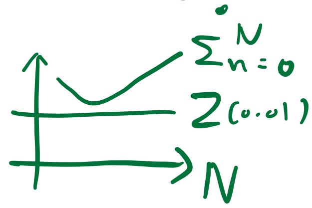
$$Z(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int x^{4n} e^{-x^2} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \frac{(4n-1)!! \sqrt{\pi}}{2^{2n}} \quad \text{div } \forall \lambda > 0$$

$$Z(0.01) \approx 1.7597$$

$$\sum_{n=0}^{1} = 1.7725 - 0.0133 = 1.7592 \quad \text{close!}$$

(but, $\sum_{n=1}^{70} \approx 0.9$ far!
as expected due to divergence.)



asymptotic: $\left| Z(\lambda) - \sum_{n=0}^{N} \right| \xrightarrow{\lambda \downarrow 0} 0$ fix N

more "useful" in practice.

- $e^{-100} = \sum_{n=0}^{\infty} \frac{(-100)^n}{n!} \approx 10^{-44}$ tiny! (convergence)

  but $\sum_{n=0}^{1} = -99 \qquad \sum_{n=0}^{2} = 4901$

- bound on $\left| Z(\lambda) - \sum_{n=0}^{N} \right|$ by, say, λ^{N+1}

Then, we know how accurate $\sum_{n=0}^{N}$ is.

Theorem: Let S^k be k-point correlation of Φ^4 measure on $\mathbb{R}^2$.

(S^k is distribution on $(\mathbb{R}^2)^k$)

Then $\forall \varphi \in S(\mathbb{R}^{2k})$

$$\langle S^k, \varphi \rangle = \sum_{n=0}^{N} \frac{\lambda^n}{n!} \langle F_n^k, \varphi \rangle + \lambda^{N+1} \langle R^k_{N+1}, \varphi \rangle$$

where: F_n^k : only dep on $(m^2-\Delta)^{-1}$ (Feynman diagrams)

$$|\langle R^k_{N+1}, \varphi \rangle| \le C$$

C indep of λ.

Rmk (1.) We prove $\mathbb{T}^2$ case. (so omit weights)

(2) IBP to generate the perturbation expansion (Step 1)

and write R^k_{N+1} as (complicated) products of Φ.

then use PDE bound on Φ. (Step 2)

(3) Dimock 74 (use techniques of Glimm, Jaffe, Spencer 74)

Φ^4_3 [Feldman-Osterwalder 76] [Magnen-Sénéor 76]

"exp cluster property" $\longrightarrow$ smoothness in λ

(observed by [Leb72] diff context)

Then Taylor remainder theorem

(4) We can also get bound e.g. $C + C\lambda^{4k+12(N+1)}$

if λ is large. but let's assume $\lambda < 1$ here.

PDE Estimates:

We will need to bound various <u>products</u> of Φ (and Y)

So L^2 estimate like last week is not good enough. $\to \underline{L^p}$

- bound $\mathbb{E}\|\Phi\|^p_{C^{0-}}$

$$\mathbb{E}\|\Phi\|^p_{C^{0-}} \lesssim \mathbb{E}\|Y\|^p_{C^{0-}} + \mathbb{E}\|Z\|^p_{C^{\beta}} \lesssim 1$$

$$\mathbb{E}\|Z\|^p_{C^{\beta}} \lesssim \lambda^p \qquad \left(0<\beta<\tfrac{1}{4p}\right)$$

- bound $\mathbb{E}\|:\Phi^2:\|^p_{C^{0-}}$ $\quad \mathbb{E}\|:\Phi^3:\|^p_{C^{0}}$

use $\Phi^2 = :Y^2: + 2YZ + Z^2$ etc.

How to bound Z ?

$$(\partial_t - m^2 - \Delta) Z = -\lambda\left(Z^3 + 3Z^2 Y + 3ZY^2 + Y^3\right)$$

<u>L^p-estimate</u>: multiply Z^{p-1}, integrate.

$$\frac{1}{p}\partial_t\|Z^p\|_{L^1} + m^2\|Z^p\|_{L^1} + \||\nabla Z|^2 Z^{p-2}\|_{L^1} + \lambda\|Z^{p+2}\|_{L^1}$$

$$(1) \quad = -\lambda\langle 3Z^{p+1}, Y\rangle - \lambda\langle 3Z^p, Y^2\rangle - \lambda\langle Z^{p-1}, Y^3\rangle$$

+ error
(IBP not exact)

proceed similarly as last week,

$$\mathbb{E}\|Z^p\|_{L^1} + \mathbb{E}\left\||\nabla Z|^2 |Z|^{p-2}\right\|_{L^1} \lesssim \lambda \qquad (A)$$

Not just $\mathbb{E}$ of such norms.

one can also bound moments of norms:

Too much below, could discuss more in TA session.

$$\begin{cases} \mathbb{E}\|Z\|_{L^2}^{2p} \lesssim \lambda^p . \\ \mathbb{E}\left(\|Z\|_{H^{1-}}^2 \|Z\|_{L^2}^{2(p-1)}\right) \lesssim \lambda^p . \end{cases} \qquad (B)$$

For example, use (1) with $p=2$

$$\frac{1}{2}\partial_t \|Z^2\|_{L^1} + m^2\|Z^2\| + \|(\nabla Z)^2\|_{L^1} + \lambda\|Z^4\|_{L^1}$$

$$\leq \lambda Q(Y, \Psi^2, \Psi^3)$$

Then calculate $\partial_t\left(\|Z\|_{L^2}^{2p}\right)$

$$\partial_t\left(\|Z\|_{L^2}^{2p}\right) = \partial_t\left(\|Z^2\|_{L^1}^p\right) = p\,\|Z^2\|_{L^1}^{p-1}\partial_t\|Z^2\|_{L^1}$$

$$\overset{\text{use (1)}}{=} \|Z^2\|_{L^1}^{p-1}\Big(-m^2\|Z^2\|_{L^1} - \|(\nabla Z)^2\|_{L^1} - \lambda\|Z^4\|_{L^1} \underbrace{+\text{RHS of (1)}}_{\text{bound them as before}}\Big)$$

i.e.

$$\partial_t\left(\|Z\|_{L^2}^{2p}\right) + m^2\|Z\|_{L^2}^{2p} + \|Z\|_{L^2}^{2(p-1)}\|\nabla Z\|_{L^2}^2 + \lambda\|Z\|_{L^4}^4\|Z\|_{L^2}^{2(p-1)}$$

$$\left(\|Z^2\|_{L^1} = \|Z\|_{L^2}^2\right)$$

$$\leq \lambda\|Z\|_{L^2}^{2(p-1)} \cdot Q(Y, \Psi^2, \Psi^3)$$

$$\leq \lambda^p Q^p + \delta\|Z\|_{L^2}^{2p} \qquad \frac{1}{p} + \frac{p-1}{p} = 1$$

and $\|Z\|_{H^{1-}} \leq \|Z\|_{L^2} + \|\nabla Z\|_{L^2}$

Cor: $0<\beta<\frac{1}{4p}$. $\quad \mathbb{E}\|Z\|_{C^\beta}^p \lesssim \lambda^p$

Pf: $(\partial_t - m^2 - \Delta) Z = -\lambda(\underbrace{Z^3 + 3Z^2 \Upsilon + 3Z\Upsilon^2 + \Upsilon^3}_{F})$

Schauder:

$$\|Z\|_{L^p_T B^\alpha_{pp}} \lesssim \|Z(0)\|_{B^{\alpha-2/p}_{pp}} + \lambda \|F\|_{L^p_T B^{\alpha-2}_{pp}}$$

So $\mathbb{E}\|Z(0)\|^p_{B^\alpha_{pp}} \overset{\text{stationary}}{=} \frac{1}{T}\mathbb{E}\|Z\|^p_{L^p_T B^\alpha_{pp}}$

$$\lesssim \frac{1}{T}\|Z(0)\|^p_{B^{\alpha-2/p}_{pp}} + \frac{\lambda^p}{T}\mathbb{E}\|F\|^p_{L^p_T B^{\alpha-2}_{pp}}$$

By embedding and stationarity,

$$\frac{1}{T}\mathbb{E}\|F\|^p_{L^p_T B^{\alpha-2}_{pp}} = \underset{\text{all at } 0}{\mathbb{E}\|F\|^p_{B^{\alpha-2}_{pp}}}$$

$$\lesssim \mathbb{E}\|Z^3\|^p_{L^p} + \mathbb{E}\|Z^2\Upsilon\|^p_{B^{-s}_{1+s,\infty}} + \mathbb{E}\|Z\Upsilon^2\|^p_{H^{-s}} + \mathbb{E}\|\Upsilon^3\|^p_{C^{-s}}$$

By (A) $\mathbb{E}\|Z^3\|^p_{L^p} \lesssim \mathbb{E}\|Z\|^{3p}_{L^{3p}} = \mathbb{E}\|Z^{3p}\|_{L^1} \lesssim \lambda$

$$\|Z^2\Upsilon\|_{B^{-s}_{1+s,\infty}} \underset{(s>\gamma)}{\lesssim} \|\Upsilon\|_{C^{-\gamma}}\|Z^2\|_{B^s_{1+s,\infty}}$$

$$\lesssim \|\Upsilon\|_{C^{-\gamma}}\|Z^2\|_{B^{3s}_{1,1}} \quad \text{(embedding)}$$

$$\lesssim \|\Upsilon\|_{C^{-\gamma}}\|Z\|_{H^{4s}}\|Z\|_{L^2} \quad \text{(product)}$$

$$\lesssim \|\Upsilon\|_{C^{-\gamma}}\|Z\|^{2\theta}_{H^{1-\sigma}}\|Z\|^{1+1-2\theta}_{L^2} \quad \left(\theta = \frac{2s}{1-\sigma}\right)$$

$$\mathbb{E}\|Z^2\Upsilon\|^p_{B^{-s}_{1+s,\infty}} \lesssim \mathbb{E}\left[\|\Upsilon\|^p_{C^{-\gamma}}\|Z\|^{2\theta p}_{H^{1-\sigma}}\|Z\|^{(2-2\theta)p}_{L^2}\right]$$

$$\lesssim \lambda^p \quad \text{By (B)}$$

$\left(2\theta p = \frac{4sp}{1-\sigma} \leq 2 \iff 2sp \leq 1-\sigma \quad \text{if } \sigma \text{ small enough}\right)$

other terms similar.

Finally, embed $B^\alpha_{pp} \hookrightarrow C^\beta$.

Integration by parts:

$$\int \frac{\partial F(\Phi)}{\partial \Phi(z)} \nu(d\Phi) \overset{\mathbb{E}\left[\frac{\partial F(\Phi)}{\partial \Phi(z)}\right]}{=} 2\mathbb{E}\left[F(\Phi)(m-\Delta)\Phi(z)\right] + \lambda \mathbb{E}\left[F(\Phi) :\Phi(z)^3:\right]$$

Convolve $C = \frac{1}{2}(m-\Delta)^{-1}$ at z

$$\int C(x-z)\,\mathbb{E}\left(\frac{\partial F(\Phi)}{\partial \Phi(z)}\right) dz$$

$$= \mathbb{E}\left[\Phi(x) F(\Phi)\right] + \lambda \int C(x-z)\, \mathbb{E}\left(F(\Phi) :\Phi(z)^3:\right) dz$$

Define k-point correlation

$$S(x_1 \cdots x_k) = \mathbb{E}\left[\Phi(x_1) \cdots \Phi(x_n)\right]$$

Dyson-Schwinger: another way to generate perturbation series.

1) Take $F(\Phi) = \Phi(y_0)$ let $x = x_0$

$$C(x_0 - y_0) = S(x_0, y_0) + \lambda \mathbb{E}\left(\Phi(y_0)\, C * :\Phi^3:(x_0)\right)$$

$\uparrow$ 0th order expansion of $S(x_0, y_0)$

2) Take $F(\Phi) = \lambda C * :\Phi^3:(x_0)$ let $x = y_0$

$$3\lambda \int C(y_0 - z)\, C(x_0 - z)\, \underbrace{\mathbb{E} :\Phi(z)^2:}\, dz + \lambda \mathbb{E}\left(\Phi(y_0)\, C * :\Phi^3:(x_0)\right)$$
$$+ \lambda^2 \mathbb{E}\left[C * :\Phi^3:(x_0)\;\; C * :\Phi^3:(y_0)\right]$$

less Φ

more Φ, more λ

(Here, $\partial F(\Phi)/\partial\Phi(z)$
$= \partial \int C(x_0 - z) :\phi^3(z): dz \,/\, \partial\Phi(z) = 3C(x_0 - z) :\Phi^2(z):$)

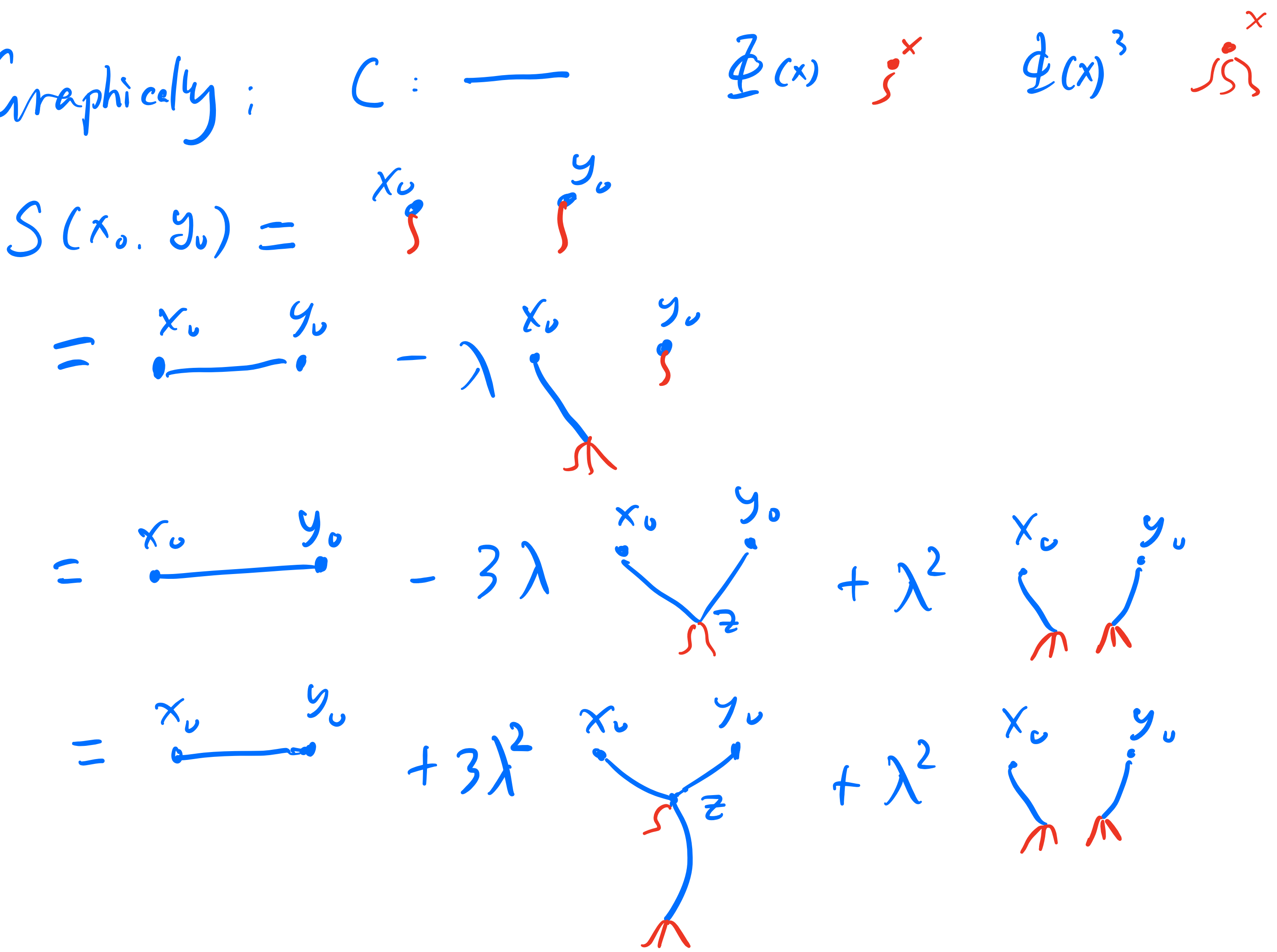
Take $F(\Phi) = \Phi(x)$

$$C(0) = E[\phi(x)^2] + \lambda \int C(x-z)\, E(\Phi(x) :\phi(z)^3:) dz$$

i.e. $E[:\Phi(x)^2:] = -\lambda E[\cancel{G} :\phi^3:(x)\, \phi(x)]$

Now $S(x_0, y_0) = C(x_0 - y_0) + \lambda \cdot 0 + O(\lambda^2)$

Graphically:

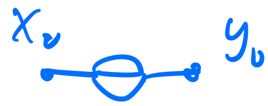


What happens is : each step of IBP

we pick a point with j

1) connect it with another j

2) create a new 

In this way, we'll get 

[when we say "deg" below, we only count <u>blue</u> lines

we also say "external", "internal" vertices]

<u>Lemma</u> : $\forall$ k-point correlation S^k. $\forall N$

$$S^k = \sum_{n=0}^{N} \frac{\lambda^n}{n!} F_n^k + \lambda^{N+1} R_{N+1}^k$$

where F_n^k : graphs with k deg=1 "external" vertices

n deg=4 "internal" vertices

R_{N+1}^k : graphs with k deg-1 "external" vertices

$N+1$ deg≤ 4 "internal" vertices.

<u>Pf</u>: induction. apply IBP to R_{N+1}^k,

which • either eliminate j (so increase deg)

(eventually $\longrightarrow F_{N+1}^k$)

• or create more internal vertex

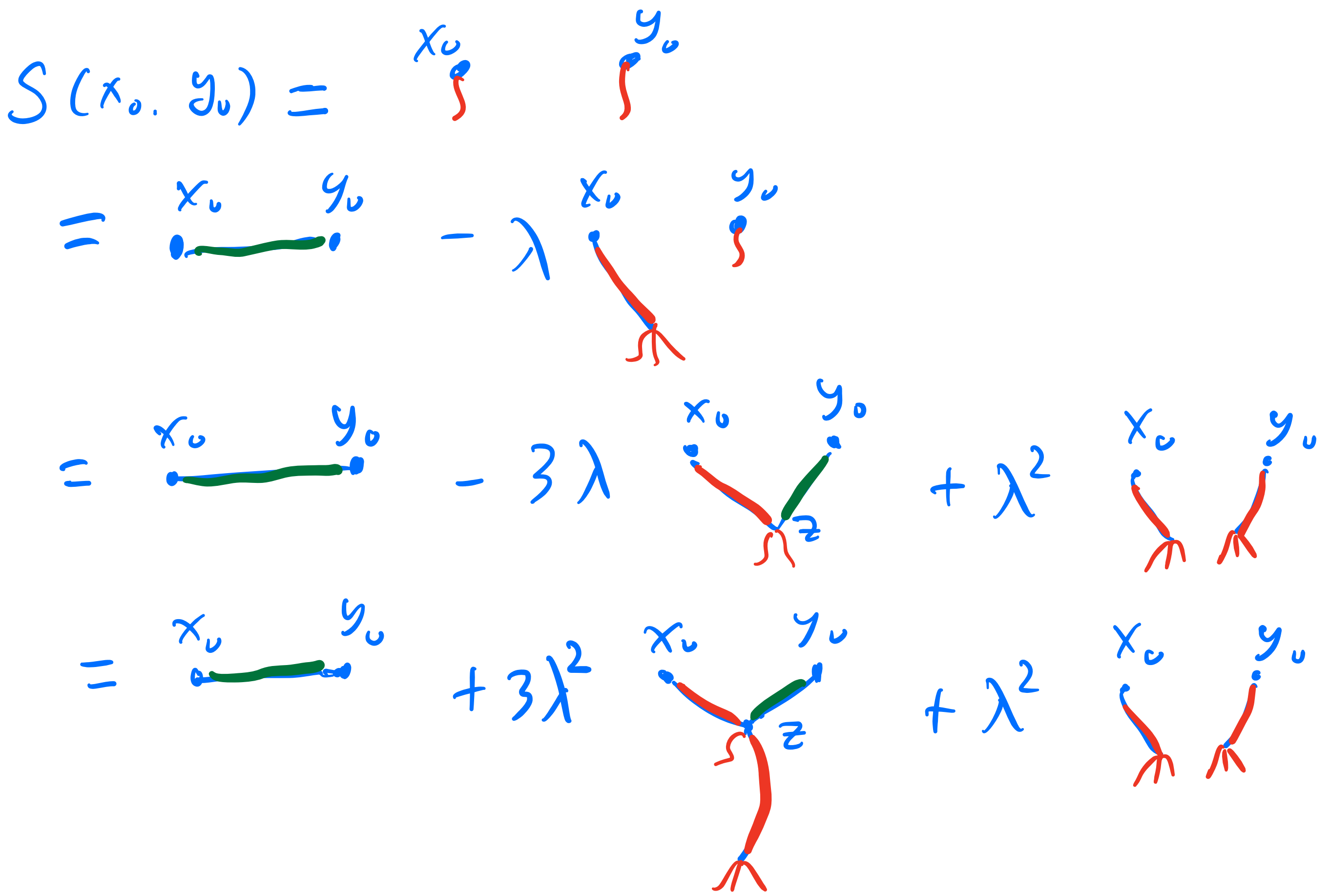
$(\longrightarrow R_{N+2}^k)$

Goal is to show $\|R_n^k\|_{C^0} \leq 1$. To prep for this,

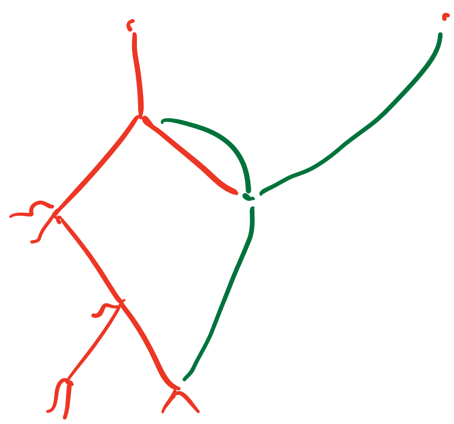
<u>Let's Color the lines:</u>

$\boxed{\text{IBP}}$: pick

- connect to another — green line
- create a new  red line



In general one could get sth like



<u>Lemma</u>

Red lines:

k disjoint rooted trees

each root = an external vertex

(exercise)

To recap:

For such a graph G, the corresponding function:

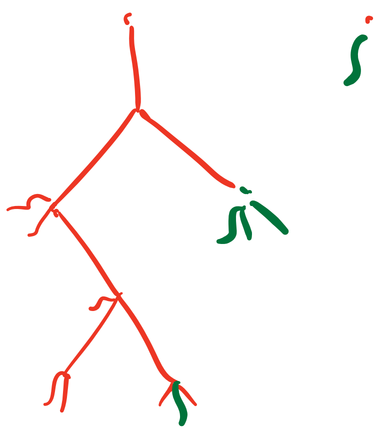
$$I_G(\underbrace{x_1 \cdots x_k}_{\text{external}}) = \int \prod_{(u,v)\in \text{Edges}} C(x_u, x_v)$$

$$\mathbb{E}\Big[\prod_{u\in \text{Ext}} \Phi(x_u)^{1-\deg(u)} \prod_{v\in \text{Int}} :\Phi^{4-\deg v}: \Big] \prod_{z\in \text{Int}} dx_z$$

First, we "cut" green lines:

$$C(x_u, x_v) = \mathbb{E}\left[Y^{(i)}_{(0,x_u)} Y^{(i)}_{(0,x_v)} \right]$$

where $Y^{(i)}$ is (an indep copy of) linear solution.



$$I_G = \mathbb{E}\Big[\prod_{i=1}^{k} F_{T_i} \Big]$$

$T_1 \ldots T_k$ trees (might be trivial)

$$F_T(x) = \int \prod_{(u,v)\in \text{Edges}} C(x_u, x_v) \prod_v f_v(x_v) \prod_v dx_v$$

where, each ext vertex, some function $f_v \in D_1 \cup D_2 \cup D_3$

Let $\Upsilon^{(i)}$ $i = 1, 2, 3 \dots$ be indep copies of Υ.

D_1 : Φ, $\Upsilon^{(i)}$

D_2 : $\Phi \Upsilon^{(i)}$, Φ^2, $\Upsilon^{(i)} \Upsilon^{(j)}$ $i \neq j$

D_3 : $\Phi^2 \Upsilon^{(i)}$, Φ^3, $\Phi \Upsilon^{(i)} \Upsilon^{(j)}$ $\Upsilon^{(i)} \Upsilon^{(j)} \Upsilon^{(k)}$

$$\boxed{\text{Key estimate: } f_i \in D_i \qquad \mathbb{E} \|f_i\|^p_{C^{-\gamma}} \lesssim 1}$$

$\Longrightarrow$ If T is trivial $\|F_T\|_{C^{0-}} \lesssim 1$.

Below, assume T are nontrivial.

$$|I_G|_{C^{0-}} \lesssim \mathbb{E}\left[\prod_{i=1}^{k} \|F_{T_i}\|_{C^{0-}} \right]$$

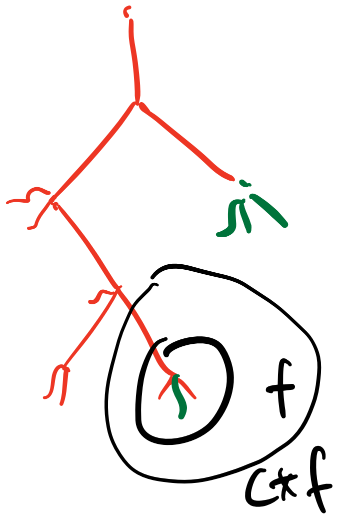
k-variables

1-variable

$$\lesssim \prod_{i=1}^{k} \left(\mathbb{E} \|F_{T_i}\|^k_{C^{0-}} \right)^{1/k}$$

Actually we have $\mathbb{E} \|F_{T_i}\|^k_{C^{2-}} \lesssim 1$

Induction from leaves:



$$\|f\|_{C^{0-}} \le 1 \quad \forall f \in D$$

$$\Rightarrow \|C * f\|_{C^{2-}} \le \|f\|_{C^{0-}}$$

$$\begin{cases} \forall g \in C^1,\ f \in D \\ \|g \cdot f\|_{C^{0-}} \le \|g\|_{C^1} \|f\|_{C^{0-}} \end{cases}$$