These are some very incomplete notes, on what I plan to discuss in the tutorials. Please be mindful that these notes are updated along the way and that they were not checked carefully for typos and mistakes; proceed at your own risk and when in doubt refer to the cited material.

General Background on Littlewood–Paley theory

Notation

- $S(\mathbb{R}^d), S'(\mathbb{R}^d)$ for the Schwartz space of rapidly decreasing smooth functions and their dual space of tempered distributions

- For $\varphi \in S(\mathbb{R}^d)$ $\mathcal{F}(\varphi) = \hat{\varphi}$, $\mathcal{F}^{-1}(\varphi) = \check{\varphi} = (\varphi)^\vee$ the Fourier and inverse Fourier transform with the usual extension to $S'(\mathbb{R}^d)$ via duality.

- I write $A \lesssim B$ if there is a constant $C > 0$ s.t. $A \leqslant CB$; and $A \lesssim_\kappa B$ to emphasise that the constant depends on κ. I write $A \sim B$ if there are constants $c, C > 0$ such that $cA \leqslant B \leqslant CA$.

- I always assume $\rho(x) = \langle x \rangle^{-\ell}$ for some ℓ, where $\langle x \rangle = (1 + |x|^2)^{1/2}$ and note that

$$\langle x \rangle^{-\ell} \langle y \rangle^{\ell} \lesssim \langle x - y \rangle^{|\ell|}, \tag{1}$$

  We will fix $\ell > d$ for the remainder of these notes.

- I use the convention

$$\|f\|_{L^p(\rho)} := \|\rho f\|_{L^p} = \int |\rho f|^p(x) \mathrm{d}x,$$

  which has the advantage that it naturally extends to the case $p = \infty$, that is $\|f\|_{L^\infty(\rho)} = \sup_{x \in \mathbb{R}^d} |\rho f|(x)$.

Things I will give without proof:

- Result about Fourier multipliers (2); I only include the Bernstein Lemma (Lemma 1)

- Existence of the partition of unity with the required properties

- Independence of the partition of unity (though this is partially resolved in Lemma 17) and the fact that the Besov spaces are Banach

- compactness of the embedding in the finite volume/with appropriate weights (Remark 8)

- Difference characterisation of Besov spaces see (6), I do the example for the L^2–based spaces and maybe for $p = q = \infty$.

- Interpolation of Besov spaces (may be sketched if time allows/ people are interested)

- Uniform bounds for the extension operator (requires more background than I have time for); I only discuss general properties and how we may come up with this

- In week 2, I might discuss the proofs of some Lemmas in [3], some possible options are the interpolation Lemma A.5 ([ref]), the wavelet representation Lemma A.9 ([ref]), the estiamtes for the localisers Lemma A.12 ([ref]) and one of the commutators from section A.3.

1 Functions with compactly supported Fourier transform.

As a basic reference for this section, see e.g. [1, Section 2.1].

For functions with compact support in Fourier space, derivatives act like dilations. In particular, for functions spectrally supported on an annulus, we have equivalence of the norms $\|D^k u\|_{L^p}$ for all k, made more precise in the following Lemma.

Lemma 1. *[1, Lemma 2.1] Let $\mathcal{C}$ be an annulus, B be a ball in $\mathbb{R}^d$. Then there is a constant $C>0$ such that for any $k\in\mathbb{N}_{\geqslant 0}$ and $1\leqslant p\leqslant q\leqslant\infty$, and $u\in L^p(\mathbb{R}^d)$ it holds that*

- *If* $\operatorname{supp}(\hat{u})\subset\lambda B$, *then* $\|D^k u_\lambda\|_{L^q} := \sup_{|\alpha|=k}\|\partial^\alpha u_\lambda\|_{L^q}\leqslant C^{k+1}\lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\|u_\lambda\|_{L^p}$,
- *If* $\operatorname{supp}(\hat{u})\subset\lambda\mathcal{C}$, *then* $C^{-k-1}\lambda^k\|u_\lambda\|_{L^p}\leqslant\|D^k u_\lambda\|_{L^p}\leqslant C^{k+1}\lambda^k\|u_\lambda\|_{L^p}$.

Proof. Let $\phi\in C_c^\infty(\mathbb{R}^d)$ be such that $\phi\equiv 1$ on B, then $\hat{u}(\xi)=\phi(\lambda^{-1}\xi)\hat{u}(\xi)$ and therefore

$$\partial^\alpha u = \partial^\alpha f_\lambda * u, \qquad \text{where } f_\lambda := \mathcal{F}^{-1}(\phi(\lambda^{-1}\cdot)), \qquad f := f_1.$$

Thus, from Young's inequality for any $\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}$,

$$\|\partial^\alpha u_\lambda\|_{L^q}\leqslant\|\partial^\alpha f_\lambda\|_{L^r}\|u_\lambda\|_{L^p}.$$

We compute

$$f_\lambda(x)=\mathcal{F}^{-1}(\phi(\lambda^{-1}\cdot))(x)=\lambda^d\mathcal{F}^{-1}(\phi)(\lambda x)=\lambda^d f(\lambda x),$$

so then for any $g\in C_c^\infty(\mathbb{R}^d)$, p, $k\in\mathbb{N}_{\geqslant 0}$, and $|\alpha|=k$

$$\|\partial^\alpha g(\lambda x)\|_{L^p}^p=\int|\partial^\alpha g(\lambda\cdot))(x)^p\mathrm{d}x=\lambda^{kp}\int|\partial^\alpha g(\lambda x))|^p\mathrm{d}x=\lambda^{kp-d}\int|\partial^\alpha g(y)|^p\mathrm{d}y=\lambda^{kp-d}\|\partial^\alpha g\|_{L^p}^p.$$

Therefore,

$$\|\partial^\alpha f_\lambda\|_{L^r}=\lambda^d\|\partial^\alpha(f(\lambda\cdot))(x)\|_{L^r(\mathrm{d}x)}=\lambda^{\left(1-\frac{1}{r}\right)d+k}\|\partial^\alpha f\|_{L^r}$$

and by Hölder's and Young's inequality for any $r>1$

$$\|g\|_{L^r}^r\leqslant\frac{r-1}{r}\|g\|_{L^\infty}^{r-1}\|g\|_{L^1}\leqslant\|g\|_{L^\infty}^r+\|g\|_{L^1}^r\leqslant(\|g\|_{L^\infty}+\|g\|_{L^1})^r.$$

Combined with $\|g\|_{L^1}\leqslant\|(1+|x|^2)^d g\|_{L^\infty}\|(1+|x|^2)^{-d}\|_{L^1}$, this yields for $g=\partial^\alpha f_\lambda$,

$$\|\partial^\alpha f_\lambda\|_{L^r}\leqslant\lambda^{\left(1-\frac{1}{r}\right)d+k}C\|(1+|x|^2)^d\partial^\alpha f\|_{L^\infty}\leqslant\lambda^{\left(1-\frac{1}{r}\right)d+k}C\|(1+\Delta)^d((\cdot)^\alpha\phi)\|_{L^1},$$

and since ϕ is compactly supported, $\|(1+\Delta)^d((\cdot)^\alpha\phi)\|_{L^1} \leqslant C^k$ for some $C>0$. Inserting the condition $1-\frac{1}{r}=\frac{1}{p}-\frac{1}{q}$ yields the first claim.

For the second claim, proceed similarly, write (possible for any positive even degree monomial)

$$|\xi|^{2k} = \sum_{|\alpha|=k} A_\alpha \xi^{2\alpha} \qquad A_\alpha \in \mathbb{N}^d_{\geqslant 0}$$

and writing $\tilde{\phi} \in C_c^\infty(\mathbb{R}^d)$ with $\tilde{\phi} \equiv 1$ on $\mathcal{C}$ and $\tilde{\phi}(B_\varepsilon(0)) \equiv 0$ for some $\varepsilon>0$ sufficiently small. Then, we can proceed as before, with

$$u = \sum_{|\alpha|=k} \tilde{f}_\alpha * \partial^\alpha u, \qquad \tilde{f}_\alpha = A_\alpha \mathcal{F}^{-1}\left((-i\xi)^\alpha \frac{1}{|\xi|^{-2k}} \tilde{\phi}(\xi)\right)$$

so that

$$\mathcal{F}(\tilde{f}_\alpha * \partial^\alpha) = A_\alpha (i\xi)^\alpha \frac{1}{|\xi|^{-2k}} (i\xi)^\alpha = A_\alpha \frac{|\xi|^{2\alpha}}{|\xi|^{-2k}}$$

and thus by definition of $(A_\alpha)_\alpha$,

$$\sum_{|\alpha|=k} A_\alpha \frac{|\xi|^{2\alpha}}{|\xi|^{-2k}} = 1$$

which brings us to the same situation as before. □

More generally, also Fourier multipliers behave as homogeneous functions see e.g. [1, Lemma 2.2], i.e. if for any $|\alpha|<k$ there are constants C_α such that

$$|\partial^\alpha \sigma(\xi)| \leqslant C_\alpha |\xi|^{m-|\alpha|},$$

then with $\sigma(D)u := \mathcal{F}^{-1}(\sigma(\xi)\hat{u}(\xi))$ it holds for $\operatorname{supp}(\hat{u}) \subset \lambda\mathcal{C}$,

$$\|\sigma(D)u\|_{L^p} \leqslant C\lambda^m \|u\|_{L^p}. \tag{2}$$

Besov spaces make use of these scaling properties by decomposing distributions into functions with compact support in Fourier space. Fourier multipliers commute, i.e.

$$\sigma_1(D)\sigma_2(D)u = \sigma_2(D)\sigma_1(D)u.$$

We use the shorthand $(1-\Delta)^s = (1-D^2)^s$.

2 Littlewood–Paley decomposition and Besov spaces

We first define everything in the continuous space and then briefly mention how to adapt the definition to the discrete setting. We will discuss weighted Besov spaces in week 2 only.

To localise functions in frequency space, we require a partition of unity. That such a partition exists can be found in [1], we take this for granted here.

Proposition 2. *There are smooth functions $\chi, \varphi: \mathbb{R}^d \longrightarrow [0,1]$ and an annulus $\mathcal{C}$ such that*

i. $\operatorname{supp}(\varphi) \subset \mathcal{C}$, $\operatorname{supp}(\chi) \subset B = \{|x| < 1/2\}$.

ii. $\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1$, *for all* $\xi \in \mathbb{R}^d$

iii. Whenever $|j - j'| \geq 2$, *it holds that* $\operatorname{supp}(\varphi(2^{-j})) \cap \operatorname{supp}(\varphi(2^{-j'})) = \emptyset$

iv. for any $j \geq 1$, $\operatorname{supp}(\varphi(2^{-j})) \cap \operatorname{supp}(\chi) = \emptyset$.

Definition 3. *Given a partition of unity as in Proposition 2, we define the associated Littlewood–Paley–projectors as*

$$\Delta_{-1} := \chi(D), \qquad \Delta_j = \varphi(2^{-j}D), \qquad j \geq 0.$$

We also write

$$K_{-1} = \mathcal{F}^{-1}(\chi), \qquad K_j = \mathcal{F}^{-1}(\varphi(2^{-j}D)), \qquad j \geq 0$$

for the resulting convolution kernels in position space.

For any distribution we can then write for any $u \in S'(\mathbb{R}^d)$,

$$u = \sum_{j \geq -1} \Delta_j u = \sum_{j \geq -1} K_j * u. \tag{3}$$

where convergence is to be understood in $S'(\mathbb{R}^d)$.

Note that $\Delta_j u$ is now a function supported on a ball (for $j = -1$) or on an annulus (for $j \geq 0$), which enables the use of the scaling properties from Lemma 1.

We readily verify by properties of the Fourier transform and interpolation,

$$K_j = 2^{jd} K_0(2^j x), \qquad \|K_j\|_{L^p(\langle x \rangle^k)} \lesssim 2^{id\frac{(p-1)}{p}}. \tag{4}$$

From (3) we finally define the Besov spaces.

Definition 4. *For any* $s \in \mathbb{R}$, $p, q \in [1, \infty]$ *and* $u \in S'(\mathbb{R}^d)$ *define*

$$\|u\|_{B^s_{p,q}(\mathbb{R}^d)} = \|\{2^{sj}\|\Delta_j u\|_{L^p(\mathbb{R}^d)}\}_{j \geq -1}\|_{\ell^q(\mathbb{Z})} = \left(\sum_{j \geq -1} 2^{qsj}\|\Delta_j u\|^q_{L^p(\mathbb{R}^d)}\right)^{1/q}, \tag{5}$$

and the Besov space $B^s_{p,q}(\mathbb{R}^d) = \{u \in S'(\mathbb{R}^d);\ \|u\|_{B^s_{p,q}(\mathbb{R}^d)} < \infty\}$.

Remark 5.

- The specific norm $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^d)}$ depends on the partition of unity, but all such norms are equivalent (see also Lemma 17) and the space $B^s_{p,q}(\mathbb{R}^d)$ does not depend on the specific choice.

- The space $\left(B^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{B^s_{p,q}(\mathbb{R}^d)}\right)$ is a Banach space.

Let us quickly note that the parameter $s \in \mathbb{R}$ indeed measures regularity in the usual sense.

Theorem 6. *For any* $s, \alpha \in \mathbb{R}$ *and* $u \in S'(\mathbb{R}^d)$

$$\|u\|_{B^s_{p,q}} \sim \|(1-\Delta)^{\alpha/2} u\|_{B^{s-\alpha}_{p,q}}.$$

Proof. Recall that the projectors Δ_j are Fourier multipliers and as such commute with $(1-\Delta)^{\alpha/2}$. Moreover,

$$\partial^\beta (|\xi|^2+1)^m \leqslant C_\beta (|\xi|^2+1)^{m-|\beta|}.$$

Since $\operatorname{supp}(\mathcal{F}(\Delta_{-1}u)) \subset B$, it follows from Lemma 1,

$$\|\Delta_{-1}(1-\Delta)^{\alpha/2} u\|_{L^p} = \|(1-\Delta)^{\alpha/2}(\Delta_{-1}u)\|_{L^p} \lesssim \|\Delta_{-1}u\|_{L^p}.$$

For $j \geqslant 0$, $\operatorname{supp}(\mathcal{F}(\Delta_j u)) \subset 2^j \mathcal{C}$ so that by (2), with $(1-\Delta)^{\alpha/2} = \sigma(D)$ for $\sigma(x) = (1+|x|^2)^{\alpha/2}$

$$\|(1-\Delta)^{\alpha/2}(\Delta_j u)\|_{L^p} \lesssim 2^{j\alpha} \|\Delta_j u\|_{L^p}.$$

Inserting this in the definition of the norms yields

$$\|(1-\Delta)^{\alpha/2}\|^q_{B^{s-\alpha}_{p,q}} \lesssim \sum_{j \geqslant -1} 2^{(s-\alpha)qj} (2^{\alpha j} \|\Delta_j u\|_{L^p})^q = \|u\|_{B^s_{p,q}}$$

and the other inequality follows from $(1-\Delta)^{-\alpha/2}(1-\Delta)^{\alpha/2} = 1$ and thus applying the previous result to $(1-\Delta)^{\alpha/2} u$,

$$\|u\|_{B^s_{p,q}} = \|(1-\Delta)^{-\alpha/2}(1-\Delta)^{\alpha/2} u\|_{B^s_{p,q}} \lesssim \|(1-\Delta)^{\alpha/2} u\|_{B^{s-\alpha}_{p,q}}.$$ □

Embeddings.

Since $\ell^{q_1}(\mathbb{Z}) \hookrightarrow \ell^{q_2}(\mathbb{Z})$ for $q_1 \leqslant q_2$, we directly see $B^s_{p,q_1}(\mathbb{R}^d) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^d)$.

Moreover, for $s_2 \leqslant s_1$, $2^{qs_2j} \leqslant 2^{qs_1j}$ so that $B^{s_1}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^d)$.

Motivated by the Sobolev embeddings (trading integrability for regularity), can look for more embeddings:

Lemma 7. *Let* $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$. *If* $s_2 \leqslant s_1 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, *then*

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^d).$$

Proof. From Lemma 1, since $\Delta_j u$ is supported on a ball of radius 2^j,

$$\|\Delta_j u\|_{L^{p_2}} \leqslant C 2^{jd\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|\Delta_j u\|_{L^{p_1}}.$$

By direct computation

$$\|u\|^{q_2}_{B^{s_2}_{p_2,q_2}} = \sum_j 2^{jq_2 s_2} \|\Delta_j u\|^{q_2}_{L^{p_1}} \leqslant \sum_j 2^{jq_2\left(s_2 + d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right)} \|\Delta_j u\|^{q_2}_{L^{p_1}}$$

$$\leqslant \sup_j 2^{-jq_2\left(s_1 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - s_2\right)} \sum_j 2^{jq_2 s_1} \|\Delta_j u\|^{q_2}_{L^{p_1}} \leqslant \|u\|^{q_2}_{B^{s_1}_{p_1,q_2}(\mathbb{R}^d)},$$

provided $s_2 \leqslant s_1 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Now the claim follows from $\|u\|_{B^{s_1}_{p_1,q_2}(\mathbb{R}^d)} \lesssim \|u\|_{B^{s_1}_{p_1,q_1}(\mathbb{R}^d)}$. □

Remark 8. On a finite volume (or appropriate weighted spaces), then the embedding is compact whenever the inequalitys are strict; that is whenever

$$s_2 < s_1, \qquad \text{and} \qquad s_2 - \frac{d}{p_2} < s_1 - \frac{d}{p_1}.$$

————————————End of Tuesday's session————————————

2.1 Relation to Sobolev and Hölder spaces

Recall the definitions of the Sobolev norms

$$\|u\|^2_{H^s} := \|(1-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^d)}, \qquad s \in \mathbb{R}$$

and

$$\|u\|^p_{W^{p,k}} := \sum_{|\alpha| \leqslant k} \|\partial^\alpha u\|^p_{L^p(\mathbb{R}^d)}, \qquad k \in \mathbb{N}.$$

One can check that $W^{2,s} = H^s$. We will see that Besov spaces "fill the gap" between the Hölder spaces $\mathcal{C}^\alpha$ for $\alpha \in \mathbb{R}$ and the Sobolev spaces $W^{p,k}$, for $p \in [1, \infty]$ and $k \in \mathbb{N}$.

We don't prove these relations; see [1, Theorem 2.36] for details.

More precisely, for any $s\in(0,1)$, the norms satisfy the following difference characterisations

$$\|u\|_{B^s_{p,q}}^q \sim \|u\|_{L^p}^q + \int_{|y|\leq 1}\left(\frac{\|u(y+\cdot)-u\|_{L^p(\mathbb{R}^d)}}{|y|^s}\right)^q \frac{\mathrm{d}y}{|y|^d} \tag{6}$$

with the obvious modification for $q=\infty$. In particular,

$$B^s_{\infty,\infty}(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d), \qquad s\in\mathbb{R}\setminus\mathbb{Z},$$

where for $s\notin(0,1)$, we rely on Theorem 6. By a direct computation

$$B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d), \tag{7}$$

and for $s\notin\mathbb{Z}$,

$$B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d).$$

Remark 9. For $s\in\mathbb{Z}$, there is no equality but $W^{s,p}(\mathbb{R}^d)\subset B^s_{p,p}(\mathbb{R}^d)$. Take e.g. the space $B^1_{\infty,\infty}=\mathcal{C}^1$ which contains all Lipschitz continuous functions, which are only almost everywhere differntiable compared to the space $W^{1,\infty}$ which requires the existence of a weak derivative.

Example. We sketch how to show $B^0_{2,2}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Combined with Theorem 6, this will imply (7). Since $S(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ it is sufficient to consider $u\in S(\mathbb{R}^d)$, in which case the following equality holds in $L^2(\mathbb{R}^d)$,

$$\sum_{j\geq -1} \Delta_j u = u.$$

Hence,

$$\|u\|^2_{L^2(\mathbb{R}^d)} = \sum_{j,j'\geq -1} \langle \Delta_j u, \Delta_{j'} u\rangle_{L^2(\mathbb{R}^d)} = \sum_{j\geq -1} \|\Delta_j u\|^2_{L^2(\mathbb{R}^d)} + \sum_{\substack{j'\neq j \\ j,j'>-1}} \langle \Delta_j u, \Delta_{j'} u\rangle_{L^2(\mathbb{R}^d)}.$$

By definition, $\langle \Delta_j u, \Delta_{j'} u\rangle_{L^2(\mathbb{R}^d)} = 0$ if $|j-j'|\geq 2$, so by Hölder's and Young's inequality,

$$\sum_{\substack{j'\neq j \\ j,j'>-1}} \langle \Delta_j u, \Delta_{j'} u\rangle_{L^2(\mathbb{R}^d)} = \sum_{|j-j'|=1} \langle \Delta_j u, \Delta_{j'} u\rangle_{L^2(\mathbb{R}^d)}$$

$$=2\sum_{j\geq -1} \langle \Delta_j u, \Delta_{j+1} u\rangle_{L^2(\mathbb{R}^d)} \leq 2\sum_{j\geq -1} \|\Delta_j u\|_{L^2}\|\Delta_{j+1}u\|_{L^2}$$

$$\leq \sum_{j\geq -1} \|\Delta_j u\|^2_{L^2} + \sum_{j\geq -1} \|\Delta_{j+1} u\|^2_{L^2}$$

Thus,

$$\|u\|^2_{L^2(\mathbb{R}^d)} \leq 3\sum_{j\geq -1} \|\Delta_j u\|^2_{L^2} \lesssim \|u\|^2_{B^0_{2,2}}.$$

For the opposite inequality, by monotone convergenve and since $\sum_{j\geqslant -1}\varphi_j = 1$

$$\|u\|_{B^0_{s,s}} \lesssim \sum_{j\geqslant -1} \int_{\mathbb{R}^d} |\varphi_j(\xi)|^2 |\hat{u}(\xi)|^2 \mathrm{d}\xi$$

$$\leqslant \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \sum_{j\geqslant -1} |\varphi_j(\xi)|^2 \mathrm{d}\xi \leqslant \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \mathrm{d}\xi = \|u\|_{L^2(\mathbb{R}^d)}.$$

2.2 Interpolation.

Besov spaces are the natural interpolation spaces for the Sobolev spaces $W^{s,p}(\mathbb{R}^d)$.

Lemma 10. *[3, Lemma A.3]For $i = 1, 2$ let $s_i \in \mathbb{R}$, $p_i, q_i \in [1, \infty]$. With*

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \qquad s_\theta = \theta s_1 + (1-\theta) s_2,$$

it holds that

$$\|f\|_{B^{s_\theta}_{p_\theta, q_\theta}} \leqslant \|f\|^\theta_{B^{s_1}_{p_1, q_1}} \|f\|^{1-\theta}_{B^{s_2}_{p_2, q_2}}.$$

Proof. Assume $p_i, q_i \in [1, \infty)$ (with essentially only notational differences for the case $p_i, q_i = \infty$). Applying Hölder's inequality to the conjugate exponents $\frac{p_1}{\theta p}$ and $\frac{p_2}{(1-\theta)p}$ we have

$$\|\Delta_j f\|^p_{L^p} = \int |\Delta_j f(x)|^p \mathrm{d}x = \int |\Delta_j f(x)|^{\theta p} |\Delta_j f(x)|^{(1-\theta)p} \mathrm{d}x$$

$$\leqslant \left(\int |\Delta_j f(x)|^{p_1} \mathrm{d}x \right)^{\theta p / p_1} \left(\int |\Delta_j f(x)|^{p_2} \right)^{(1-\theta) p / p_2}$$

so then again by Hölder now for $\frac{q_1}{\theta q}$ and $\frac{q_2}{(1-\theta) q}$,

$$\|f\|^q_{B^s_{p,q}} = \sum_{j\geqslant -1} 2^{isq} \|\Delta_j f\|^q_{L^p}$$

$$\leqslant \sum_{j\geqslant -1} 2^{isq} \left(\int |\Delta_j f(x)|^{p_1} \mathrm{d}x \right)^{\theta q} \left(\int |\Delta_j f(x)|^{p_2} \right)^{(1-\theta) q}$$

$$= \sum_{j\geqslant -1} (2^{isq} \|\Delta_j f\|^q_{L^{p_1}})^\theta (2^{isq} \|\Delta_j f\|^q_{L^{p_2}})^{1-\theta}$$

$$\leqslant \left(\sum_{j\geqslant -1} 2^{isq_1} \|\Delta_j f\|^{q_1}_{L^{p_1}} \right)^{\theta q / q_1} \left(\sum_{j\geqslant -1} 2^{isq_2} \|\Delta_j f\|^{q_2}_{L^{p_2}} \right)^{(1-\theta) q / q_1}.$$

□

Remark 11.

- The interpolation in s corresponds directly to the interpolation of the weights $(1 + |\xi|^2)^{s_1/2}$ and $(1 + |\xi|^2)^{s_2/2}$.

- Note that the interpolation result does not hold in this generality for the Sobolev spaces $W^{p,k}$, see e.g. [2].

- We might also have time next week to show [3, Lemma A.5], which states that

$$\|f\|_{B^{1-s}_{p,q}} \lesssim \|f\|_{B^{-s}_{p,q}} + \|\nabla f\|_{B^{-s}_{p,q}}.$$

2.3 Duality

The Besov spaces $B^{-s}_{p,q}(\mathbb{R}^d)$ for $s \geqslant 0$, are genuine distribution spaces, so that we cannot define point evaluations $x \mapsto u(x)$ for $u \in B^s_{p,q}(\mathbb{R}^d)$. However, since $B^s_{p,q}(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$, testing against nice test functions, $\varphi \in S(\mathbb{R}^d)$ is well defined via the dual pairing

$$u(\varphi) := {}_{S'(\mathbb{R}^d)}\langle u, \varphi \rangle_{S(\mathbb{R}^d)}.$$

It turns out that asking for φ to be a Schwartz function is more restrictive than necessary, and that we can define the $u(f)$ already for $f \in B^s_{p,q}(\mathbb{R}^d)$.

Lemma 12. *For $p, q \in [1, \infty]$ with Hölder conjugates $p', q' \in [1, \infty]$, the bilinear map*

$$\langle \cdot, \cdot \rangle : B^{-s}_{p,q} \times B^s_{p',q'} \longrightarrow \mathbb{R}; \quad (u, f) \mapsto \sum_{|j-j'| \leqslant 1} {}_{S'(\mathbb{R}^d)}\langle \Delta_j u, \Delta_{j'} f \rangle_{S(\mathbb{R}^d)}$$

is continuous in the sense that

$$|\langle u, f \rangle| \lesssim \|u\|_{B^{-s}_{p,q}(\mathbb{R}^d)} \|f\|_{B^s_{p',q'}(\mathbb{R}^d)}.$$

In particular for $p, q \notin \{1, \infty\}$, we have

$$(B^s_{p,q})^* = B^{-s}_{p',q'}.$$

Proof. The proof is again by Hölder's inequality, see e.g. [1, Proposition 2.29] □

Remark 13. We can also understand the Lemma as follows: In order to define the "mean" (i.e. the integral) of a product fu, it is sufficient that $f \in B^{s_1}_{p,q}$ and $u \in B^{s_2}_{p',q'}$ for some $s_1 + s_2 \geqslant 0$.

To define a pointwise product (the paraproduct), we will later require $s_1 + s_2 > 0$.

3 Besov spaces on the Lattice

To deal with the small scale problems, it is convenient to first work on a finite lattice, so we need to adapt the Littlewood–Paley theory to the discrete setting, in such a way that we recover the continuous theory as the lattice spacing vanishes and the volume cut–off is removed.

For some $N \in \mathbb{N}$, $N \geqslant N_0$, let $\varepsilon = 2^{-N}$ and define the (infinite) lattice $\Lambda_\varepsilon = \varepsilon \mathbb{Z}^d$, and for $M > 0$ such that $M / 2\varepsilon \in \mathbb{N}$, the finite lattice $\Lambda_{\varepsilon,M} = \Lambda_\varepsilon \cap \mathbb{T}^d_M$ where $\mathbb{T}^d_M = \frac{1}{2}[-M, M)^d$ is the d–dimensional Torus of size M. We emphasise that all constants (unless explicitely stated otherwise) are independent of both ε, M.

The discrete Littlewood–Paley projectors

Recall the notation introduced on Monday: Let $\hat{\Lambda}_\varepsilon := \mathcal{F}(\Lambda_\varepsilon) = (\varepsilon^{-1}\mathbb{T})^d$, define a modified partition of unity from χ, φ via

$$\varphi_j^\varepsilon(\xi) = \begin{cases} \varphi_j(\xi), & j < N - \mathcal{J} \\ 1 - \sum_{j < N - \mathcal{J}} \varphi_j(\xi) & j = N - \mathcal{J} \end{cases}$$

where here $\mathcal{J} \in \mathbb{N}$ is to be chosen such that

- $0 \leqslant N - \mathcal{J} \leqslant \mathcal{J}_\varepsilon := \inf\left\{ j \in \mathbb{Z};\ \mathrm{supp}(\varphi_j) \not\subset \left[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}\right)^d \right\}$. Note that $\mathcal{J}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \infty$ and that $\mathcal{J}_\varepsilon = N - \ell$ for some $\ell \in \mathbb{Z}$ (since $\varepsilon = 2^{-N}$)
- Cannot take $\mathcal{J} = \mathcal{J}_\varepsilon$ because the largest frequency is not exactly 2^j but $2^j |\mathcal{C}|$
- Note also that $0 \leqslant N - \mathcal{J}$ means that $\varepsilon > 0$ cannot be too large, i.e. we need to choose $N \geqslant N_0$.

From there, can define the discrete Littlewood–Paley blocks as before

$$\Delta_j^\varepsilon f = \mathcal{F}_{\Lambda_\varepsilon}^{-1}(\varphi^\varepsilon(2^{-j} \cdot) \hat{f})$$

where we recall the definition of the discrete Fourier transform

$$(\mathcal{F}_{\Lambda_\varepsilon} f)(x) = \sum_{k \in \Lambda_\varepsilon} f(k) \mathrm{e}^{-2\pi i \langle x, k \rangle}, \qquad (\mathcal{F}_{\Lambda_\varepsilon}^{-1} f)(k) = \int_{\hat{\Lambda}_\varepsilon} f(x) \mathrm{e}^{2\pi i \langle \xi, k \rangle} \mathrm{d}x.$$

The discrete Besov–norms are then defined in complete analogy to (5),

$$\|f\|_{B_{p,q}^{s,\varepsilon}(\mathbb{R}^d)} = \left(\sum_{-1 \leqslant j \leqslant N - \mathcal{J}} 2^{sjq} \|\Delta_j^\varepsilon f\|_{L^{p,\varepsilon}}^q \right)^{1/q},$$

with

$$\|f\|_{L^{p,\varepsilon}} = \left(\varepsilon^d \sum_{x \in \Lambda_\varepsilon} |f(x)|^p \right)^{1/p} \sim \left(\int \sum_{y \in \Lambda_\varepsilon} \frac{1}{|\Lambda_\varepsilon|} \delta_y(\mathrm{d}x)\, |f(x)|^p \right)^{1/p}, \tag{8}$$

with the obvious definition for the Sobolev and Hölder spaces using the discrete gradient.

Remark 14. The results for the continuous Besov spaces (interpolation, embedding, relation to Sobolev and Hölder spaces) transfer to these discrete spaces, with constants uniformly in the lattice spacing $\varepsilon > 0$.

———————————End of Friday's session———————————

4 Paradifferential calculus

Some references for this part include [1, 4] for the general theory and [7] for results on the weighted spaces (however note the different convention $\|f\|_{L^p(\rho)} = \int |f|^p \mathrm{d}\rho = \int |f|^p \rho(x) \mathrm{d}x$ used there).

We cannot hope to define a meaningful product on all distributions, consider e.g.

$$1 \cdot \delta = \delta \qquad \delta \cdot x = 0 \qquad x \cdot \mathrm{PV}\left(\frac{1}{x}\right) = 1$$

where $\mathrm{PV}(\varphi) = \lim_{\varepsilon \to 0} \int_{|x|<\varepsilon} \frac{\varphi(x)}{|x|} \mathrm{d}x$ is the principal value distribution. If we want all of these to be true, then

$$0 = (\delta \cdot x) \cdot \mathrm{PV}\left(\frac{1}{x}\right) \neq \delta \cdot \left(x \cdot \mathrm{PV}\left(\frac{1}{x}\right)\right) = \delta \cdot 1 = \delta$$

so any such product cannot be associate. More precisely one can show the Schwartz impossibility theorem: There is no associative algebra $\mathcal{A}$ over $\mathbb{R}$ sucht that

- $S'(\mathbb{R}^d) \subset \mathcal{A}$ is a subvector space
- $1_{\mathcal{A}} = 1_{S'(\mathbb{R}^d)}$
- there is a linear operator acting like the differential on $S'(\mathbb{R}^d)$
- for functions, the multiplication reduces to the usual pointwise multiplication; that is for any $f, g \in S(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$, $f \cdot_{\mathcal{A}} g = (fg) := (x \mapsto f(x)g(x))$

While this means we cannot expect to define a useful product for all the distributions, certain products are still possible: The more singular one factor, the more regular we need the other factor to be.

For $u \in S(\mathbb{R}^d)$ and $f \in S'(\mathbb{R}^d)$ we can try to find conditions under which the (formal) rhs of

$$f \cdot u = \sum_{j,j'} \Delta_j f \Delta_{j'} u$$

is meaningful; importantly: the product $\Delta_j f \Delta_{j'} u$ is always well–defined as the product of analytic functions; however their sum might fail to converge.

Due to the support properties of the Littlewood–Paley projectors, (since $2^{j-1}B + 2^j\mathcal{C} \subset 2^j\tilde{\mathcal{C}}$) it makes sense to decompose the product in the overlapping and disjoint frequency parts

$$fu = \sum_{|j-j'| \leq 1} \Delta_j f \Delta_{j'} u + \sum_{|j-j'| \geq 2} \Delta_j f \Delta_{j'} u$$

$$= \sum_{|j-j'| \leq 1} \Delta_j f \Delta_{j'} u + \sum_{j \geq -1} S_{j-1} f \Delta_j u + \sum_{j \geq -1} S_{j-1} u \Delta_j f$$

where

$$S_j g := \sum_{-1 < j' < j} \Delta_{j'} g = 2^{jd} K_{-1}(2^j \cdot) * g$$

which is well defined. This suggests the following decomposition of the product

$$fu = f \prec u + f \circ u + f \succ u \tag{9}$$

where

$$f \prec u := \sum_{j \geq -1} S_{j-1} f \Delta_j u, \qquad f \succ u := u \prec f, \qquad f \circ u := \sum_{|j-j'| \leq 1} \Delta_j f \Delta_{j'} u. \tag{10}$$

We call $f \bowtie u := f \prec u + f \succ u$ the *paraproduct* and $f \circ u$ the *resonant product*.

Theorem 15. *For any* $q \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ *and* $s = s_1 + s_2 \in \mathbb{R}$ *it holds*

i. $\|f \prec u\|_{B^{s_2}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|u\|_{B^{s_2}_{p_2,q_2}}$,

ii. $\|f \prec u\|_{B^{s}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q_1}} \|u\|_{B^{s_2}_{p_2,q_2}}$, *whenever* $s_1 < 0$,

iii. $\|f \circ u\|_{B^{s}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q_1}} \|u\|_{B^{s_2}_{p_2,q_2}}$, *whenever* $s = s_1 + s_2 > 0$.

In particular, for $f \in B^{s_1}_{p_1,q_1}(\mathbb{R}^d)$ *and* $u \in B^{s_2}_{p_2,q_2}(\mathbb{R}^d)$ *where* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} \leqslant 1$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \leqslant 1$ *and* $s_1 + s_2 > 0$, *the product defined by (9) is well–defined and with* $s := s_1 \wedge s_2$ *we have*

$$\|f u\|_{B^{s}_{p,q}} \lesssim \|f\|_{B^{s_1}_{p_1,q_1}} \|u\|_{B^{s_2}_{p_2,q_2}}.$$

Remark 16.

- In other words, theorem 15 states that the paraproduct $f \bowtie u$ is always well–defined with a regularity no worse than $s_1 \wedge s_2$. The product $f \prec u$ behaves like u at the large frequencies (so the regularity is the regularity of u) while f only provides a frequency modulation; see the figure I steal from Max's lectures below

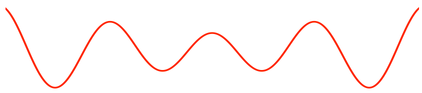

Figure 1.

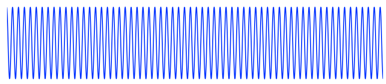

Figure 2.

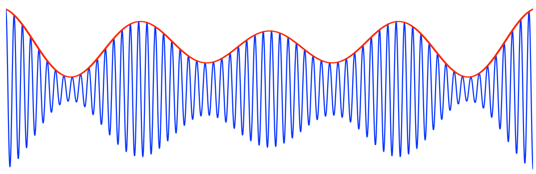

Figure 3.

- The resonant part $f \circ u$ can only be defined in general if $s_1 + s_2 > 0$, which is essentially sharp (at least in this generality); e.g. $B_t \circ \partial_t B_t$ where B_t is a Brownian motion cannot be defined as a continuous bilinear operation on a distribution space (see [5, Proposition 1.29]). This shows that this restriction on the regularity is indeed required and not a technical issue of this particular way to extend the usual product.

- The reason why we include all j, j' sucht that $|j - j'| \leqslant 1$ in the "diagonal"/resonant part is related to the partition of unity. The only a priori information we have is that there is a ball B and an annulus $\mathcal{C}$ such that

$$\operatorname{supp}(\Delta_j f \Delta_{j'} u) \subset \begin{cases} 2^j \mathcal{C}, & |j - j'| > 1 \\ 2^j B, & |j - j'| \leqslant 1. \end{cases}$$

In other words, in the resonant part, the microscopic (i.e. high–frequency) oscillations of both f and u can accummulate and contribute to the large scale (i.e. low–frequency) behavior, possibly causing preventing the defining series from converging.

- In terms of the $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ spaces, the theorem reads as follows: For $\alpha, \beta \in \mathbb{R}$,
    - $\|f \prec g\|_\beta \lesssim \|f\|_{L^\infty} \|g\|_\beta$ for any $f \in L^\infty, g \in \mathcal{C}^\beta$,
    - $\|f \succ g\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta$ for $\beta < 0$, $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$,
    - $\|f \circ g\|_{\alpha+\beta} \lesssim \|f\|_\alpha \|g\|_\beta$ for $\alpha + \beta > 0$ $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$,

  Note that the first paraproduct in this expansion is the most singular term whenever $\beta < 0$ as there is no improvement in regularity coming from f.

- One can check that while $f \succ u$, $f \prec u$ and $f \circ u$ depend on the specific choice of the partition of unity, the product fu does not; e.g. by approximation with smooth functions.

- For this definition to be reasonable, it should satisfy the Leibniz/product rule for differentials whenever the product is defined. This can be checked easily using the facts that the derivative is a continuous operator $S'(\mathbb{R}^d) \longrightarrow S'(\mathbb{R}^d)$ and the (partial) sums converge in $S'(\mathbb{R}^d)$.

- If $\alpha > 0$, then the map $C_c^\infty(\mathbb{R}^d) \times C_c^\infty(\mathbb{R}^d) \longrightarrow C_c^\infty(\mathbb{R}^d) : (f, g) \mapsto fg = f \prec g + f \circ g + f \succ g$ has a continuous extension as a map $B^\alpha_{p_1,q_1} \times B^\alpha_{p_2,q_2} \longrightarrow B^\alpha_{p,q}$. (i.e. the product we defined extends the usual pointwise product)

Observe that $S_{j-1} f \Delta_j u$, $S_{j-1} u \Delta_j f$ are spectrally supported on an annulus of radius $2^j \mathcal{C}$ while $\Delta_j f \Delta_{j'} u$ for $|j - j'| \leqslant 1$ has spectral support on a ball $2^j \tilde{B}$. Hence, the terms in (10) will always be of the form $u = \sum_{j \geqslant -1} u_j$ for some u_j with compact support in Fourier space. To this end we need some auxilliary results.

Lemma 17. *Let $\{u_j\}_{j \geqslant -1}$ be a family of distributions such that for some ball B and an annulus $\mathcal{A}$ it holds that*

$$\operatorname{supp}(\mathcal{F}(u_{-1})) \subset B \qquad \operatorname{supp}(\mathcal{F}(u_j)) \subset 2^j \mathcal{A}. \tag{11}$$

If for some $p, q \in [1, \infty]$,

$$\|\{2^{js} \|u_j\|_{L^p}\}_{j \geqslant -1}\|_{\ell^q} = \left(\sum_{j \geqslant -1} 2^{jsq} \|u_j\|^q_{L^p} \right)^{1/q} < \infty,$$

then $u := \sum_{j \geqslant -1} u_j \in B^s_{p,q}(\mathbb{R}^d) \subset S'(\mathbb{R}^d)$ and

$$\|u\|_{B^s_{p,q}} \lesssim \|\{2^{js} \|u_j\|_{L^p}\}_{j \geqslant -1}\|_{\ell^q}.$$

Proof. The assumptions on the spectral support of u_j imply that there is a an $N \in \mathbb{N}$ such that

$$|i - j| \geqslant N \Longrightarrow \Delta_i u_j = 0.$$

Therefore,

$$\|\Delta_i u\|_{L^p} = \left\| \sum_{|j-i| < N} \Delta_i u_j \right\|_{L^p} \leqslant \sum_{|j-i|<N} \|\Delta_i u_j\|_{L^p} \leqslant \sum_{|j-i|<N} \|u_j\|_{L^p} \|K_i\|_{L^1} \lesssim \sum_{|j-i|<N} \|u_j\|_{L^p}$$

where we used again the representation of the Littlewood–Paley–projectors in terms of convolution kernels,

$$K_j = 2^{jd} K_0(2^j x), \qquad \|K_j\|_{L^p(\langle x\rangle^k)} \lesssim 2^{jd\frac{(p-1)}{p}} \qquad k \in \mathbb{R}.$$

By the computation above

$$\|u\|_{B^s_{p,q}}^q = \sum_{i\geq -1} 2^{isq} \|\Delta_i u\|_{L^p}^q \lesssim \sum_{i\geq -1} 2^{isq} \Bigg(\sum_{|j-i|<N} \|u_j\|_{L^p}\Bigg)^q$$

$$\leq \sum_{i\geq -1} \Bigg(\sum_{|j-i|<N} 2^{(i-j)s} 2^{js} \|u_j\|_{L^p}\Bigg)^q$$

$$\leq 2^{sN} \|a * b\|_{\ell^q(\mathbb{Z})}^q,$$

where

$$a_i := \begin{cases} 2^{js}\|u_j\|_{L^p}, & j \geq -1 \\ 0 & j < -1 \end{cases} \qquad \text{and} \qquad b_i := \begin{cases} 1 & |i| < N \\ 0 & |i| \geq N \end{cases},$$

so that

$$\sum_{|j-i|<N} \|u_j\|_{L^p} = \Bigg(\sum_i \mathbb{1}_{\{|i-j|<N\}} \|u_j\|_{L^p}\Bigg) = \sum_i b_{j-i} a_i =: b * a.$$

By Young's convolution inequality

$$\|a * b\|_{\ell^q(\mathbb{Z})} \leq \|a\|_{\ell^q} \|b\|_{\ell^1} \lesssim \|a\|_{\ell^q} = \|\{2^{js}\|u_j\|_{L^p};\ j \geq -1\}\|_{\ell^q}.$$

which gives the claim. □

Lemma 18. *For $s < 0$, $\|u\|_{B^s_{p,q}} \sim \|\{2^{js}\|S_j u\|_{L^p}\}_{j\geq -1}\|_{\ell^q}$.*

Proof. Writing $\Delta_j u = S_{j+1} u - S_j u$ we have

$$\|\Delta_j u\|_{L^p} \leq \|S_{j+1} u\|_{L^p} + \|S_j u\|_{L^p}$$

so that

$$\|u\|_{B^s_{p,q}} \lesssim \|\{2^{js}\|S_j u\|_{L^p}\}_{j\geq -1}\|_{\ell^q}$$

holds for any $s \in \mathbb{R}$. For the missing (and more interesting) inequality, using $s < 0$ so that $2^{s(j'-j)} \leq 1$ for $j' \leq j$,

$$\|S_j u\|_{L^p} \leq \sum_{j'\leq j} \|\Delta_{j'} u\|_{L^p} \leq \sum_{j'\leq j} 2^{s(j'-j)} \|\Delta_{j'} u\|_{L^p}.$$

Therefore,

$$\sum_{j\geq -1} 2^{jqs} \|S_j u\|_{L^p}^q \leq \sum_{j\geq -1} \Bigg(\sum_{j'\geq -1} \mathbb{1}_{\{j'-j\leq 0\}} 2^{sj'} \|\Delta_{j'} u\|_{L^p}\Bigg)^q =: \|a * b\|_{\ell^q},$$

where $a_j = 2^{sj'} \|\Delta_{j'} u\|_{L^p}$ and $b_j = \mathbb{1}_{\{j \leq 0\}}$ and $(a * b)_j = \sum_{j' \geq -1} a_{j'} b_{j'-j}$. So by Young's inequality

$$\|a * b\|_{\ell^q} \leq \|a\|_{\ell^q} \|b\|_{\ell^1} \lesssim \|a\|_{\ell^q} = \|\{2^{sj} \|\Delta_j u\|_{L^p}\}_j\|_{\ell^q}. \qquad \square$$

Remark 19. Lemma 17 also shows that the definition of the Besov norm is independent of the specific choice for the partition of unity.

Proof of Theorem 15. We claim that there is a ball B and an annulus $\tilde{\mathcal{C}}$ such that for any $j \geq -1$ and $j' \geq -1$ such that $|j - j'| \leq 1$,

$$\operatorname{supp}(\mathcal{F}(S_{j-1} f \Delta_j u)) \subset 2^j \tilde{\mathcal{C}}, \qquad \operatorname{supp}(\mathcal{F}(\Delta_j f \Delta_{j'} u)) \subset 2^j \tilde{B}. \tag{12}$$

Indeed, by properties of the Fourier transform

$$\mathcal{F}(S_{j-1} f \Delta_j u) = \mathcal{F}(S_{j-1} f) * \mathcal{F}(\Delta_j g)$$

so that using $\operatorname{supp}(\varphi * \psi) \subset \operatorname{supp}(\varphi) + \operatorname{supp}(\psi)$ we have

$$\operatorname{supp}(\mathcal{F}(S_{j-1} f \Delta_j u)) \subset 2^{j-1} B + 2^j \mathcal{C} \subset 2^j \tilde{\mathcal{C}}.$$

Here, we used by definition of the partition of unity, there is an annulus $\tilde{\mathcal{C}}$ such that $\frac{1}{2} B + \mathcal{C} \subset \tilde{\mathcal{C}}$. The second estimate follows in the same way.

Starting with i), thanks to the support properties just shown,

$$\|f \prec u\|_{B^{s_2}_{p,q}}^q = \left\| \sum_{j \geq -1} (S_{j-1} f \Delta_j u) \right\|_{B^{s_2}_{p,q}}^q \overset{\text{Lemma 17}}{\lesssim} \sum_{j \geq -1} 2^{j s_2 q} \|S_{j-1} f \Delta_j u\|_{L^p}^q.$$

To estimate the rhs, apply Hölder's inequality,

$$\|S_{j-1} f \Delta_j u\|_{L^p} \leq \|S_{j-1} f\|_{L^{p_1}} \|\Delta_j u\|_{L^{p_2}}$$

and again the convolution kernel representation of the projectors,

$$\|S_{j-1} f\|_{L^{p_1}} \leq \|K_{-1}\|_{L^1} \|f\|_{L^{p_1}} \lesssim \|f\|_{L^{p_1}}.$$

Combined,

$$\|f \prec u\|_{B^{s_2}_{p,q}}^q \lesssim \sum_{j \geq -1} 2^{jsq} \|S_{j-1} f \Delta_j u\|_{L^p}^q \leq \|f\|_{L^{p_1}}^q \sum_{j \geq -1} 2^{jsq} \|\Delta_j u\|_{L^{p_2}}^q = \|f\|_{L^{p_1}}^q \|\Delta_j u\|_{B^s_{p_2,q}}^q,$$

Regarding (ii), we argue in the same way, now instead using Lemma 18 with $s_1 < 0$,

$$\sum_{j \geq -1} 2^{sjq} \|S_{j-1} f \Delta_j u\|_{L^p}^q \leq \sum_{j \geq -1} (2^{s_1 j} \|S_{j-1} f\|_{L^{p_1}})(2^{j s_2} \|\Delta_j u\|_{L^{p_2}})^q$$

$$\lesssim \left(\sum_{j \geq -1} 2^{s_1 q_1 j} \|S_{j-1} f\|_{L^{p_1}}^{q_1} \right)^{q/q_1} \left(\sum_{j \geq -1} 2^{s_2 q_2 j} \|\Delta_j u\|_{L^{p_2}}^{q_2} \right)^{q/q_2}$$

$$\lesssim \|f\|_{B^{s_1}_{p_1,q_1}} \|u\|_{B^{s_2}_{p_2,q_2}}.$$

Finally, the estimate on the resonant term follows using (12), so that

$$j' \vee j \leq \ell - N \qquad \Longrightarrow \qquad \Delta_\ell(\Delta_{j'} f \Delta_j u) = 0.$$

Expanding the definitions

$$(f \circ u) = \sum_{\ell \geq -1} \Delta_\ell (f \circ u) = \sum_{\ell \geq -1} \sum_{j \geq -1} \sum_{|k| \leq 1} \Delta_\ell (\Delta_{j-k} f \Delta_j u),$$

So then we estimate,

$$\begin{aligned} \|\Delta_\ell (f \circ u)\|_{L^p} &\lesssim \sum_{\substack{|k| \leq 1 \\ j \geq \ell - N}} \|\Delta_{j-k} f\|_{L^{p_1}} \|\Delta_j u\|_{L^{p_2}} \\ &\lesssim \sum_{\substack{|k| \leq 1 \\ j \geq \ell - N}} 2^{-j(s_1+s_2)} (2^{js_1} \|\Delta_{j-k} f\|_{L^{p_1}}) (2^{js_2} \|\Delta_j u\|_{L^{p_2}}) \\ &\lesssim 2^{-\ell s} \sum_{j \geq \ell - N} 2^{-s|j-\ell|} \left(\sum_{|k| \leq 1} 2^{js_1} \|\Delta_{j-k} f\|_{L^{p_1}} \right) (2^{js_2} \|\Delta_j u\|_{L^{p_2}}) \end{aligned}$$

where we used $j - \ell \geq -N$ and $s \geq 0$ in the second to last line. To again apply Young's inequality, define

$$a_j = \mathbb{1}_{j \geq -1} 2^{js_2} \|\Delta_j u\|_{L^{p_2}} \sum_{|k| \leq 1} 2^{js_1} \|\Delta_{j-k} f\|_{L^{p_1}}, \qquad b_j = \mathbb{1}_{j \geq -N} 2^{-|j|s},$$

so that the previous computation shows with $\|b\|_{\ell^1} < \infty$ for $s < 0$,

$$\|\{2^{\ell s} \|\Delta_\ell (f \circ u)\|_{L^p}\}\|_{\ell^q} \lesssim \|a * b\|_{\ell^q} \lesssim \|a\|_{\ell^q} \|b\|_{\ell^1} \lesssim \|a\|_{\ell^q} \lesssim \|f\|_{B^{s_1}_{p_1,q_1}} \|g\|_{B^{s_2}_{p_2,q_2}}.$$ □

Remark 20. If $q, q_1, q_2 \neq \infty$, modulo keeping track of more indices, the argument can be used verbatim to show that the sequences of partial sums

$$\left\{ \sum_{j=-1}^{N} S_{j-1} f \Delta_j u \right\}_{N \in \mathbb{N}}, \qquad \left\{ \sum_{j=-1}^{N} \sum_{|k| \leq 1} \Delta_k f \Delta_j u \right\}_{N \in \mathbb{N}},$$

are Cauchy and thus convergent in the same Besov space.

———————————End of Monday's session———————————

5 Weighted Besov spaces

Since the GFF grows logarithmically at (spacial) infinity, control in the infinite volume requires weighted spaces. Here, we just briefly explain how the previous unweighted results can be adapted to the weighted setting. All properties we discussed previously for the Besov spaces transfer verbatim, replacing Hölder's inequality in L^p by Hölder's inequality in the weighted $L^p(\rho)$ spaces.

Define the weighted Besov space

$$B_{p,q}^{s}(\rho) := \{f \in \mathcal{S}'(\mathbb{R}^d);\ \|f\|_{B_{p,q}^{s}(\rho)} < \infty\},$$

where

$$\|f\|_{B_{p,q}^{s}(\rho)}^{q} := \sum_{j \geqslant -1} 2^{sjq} \|\Delta_j f\|_{L^p(\rho)}^{q} = \sum_{j \geqslant -1} 2^{sjq} \|\rho \Delta_j f\|_{L^p}^{q}.$$

The difference characterisation (6) transfers verbatim, that is for $s \in (0,1)$,

$$\|f\|_{B_{p,q}^{s}(\rho)}^{q} \sim \|\rho f\|_{L^p}^{q} + \int_{|y|<1} \frac{\|\rho(f(y+\cdot) - f)\|_{L^p}^{q}}{|y|^{n+sq}} \mathrm{d}y.$$

Adapting the proof of Theorem 15 to this situation we can then show for any $\rho_1 \rho_2 = \rho$ essentially verbatim,

- $\|f \prec u\|_{B_{p,q}^{s_2}(\rho)} \lesssim \|f\|_{L^{p_1}(\rho_1)} \|u\|_{B_{p_2,q}^{s_2}(\rho_2)}$
- $\|f \prec u\|_{B_{p,q}^{s}(\rho)} \lesssim \|f\|_{B_{p_1,q_1}^{s_1}(\rho_1)} \|u\|_{B_{p_2,q_2}^{s_2}(\rho_2)}$ whenever $s_1 < 0$,
- $\|f \circ u\|_{B_{p,q}^{s}(\rho)} \lesssim \|f\|_{B_{p_1,q_1}^{s_1}(\rho_1)} \|u\|_{B_{p_2,q_2}^{s_2}(\rho_2)}$, whenever $s_1 + s_2 > 0$.

In the same way, Lemma 12 becomes for any admissible weight ρ,

$$|\langle u, f \rangle| \lesssim \|u\|_{B_{p,q}^{-s}(\rho)} \|f\|_{B_{p',q'}^{s}(\rho^{-1})}.$$

With $\rho_i = \langle x \rangle^{\ell_i}$ for $i = 1, 2$, the embedding (c.f. Lemma 7) remains intact provided $\ell_1 \geqslant \ell_2$, that is

$$B_{p_1,q_1}^{s_1}(\rho_1) \hookrightarrow B_{p_2,q_2}^{s_2}(\rho_2) \qquad \text{whenever} \quad s_1 - \frac{d}{p_1} > s_2 - \frac{d}{p_2}.$$

The embedding is compact if moreover $s_1 > s_2$, $\ell_1 < \ell_2$. (Since all weighted Besov spaces are equivalent on the torus, this also implies that the embeddings are compact for $\mathbb{R}^d$ replaced by $\mathbb{T}^d$).

In computations, we may prefer to work with $\|\rho f\|_{B_{p,q}^{s}}$ instead of $\|f\|_{B_{p,q}^{s}(\rho)}$. It turns out that both ways to include the weights are equivalent.

Lemma 21. $\|f\|_{B_{p,q}^{s}(\rho)}^{q} \sim \|\rho f\|_{B_{p,q}^{s}}^{q}$.

Proof. Let $n \in \mathbb{N}$ such that $n > |s|$. Writing $\rho \succcurlyeq f := \rho \succ f + \rho \circ f$ and splitting the product $\rho f = \rho \prec f + \rho \succcurlyeq f$, we use Theorem 15 with $1 = \rho \rho^{-1}$ to estimate both terms separately

$$\|\rho \prec f\|_{B_{p,q}^{s}} \lesssim \|\rho\|_{L^\infty(\rho^{-1})} \|f\|_{B_{p,q}^{s}(\rho)} \leqslant \|f\|_{B_{p,q}^{s}(\rho)}.$$

while since $s + n \geqslant 0$ and $s - n < s \wedge 0$,

$$\|\rho \succcurlyeq f\|_{B_{p,q}^{s}} \lesssim \|\rho \succcurlyeq f\|_{B_{p,q}^{s+n}} \lesssim \|f\|_{B_{p,q}^{s-n}(\rho)} \|\rho\|_{B_{\infty,q}^{2n}(\rho^{-1})} \lesssim \|f\|_{B_{p,q}^{s}(\rho)} \|\rho\|_{\mathcal{C}^{2n}(\rho^{-1})} \lesssim \|f\|_{B_{p,q}^{s}(\rho)}$$

Combined this shows

$$\|\rho f\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p,q}(\rho)}.$$

The converse follows in the same way with $f = \rho^{-1}(\rho f) = \rho^{-1} \prec (\rho f) + \rho^{-1} \succcurlyeq (\rho f)$, from which we get

$$\|f\|_{B^s_{p,q}(\rho)} \leqslant \|\rho^{-1} \prec (\rho f)\|_{B^s_{p,q}(\rho)} + \|\rho^{-1} \succcurlyeq (\rho f)\|_{B^s_{p,q}(\rho)}$$

$$\lesssim \|\rho^{-1}\|_{L^\infty(\rho)} \|\rho f\|_{B^s_{p,q}(\rho)} + \|\rho f\|_{B^s_{p,q}} \|\rho^{-1}\|_{\mathcal{C}^n(\rho)} \lesssim \|\rho f\|_{B^s_{p,q}(\rho)}. \qquad \square$$

The Extension operator.

A detailed reference for this part is [6].

We construct the theory on the lattice first, but in the end, we are interested in the continum limit. This means that we need a way of comparing discrete and continuous distributions, so we require an operator to bring all the lattice theories $\{\Lambda_\varepsilon\}_{\varepsilon \in \mathcal{A}}$ to the full space $\mathbb{R}^d$.

Given any distribution on the lattice, we can define $(\cdot)_{\text{ext}}: S'((\varepsilon^{-1}\mathbb{T})^d) \longrightarrow S'(\mathbb{R}^d)$ as the usual periodic extension, that is for $g \in S'((\varepsilon^{-1}\mathbb{T})^d)$ we define for any $\varphi \in S(\mathbb{R}^d)$

$$g_{\text{ext}}(\varphi) := g\left(\sum_{k \in (\varepsilon^{-1}\mathbb{Z})^d} \varphi(\cdot - k)\right).$$

Conversely, for a periodic function, we can define the restriction operator $(\cdot)_{\text{res}}: S'(\mathbb{R}^d) \longrightarrow S'(\Lambda_\varepsilon)$ via

$$g_{\text{res}}(\varphi) := (\psi g)(\varphi_{\text{ext}}) = g(\psi \varphi_{\text{ext}}),$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ is any function statisfying $\sum_{k \in (\varepsilon^{-1}\mathbb{Z})^d} \psi(\cdot - k) = 1$ (e.g. $\psi = 1_{\Lambda_\varepsilon} * \eta$ for some $\eta \in C_c^\infty(\mathbb{R}^d)$). A direct computation shows $(g_{\text{ext}})_{\text{rest}} = g$ and that the definition does not depend on the choice for ψ.

Similarly, a simple way to lift a discrete distribution $f \in S'(\Lambda_\varepsilon)$ to a continuous distribution $f_{\text{dir}} \in S'(\mathbb{R}^d)$ is via a Dirac–comb, i.e.

$$f_{\text{dir}} := |\Lambda_\varepsilon| \sum_{k \in \Lambda_\varepsilon} f(k) \delta(\cdot - k). \tag{13}$$

This gives rise to the following relations

$$\begin{array}{ccc} S'(\Lambda_\varepsilon) & \xrightarrow{\mathcal{F}_{\Lambda_\varepsilon}} & S'(\hat{\Lambda}_\varepsilon) \\ (\cdot)_{\text{dir}} \big\downarrow & & \big\downarrow (\cdot)_{\text{ext}} \\ S'(\mathbb{R}^d) & \xrightarrow{\mathcal{F}_{\mathbb{R}^d}} & S'(\mathbb{R}^d) \end{array}$$

and a direct computation shows

$$(\mathcal{F}_{\mathbb{Z}^d}(f))_{\mathrm{ext}} = \mathcal{F}_{\mathbb{R}^d}(f_{\mathrm{dir}}), \tag{14}$$

i.e. that the diagram commutes. The biggest problem with the extension from $f \mapsto f_{\mathrm{dir}}$ as defined in (13) is the low regularity of δ, which would mean that the extension f_{dir} can only be controlled in spaces less regular than we would naturally expect. Therefore, we instead work with the following extension operator defined already in the lecture

$$\mathcal{E}^\varepsilon f := \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon(\mathcal{F}_{\Lambda_\varepsilon} f)_{\mathrm{ext}}), \qquad f \in S'(\Lambda_\varepsilon),$$

which still statisfies (14), provided ψ is a smear function (as in the definition of $(\cdot)_{\mathrm{res}}$) in the sense of [6]; that is

- $\sum_{k \in \mathbb{Z}^d} \psi(\cdot - k) = 1$
- $\psi \equiv 1$ on $\operatorname{supp} \varphi_j$ for $j \leqslant N - \mathcal{J}$
- technical smallness property of the support of ψ, which ensures that

$$\Delta_j \mathcal{E}^\varepsilon f = \mathcal{E}^\varepsilon \Delta_j^\varepsilon f \qquad j < N - \mathcal{J}.$$

In contrast to f_{dir}, the extended function $\mathcal{E}^\varepsilon f$ has compact spectral support and thus $\mathcal{E}^\varepsilon f \in C^\infty(\mathbb{R}^d) \cap S'(\mathbb{R}^d)$ with $f(k) = \mathcal{E}^\varepsilon(k)$ whenever $k \in \varepsilon^{-1}\mathbb{Z}^d$. Indeed, using the properties of the Fourier transform and (14) we have

$$\begin{aligned} \mathcal{E}^\varepsilon(f) &= \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon) *_{\mathbb{R}^d} \mathcal{F}^{-1}_{\mathbb{R}^d}(\mathcal{F}_{\Lambda_\varepsilon} f)_{\mathrm{ext}} = \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon) *_{\mathbb{R}^d} f_{\mathrm{dir}(\Lambda_\varepsilon)} = \mathcal{F}^{-1}_{\mathbb{R}^d}(\psi^\varepsilon) *_{\Lambda_\varepsilon} f \\ &= |\Lambda_\varepsilon| \sum_{k \in \Lambda_\varepsilon} (\mathcal{F}^{-1}_{\mathbb{R}^d} \psi^\varepsilon)(\cdot - k) f(k) \in C^\infty(\mathbb{R}^d). \end{aligned}$$

Another advantage is that this definition interplays nicely with the Besov spaces, which we cannot prove here.

Lemma 22. *[6, Lemma 2.24]For $s \in \mathbb{R}$, $p, q \in [1, \infty]$, the operators*

$$\mathcal{E}^\varepsilon : B^{s,\varepsilon}_{p,q}(\rho) \longrightarrow B^s_{p,q}(\rho)$$

are bounded uniformly in ε.

More specific material for Φ^4_3

Here we discuss some additional background required to close the energy estimates for $\varphi = X + Y + \phi$, where (mod UV and IR cut–off),

$$\mathscr{L} X = \xi \qquad \mathscr{L} Y = -[\![X^3]\!] - 3\lambda \mathcal{U}_{>}([\![X^2]\!]) \succ Y$$

and for some Ξ which contains unproblematic terms only, the remainder ϕ satisfies

$$\mathscr{L}\phi + \lambda\phi^3 = -3\lambda[\![X^2]\!] \succ \phi - 3\lambda[\![X^2]\!] \circ \phi + 3\lambda^2 b\phi + \Xi.$$

Since all constants in this section are uniform in $\varepsilon > 0$ and $M < \infty$ I drop drop the dependency in the notation to avoid clutter.

1 Wavelet representation and localisation

We are mainly concerned with the localisation operators $\mathcal{U}_{>}$ which are defined in terms of wavelets. Given a function $f \in B^{s,\varepsilon}_{p,q}(\rho)$, we can define

$$\lambda_{j,m}(f) := \Delta^{\varepsilon}_{j} f(2^{-j-\mathcal{J}} m), \qquad -1 \leqslant j \leqslant N - \mathcal{J} \qquad m \in \mathbb{Z}^d.$$

One can show [3, Lemma A.9], that is there is a $\mathcal{J}$ such that for $\lambda = (\lambda_{j,m})_{-1 \leqslant j \leqslant N-\mathcal{J}, m \in \mathbb{Z}^d}$ as defined above, it holds uniformly in ε,

$$\|f\|_{B^{s,\varepsilon}_{\infty,\infty}(\rho)} \sim \|\lambda(f)\|_{b^{s,\varepsilon}_{\infty,\infty}(\rho)} := \sup_{-1 \leqslant j \leqslant N-\mathcal{J}} 2^{sj} \sup_{m \in \mathbb{Z}^d} \rho(2^{-j-\mathcal{J}}|\lambda_{j,m}|), \tag{15}$$

and the function f can be recovered from λ via the *wavelet representation*

$$f = \sum_{-1 \leqslant j \leqslant N-\mathcal{J}} \mathcal{F}^{-1}_{\Lambda_\varepsilon}(\mathcal{F}_{2^{-j-\mathcal{J}}\mathbb{Z}^d}(\lambda_j(f))). \tag{16}$$

From (16) and (15), there is a one–to–one correspondence between sequences $\{\lambda_{j,m}(f)\}_{-1 \leqslant j \leqslant N-\mathcal{J}, m \in \mathbb{Z}^d} \in b^{s,\varepsilon}_{\infty,\infty}(\rho)$ and distributions $f \in B^{s,\varepsilon}_{\infty,\infty}(\rho)$. We recall the definition of the localisers $\mathcal{U}_{\leqslant}$ and $\mathcal{U}>$ in terms of the sequence λ,

$$\mathcal{U}^{\varepsilon}_{>} f := (\lambda_{j,m}(\mathcal{U}^{\varepsilon}_{>} f))_{-1 \leqslant j \leqslant N-\mathcal{J}, m \in \mathbb{Z}^d} \qquad \mathcal{U}^{\varepsilon}_{\leqslant} f := (\lambda_{j,m}(\mathcal{U}^{\varepsilon}_{\geqslant} f))_{-1 \leqslant j \leqslant N-\mathcal{J}, m \in \mathbb{Z}^d}$$

where for some positive sequence $(L_k)_{k \geqslant -1}$ to be chosen indepent of $\varepsilon > 0$,

$$\begin{aligned} \lambda_{j,m}(\mathcal{U}^{\varepsilon}_{>} f) &:= \begin{cases} \lambda_{j,m}(f), & \text{if} |m| \sim 2^k \text{ and } j > L_k \text{ for some } k \in \mathbb{Z}_{\geqslant -1} \\ 0, & \text{otherwise} \end{cases} \\ \lambda_{j,m}(\mathcal{U}^{\varepsilon}_{\leqslant} f) &:= \begin{cases} \lambda_{j,m}(f), & \text{if} |m| \sim 2^k \text{ and } j \leqslant L_k \text{ for some } k \in \mathbb{Z}_{\geqslant -1} \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{17}$$

Heuristically, the sequence L_k defines a notion of “low” and “high” frequencies. The high frequency part $\mathcal{U}^{\varepsilon}_{>} f$ will be irregular, but have better spacial decay, while the regular low frequency will have worse decay at ∞. Tuning the choice of L_k, we can quantify this behavior more precisely.

Lemma 23. *[3, Lemma A.12] Fix $L > 0$. For real numbers $\alpha < \beta < \gamma$ and $a < b < c$ such that*

$$r := \frac{b-a}{\beta - \alpha} = \frac{c-b}{\gamma - \beta} > 0,$$

and there is a sequence $(L_k)_{k\geq -1}$ depending only on L, ρ and r such that the localisers defined in (17) satisfy

$$\begin{aligned}\|\mathcal{U}^\varepsilon_> f\|_{B^{\alpha,\varepsilon}_{\infty,\infty}(\rho^a)} &\lesssim 2^{-(\beta-\alpha)L}\|f\|_{B^{\beta,\varepsilon}_{\infty,\infty}(\rho^b)},\\ \|\mathcal{U}^\varepsilon_\leq f\|_{B^{\gamma,\varepsilon}_{\infty,\infty}(\rho^c)} &\lesssim 2^{(\gamma-\beta)L}\|f\|_{B^{\beta,\varepsilon}_{\infty,\infty}(\rho^b)}.\end{aligned} \tag{18}$$

Proof. Using the equivalence of the norm (15),

$$\begin{aligned}&\|\mathcal{U}^\varepsilon_> f\|_{B^{\alpha,\varepsilon}_{\infty,\infty}(\rho^a)}\\ \lesssim\ & \sup_{-1\leq j\leq N-\mathcal{J}} 2^{\alpha j}\sup_{m\in\mathbb{Z}^d}\rho^\alpha(2^{-j-\mathcal{J}}m)|\lambda_{j,m}(\mathcal{U}^\varepsilon_> f)| && \text{equivalence of norms}\\ =\ & \sup_{k\geq -1}\sup_{\substack{m\sim 2^k\\ L_k<j\leq N-\mathcal{J}}} 2^{-(\beta-\alpha)j}\rho^{a-b}(2^{-j-\mathcal{J}}m)2^{\beta j}\rho^b(2^{-j-\mathcal{J}}m)|\lambda_{j,m}(f)| && \text{definition of the localisation}\\ \lesssim\ & \|f\|_{B^{\beta,\varepsilon}_{\infty,\infty}(\rho^b)}\sup_{k\geq -1}\sup_{\substack{m\sim 2^k\\ L_k<j\leq N-\mathcal{J}}} 2^{-(\beta-\alpha)j}\rho^{a-b}(2^{-j-\mathcal{J}}m). && \text{equivalence of norms}\end{aligned}$$

Now for any $m\sim 2^k$ since $\alpha<\beta$, $a<b$, $j>L_k$ we have using the fact that weights ρ are non–decreasing

$$2^{-(\beta-\alpha)j}\leq 2^{-(\beta-\alpha)L_k} \qquad \text{and} \qquad \rho^{a-b}(2^{-j-\mathcal{J}}m)\lesssim \rho^{a-b}(2^{-L_k-\mathcal{J}}2^k)\lesssim\rho^{a-b}(2^k).$$

Combined,

$$\|\mathcal{U}^\varepsilon_> f\|_{B^{\alpha,\varepsilon}_{\infty,\infty}(\rho^a)}\lesssim\|f\|_{B^{\beta,\varepsilon}_{\infty,\infty}(\rho^b)}\sup_{k\geq -1}2^{-(\beta-\alpha)L_k}\rho^{a-b}(2^k)\leq\|f\|_{B^{\beta,\varepsilon}_{\infty,\infty}(\rho^b)}\sup_{k\geq -1}2^{-(\beta-\alpha)L_k+(b-a)c_k}$$

where $c_k=-\log_2(\rho(2^k))\longrightarrow\infty$. To arrive at the desired estimates (18) we arrive at the first condition on L_k,

$$(\beta-\alpha)(L_k-L)\geq(b-a)c_k \qquad \text{for almost every } k.$$

Following similar reasoning for $\mathcal{U}^\varepsilon_\leq$ we find

$$\|\mathcal{U}^\varepsilon_\leq f\|_{B^{\gamma,\varepsilon}_{\infty,\infty}(\rho^c)}\lesssim\|f\|_{B^{\beta,\varepsilon}_{\infty,\infty}(\rho^b)}\sup_{k\geq -1}2^{(\gamma-\beta)L_k-(c-b)c_k},$$

which leads to the second condition

$$(\gamma-\beta)(L_k-L)\geq(c-b)c_k.$$

For both conditions to hold simultaneously, we see that we have to choose

$$r=\frac{b-a}{\beta-\alpha}=\frac{c-b}{\gamma-\beta} \qquad \text{and} \qquad L_k=r\,c_k.$$

□

Proof of [3, Lemma A.9]. *Wavelet representation (16).* Let us first fix $\mathcal{J}$: With $B_j=\frac{1}{2}[-2^{j+\mathcal{J}},2^{j+\mathcal{J}}]^d$ we choose $\mathcal{J}$ such that

$$\operatorname{supp}(\varphi^\varepsilon_j)\subset B_j \qquad j\in\mathbb{N}.$$

For $j < N - \mathfrak{J}$, $B_j \sim (2^{j+\mathfrak{J}}\mathbb{T})^d \subset (2^N\mathbb{T})^d = \hat{\Lambda}_\varepsilon$ so that for any $f \in S'(\Lambda_\varepsilon)$, we understand $\varphi_j^\varepsilon \mathcal{F}_{\Lambda_\varepsilon}(f)$ as a periodic function on $(2^{j+\mathfrak{J}}\mathbb{T})^d$. Then, we can compute the coefficients for the Fourier series expansion as

$$\lambda_{j,m}(f) = \int_{B_j} (\varphi_j^\varepsilon \mathcal{F}_{\Lambda_\varepsilon} f)(y) \mathrm{e}^{2\pi i 2^{-j-\mathfrak{J}} m y} \mathrm{d}y = \mathcal{F}^{-1}(\varphi^\varepsilon \mathcal{F}_{\Lambda_\varepsilon} f)(2^{j-\mathfrak{J}} m) = \Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m),$$

to write

$$(\varphi_j^\varepsilon \mathcal{F} f)(z) = 2^{(-j-\mathfrak{J})d} \sum_{m \in \mathbb{Z}^d} \lambda_{j,m}(f) \mathrm{e}^{-2\pi i 2^{-j-\mathfrak{J}} m z} = \mathcal{F}_{2^{-j-\mathfrak{J}}\mathbb{Z}^d}(\lambda_{j,\cdot}(f))(z).$$

The boundary case $j = N - \mathfrak{J}$ follows similarly.

Equivalence of norms (15). By definition

$$\|\lambda\|_{b^{s,\varepsilon}_{\infty,\infty}(\rho)} := \sup_{-1 \leqslant j \leqslant N-\mathfrak{J}} 2^{sj} \sup_{m \in \mathbb{Z}^d} \rho(2^{-j-\mathfrak{J}} |\lambda_{j,m}|) = \sup_{-1 \leqslant j \leqslant N-\mathfrak{J}} 2^{sj} \sup_{m \in \mathbb{Z}^d} \rho(2^{-j-\mathfrak{J}} |\Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)|)$$

while

$$\|f\|_{B^{s,\varepsilon}_{\infty,\infty}(\rho)} = \sup_{-1 \leqslant j \leqslant N-\mathfrak{J}} 2^{sj} \sup_{m \in \Lambda_\varepsilon} \rho(x) |\Delta_j^\varepsilon f(x)| = \sup_{-1 \leqslant j \leqslant N-\mathfrak{J}} 2^{sj} \sup_{m \in \mathbb{Z}^d} \rho(2^{-N} m) |\Delta_j^\varepsilon f(2^{-N} m)|.$$

In other words, the difference between the two norms comes down to the resolution of the mesh (being the equal iff $j = N - \mathfrak{J}$) and we immediately see that

$$\|\lambda\|_{b^{s,\varepsilon}_{\infty,\infty}(\rho)} \leqslant \|f\|_{B^{s,\varepsilon}_{\infty,\infty}(\rho)}.$$

The other direction requires some work, starting from

$$|\Delta_j^\varepsilon f(x)| \leqslant \inf_{m \in \mathbb{Z}^d} |\Delta_j^\varepsilon f(x) - \Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)| + |\Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)|.$$

To insert the weight, we use (1) for $\ell > 0$, that is $\rho(x)\rho(z)^{-1} \leqslant \rho(x-y)^{-1}$. For $x \in 2^{-N}\mathbb{Z}^d$, we can find $z \in 2^{-j-\mathfrak{J}}\mathbb{Z}^d$ such that $|x - 2^{-j-\mathfrak{J}} m| \leqslant \sqrt{d}\, 2^{-j-\mathfrak{J}-1}$ and thus

$$\inf_{z \in 2^{-j-\mathfrak{J}}\mathbb{Z}^d} \rho(x)\rho(z)^{-1} \lesssim \inf_{z \in 2^{-j-\mathfrak{J}}\mathbb{Z}^d} \rho(x-z)^{-1} \leqslant \left(1 + \left|\sqrt{d}\, 2^{-j-\mathfrak{J}-1}\right|^2\right)^{\ell/2} \leqslant C_1. \tag{19}$$

Following the same strategy as in the proof of the Bernstein Lemma 1 (that is multiplying by a function identically 1 on the support of $\mathcal{F}(\Delta_j^\varepsilon f)$ and estimating the kernel) we find for $j < N - \mathfrak{J}$

$$\rho(x) |\Delta_j^\varepsilon f(x) - \Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)| \leqslant C_2 2^{-\mathfrak{J}-1} \|\Delta_j^\varepsilon f\|_{L^{\infty,\varepsilon}(\rho)} \tag{20}$$

putting all of this together we find

$$\begin{aligned} \rho(x) |\Delta_j^\varepsilon f(x)| &\leqslant \inf_{m \in \mathbb{Z}^d} \rho(x) |\Delta_j^\varepsilon f(x) - \Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)| + \rho(x) |\Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)| \\ &\leqslant C_2 2^{-\mathfrak{J}-1} \|\Delta_j^\varepsilon f\|_{L^{\infty,\varepsilon}(\rho)} + \rho(2^{-j-\mathfrak{J}} m) |\Delta_j^\varepsilon f(2^{-j-\mathfrak{J}} m)| \, \rho(x) \rho(2^{-j-\mathfrak{J}} m)^{-1} \\ &\leqslant C_2 2^{-\mathfrak{J}-1} \|\Delta_j^\varepsilon f\|_{L^{\infty,\varepsilon}(\rho)} + C_1 \|\lambda\|_{b^{s,\varepsilon}_{\infty,\infty}(\rho)}. \end{aligned}$$

Taking the supremum over all $x \in \Lambda_\varepsilon$ and chosing $\mathcal{J}$ sufficiently large so that $C_2 2^{-\mathcal{J}-1} < 1$, we obtain the claim. □

Proof of (20). For notational simplicity, let $u := \Delta_j^\varepsilon f$ and $\lambda = 2^j$ and let me ignore the weight, trusting that it can be dealt with using (19). We start by rewriting

$$|u(x) - u(z)| = \left\| \int_0^1 \mathrm{d}\theta \nabla u(x + \theta(z - x))(z - x) \right\|_{L^\infty} \leqslant |z - x| \, \|\nabla u\|_{L^\infty}.$$

To estimate the lhs, we proceed as in the proof of the Bernstein Lemma: There is a ball B such that $\mathcal{F}(u) \subset 2^j B$, so let $\phi \in C_c^\infty(\mathbb{R}^d)$ that is $\phi \equiv 1$ on B. Then,

$$\nabla u = \nabla G_\lambda * u \qquad G_\lambda := \mathcal{F}^{-1}(\phi(\lambda^{-1} \cdot)),$$

so

$$\|\nabla u\|_{L^\infty} \leqslant \|\nabla G_\lambda\|_{L^1} \, \|u\|_{L^\infty}$$

and it remains to estimate $\|\nabla G_\lambda\| \leqslant \sup_{|\alpha| \leqslant 1} \|\partial^\alpha G_\lambda\|_{L^1}$. Repeating the same computations as in the Bernstein Lemma we have

$$\|\nabla G_\lambda\|_{L^1} \lesssim \lambda.$$

Therefore, using that $|x - 2^{-j-\mathcal{J}} m| \leqslant \sqrt{d} \, 2^{-j-\mathcal{J}-1}$

$$|\Delta_j^\varepsilon f(x) - \Delta_j^\varepsilon f(2^{-j-\mathcal{J}} m)| \leqslant |2^{-j-\mathcal{J}} m - x| \, 2^j \|\Delta_j^\varepsilon f\|_{L^{\infty,\varepsilon}} \lesssim 2^{-\mathcal{J}-1} \|\Delta_j^\varepsilon f\|_{L^{\infty,\varepsilon}}$$ □

2 Approximate duality and commutators

[I want to discuss and prove [3, Lemma A.13] and one of the commutators in [3, Lemma A.14]].

Recall that in the energy estimate, we have to control the terms

$$\langle \rho^4 \varphi, -\lambda [\![X^2]\!] \circ \varphi \rangle + \langle \rho^4 \varphi, -\lambda [\![X^2]\!] \succ \varphi \rangle,$$

both of which are individually problematic, but thanks to an approximate duality, the combination of both terms can be controlled. Define for $f, g, h \in S(\mathbb{R}^d)$ the operator

$$C(f, g, h) := h \circ (f \prec g) - f \, (h \circ g).$$

Once can show that for $\rho = \rho_1 \rho_2 \rho_3$, $p_1, p_2, p_3 \in [1, \infty]$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\alpha + \beta + \gamma > 0$, $\beta + \gamma \neq 0$, C extends to a trilinear operator on the appropriate Besov spaces with the bounds [maybe show this bound also]

$$\|C(f, g, h)\|_{B^{\beta+\gamma}_{p,\infty}(\rho)} \leqslant \|f\|_{B^\alpha_{p_1,\infty}(\rho_1)} \|g\|_{B^\beta_{\infty,\infty}(\rho_2)} \|h\|_{B^\gamma_{p_3,\infty}(\rho_3)}.$$

Again for $f, g, h \in S(\mathbb{R}^d)$, we also define

$$D_\rho(f, g, h) = \langle \rho f, g \circ h \rangle - \langle \rho (f \prec g), h \rangle.$$

Note that D essentially measures by how much $g \succ$ fails to be the dual of $g \circ$; we want to show that this error can be controlled.

Lemma 24. *(Approximate duality) [3, Lemma A.13] Let ρ, ρ_i be as before. If α, β, γ are such that*

- $\alpha, \gamma > 0$
- $\beta + \gamma < 0$
- $\alpha + \beta + \gamma > 0,$

then D_ρ extends to a trilinear operator $H^\alpha(\rho_1) \times \mathcal{C}^\beta(\rho_2) \times H^\gamma(\rho_3) \longrightarrow \mathbb{R}$, with

$$|D_\rho(f, g, h)| \lesssim \|f\|_{H^\alpha(\rho_1)} \|g\|_{\mathcal{C}^\beta(\rho_2)} \|h\|_{H^\gamma(\rho_3)}.$$

Proof. We can rewrite the operator D_ρ in term of C_ρ defined in (2),

$$D_\rho(f, g, h) = \langle \rho, C(f, g, h) \rangle - \langle \rho, (f \prec g) \succ h \rangle - \langle \rho, (f \prec g) \prec h \rangle.$$

Then we just have to apply the estimates for C and the paraproduct estimates.

□

Sketch of Proof for (2). □

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