L^p estimates for Φ^4_2.

Ref: Shen-Zhu-Zhu paper on perturbation theory. Lemma 2.1.

Consider Φ^4_2 equation

$$\Phi = Z + Y,$$

$$(\partial_t - \Delta + m) Z = \xi$$

$$(\partial_t - \Delta + m) Y = -\frac{\lambda}{2}\left(Y^3 + 3ZY^2 + 3{:}Z^2{:}\,Y + {:}Z^3{:}\right)$$

Here and throughout, omit ε parameter. [If you want, can imagine this all takes place in the continuum, although we may not have covered the stochastic estimates for Z, $:Z^2:$, $:Z^3:$ in the continuum.)

Simplification: unit volume, so that we can avoid dealing with weights.

Goal: get estimates on Y in terms of

$$Z, \quad :Z^2:, \quad :Z^3:,$$

which we control using probability theory.

Lemma. Let $p \geq 1$. We have that

$$\frac{1}{2p} \partial_t \left(\|Y\|_{L^{2p}}^{2p} \right) = -(2p-1) \| Y^{p-1} \nabla Y \|_{L^2}^2 - \| Y^{p-2} \nabla (|Y|^2) \|_{L^2}^2$$

$$-m\|Y\|_{L^{2p}}^{2p} - \frac{\lambda}{2}\|Y\|_{L^{2(p+1)}}^{2(p+1)}$$

$$-\frac{\lambda}{2}\int_{\mathbb{T}^2} G\, Y^{2p-1} \quad ,$$

where $G := ZY^2 + :Z^2:Y + :Z^3:$.

Remark. To get estimates on Y, we will need to bound the bad terms (red) by the good terms (green). The good terms are good b/c they come with a sign.

Note if $G \equiv 0$, then we would obtain that $\|Y\|_{L^{2p}}$ is decreasing in time.

Pf. We write

$$(\partial_t - \Delta + m)Y = -\frac{\lambda}{2}Y^3 - \frac{\lambda}{2}G.$$

Thus, we compute

$$\frac{1}{2p}\partial_t(Y^{2p}) = Y^{2p-1}\partial_t Y$$

$$= Y^{2p-1}\left(\Delta Y - mY - \frac{\lambda}{2}Y^3 - \frac{\lambda}{2}G\right)$$

$$= Y^{2p-1}\Delta Y - mY^{2p} - \frac{\lambda}{2}Y^{2(p+1)} - \frac{\lambda}{2}GY^{2p-1}.$$

We have that (implicit sum over j)

$$\frac{1}{2p}\Delta(Y^{2p}) = \frac{1}{2p}\partial^j\partial_j(Y^{2p})$$

$$= \partial^j(Y^{2p-1}\partial_j Y)$$

$$= (2p-1)Y^{2p-2}\partial^j Y\partial_j Y$$

$$+ \varphi^{2p-1} \partial^j \partial_j \varphi$$

$$= (2p-1)\varphi^{2(p-1)} |\nabla \varphi|^2 + \varphi^{2p-1} \Delta \varphi .$$

Inserting this identity for $\varphi^{2p-1}\Delta\varphi$, we obtain

$$\frac{1}{2p}(\partial_t - \Delta)(\varphi^{2p}) = -(2p-1)\varphi^{2(p-1)}|\nabla\varphi|^2 - m\varphi^{2p} - \frac{\lambda}{2}\varphi^{2(p+1)} - \frac{\lambda}{2} G \varphi^{2p-1} .$$

The desired result now follows by integrating in space. □

Now, suppose z, $:z^2:$, $:z^3:$ were in L^∞. Then we could easily bound

$$\left|\int z Y^{2p+1}\right| \leq \|z\|_{L^\infty} \|Y\|_{L^{2p+1}}^{2p+1}$$

$$\leq \gamma \|Y\|_{L^{2p+2}}^{2p+2} + C_\gamma \|z\|_{L^\infty}^{2p+2}$$

(apply Young, i.e. $xy \leq \frac{x^q}{q} + \frac{y^{q'}}{q'}$ where

$$1 = \frac{1}{q} + \frac{1}{q'},$$

$$q = \frac{2p+2}{2p+1}, \quad q' = 2p+2).$$

Can handle other terms similarly, and obtain

$$\partial_t \|Y\|_{L^{2p}}^{2p} + m\|Y\|_{L^{2p}}^{2p} + p\|Y^{p-1}\nabla Y\|_{L^2}^2 + \lambda\|Y\|_{2p+2}^{2p+2} \leq C_\lambda \max\left(\|Z\|_{L^\infty}, \|{:}Z^2{:}\|_{L^\infty}, \|{:}Z^3{:}\|_{L^\infty}\right)^{2p+2}.$$

We would then have bounded the size of Y by the sizes of

$$Z, \quad {:}Z^2{:}, \quad {:}Z^3{:},$$

which was our goal.

However, are not in L^∞.

OTOH, they are "almost" in L^∞, since they lie in

$$B^{-k}_{\infty,\infty} \text{ for any } k>0.$$

Thus, we should be able to perturb our previous argument, by making use of the good term $\|Y^{p-1}\nabla Y\|_{L^2}^2$,

which controls derivatives of Y. We turn to this next. We will need the following

Analytic lemmas.

Lemma. Let $\kappa > 0$. For $f \in B^{-\kappa}_{\infty,\infty}$, $g \in B^{\kappa}_{\infty,\infty}$, we have that

$$\left| \int fg \right| \lesssim \|f\|_{B^{-\kappa}_{\infty,\infty}} \|g\|_{B^{\kappa}_{1,1}} .$$

Pf. Have that

$$\int fg = \sum_{|j-j'| \lesssim 1} \int \Delta_j f \, \Delta_{j'} g ,$$

thus

$$\left| \int fg \right| \leq \sum_{|j-j'| \lesssim 1} \|\Delta_j f\|_{L^\infty} \|\Delta_{j'} g\|_{L^1}$$

$$\lesssim \|f\|_{B^{-k}_{\infty,\infty}} \sum_{j'} 2^{j'k} \|\Delta_j g\|_{L^1}$$

$$\lesssim \|f\|_{B^{-k}_{\infty,\infty}} \|g\|_{B^{k}_{1,1}} . \quad \square$$

Lemma. For $\alpha > 0$, we have that

$$\|f\|_{B^{-\alpha}_{1,1}} \lesssim_\alpha \|f\|_{L^1} .$$

Pf. For any j, we have that

$$\|\Delta_j f\|_{L^1} \lesssim \|f\|_{L^1}$$

(by, say, Young's convolution inequality, since

$$\Delta_j f = \check{\varphi}_j * f, \text{ and}$$

$\check{e}_j$ is L^1-normalized. □

Lemma. We have that

$$\|g\|_{B^{\kappa}_{1,1}} \leq \|g\|^{\frac{1}{2}}_{B^{-\kappa}_{1,1}} \|g\|^{\frac{1}{2}}_{B^{3\kappa}_{1,1}}$$

Pf. Cauchy-Schwarz. □

Lemma. We have that

$$\|f\|_{B^{1-\gamma}_{1,1}} \lesssim \|\nabla f\|_{B^{-\gamma}_{1,1}} + \|f\|_{L^1}.$$

Pf. By Bernstein's inequality, for $j \geq 0$, we have that

$$\|\Delta_j f\|_{L^1} \lesssim 2^{-j} \|\Delta_j \nabla f\|_{L^1}.$$

Thus

$$\|f\|_{B^{1-\gamma}_{1,1}} \lesssim \|f\|_{L^1} + \sum_{j \geq 0} 2^{j(1-\gamma)} 2^{-j} \|\Delta_j \nabla f\|_{L^1}$$

$$\lesssim \|f\|_{L^1} + \|\nabla f\|_{B^{-\gamma}_{1,1}}. \quad \square.$$

With these lemmas in hand, we return to estimating Υ. First,

$$\lambda \left| \int Z\, \Upsilon^{2p+1} \right|$$

$$\lesssim \lambda \|Z\|_{B^{-\kappa}_{\infty,\infty}} \|\Upsilon^{2p+1}\|_{B^{\kappa}_{1,1}}$$

$$\lesssim \lambda \|z\|_{B^{-\kappa}_{\infty,\infty}} \|Y^{2p+1}\|^{\frac{1}{2}}_{B^{-\kappa}_{1,1}} \|Y^{2p+1}\|^{\frac{1}{2}}_{B^{3\kappa}_{1,1}}$$

$$\lesssim \lambda \|z\|_{B^{-\kappa}_{\infty,\infty}} \|Y^{2p+1}\|^{\frac{1}{2}}_{L^1} \|Y^{2p+1}\|^{\frac{1}{2}}_{B^{3\kappa}_{1,1}}.$$

$$\leq \lambda \|z\|_{B^{-\kappa}_{\infty,\infty}} \|Y\|^{\frac{2p+1}{2}}_{L^{2p+2}} \|Y^{2p+1}\|^{\frac{1}{2}}_{B^{3\kappa}_{1,1}}.$$

We estimate

$$\|Y^{2p+1}\|_{B^{3\kappa}_{1,1}} \leq \|\nabla(Y^{2p+1})\|_{L^1} + \|Y^{2p+1}\|_{L^1}$$

$$\lesssim p\|Y^{2p}\nabla Y\|_{L^1} + \|Y^{2p+1}\|_{L^1}$$

$$\lesssim p \| \psi^{p+1} \|_{L^2} \| \psi^{p-1} \nabla \psi \|_{L^2}$$

$$+ \| \psi \|_{L^{2p+2}}^{2p+1} .$$

From this, obtain

$$\lambda \left| \int Z \psi^{2p+1} \right|$$

$$\lesssim p \lambda \| Z \|_{B^{-\kappa}_{\infty,\infty}} \| \psi \|_{L^{2p+2}}^{\frac{2p+1}{2}} \| \psi \|_{L^{2p+2}}^{\frac{p+1}{2}} \| \psi^{p-1} \nabla \psi \|_{L^2}^{\frac{1}{2}}$$

$$+ \lambda \| Z \|_{B^{-\kappa}_{\infty,\infty}} \| \psi \|_{L^{2p+2}}^{2p+1} .$$

Estimate by Young

$$\le \gamma \lambda \|Y\|_{L^{2p+2}}^{2p+2} + C_\gamma \lambda \|Z\|_{B^{-\varepsilon}_{\infty,\infty}}^{2p+2} .$$

Estimate

$$= p\lambda \|Z\|_{B^{-\varepsilon}_{\infty,\infty}} \|Y\|_{L^{2p+2}}^{\frac{3p+2}{2}} \|Y^{p-1}\nabla Y\|_{L^2}^{\frac{1}{2}}$$

$$\le \gamma p \|Y^{p-1}\nabla Y\|_{L^2_4}^{2} \qquad \left(q = \tfrac{4}{3},\ q' = 4 \right)$$

$$+ C_\gamma \lambda^{\frac{4}{3}} p \|Z\|_{B^{-\varepsilon}_{\infty,\infty}}^{\frac{4}{3}} \|Y\|_{L^{2p+2}}^{2p+2-\frac{2}{3}} .$$

$$\leq \gamma p \, \| \psi^{p-1} \nabla \psi \|_{L^2}^2 \qquad \left(\begin{array}{l} q = \frac{2p+2}{2p+2-\frac{2}{3}} \\ q' = \frac{2p+2}{\frac{2}{3}} \end{array} \right)$$

$$+ \gamma \lambda \, \| \psi \|_{L^{2p+2}}^{2p+2}$$

$$+ C \, p^{3(p+1)} \lambda^{\left(\frac{4}{3} - \frac{1}{q}\right) q'} \| z \|^{\frac{4}{3}\left(\frac{2p+2}{\frac{2}{3}}\right)}$$

Using that

$$\left(\frac{4}{3} - \frac{1}{q}\right) q' = \frac{4}{3} q' - q' \left(1 - \frac{1}{q'}\right)$$

$$= \frac{4}{3} q' - q' + 1$$

$$= \frac{q'}{3} + 1$$

$$= \frac{(2p+2)}{\left(\frac{2}{3}\right)3} + 1$$

$$= p + 2,$$

we finally obtain

$$\le \gamma p \|Y^{p-1}\nabla Y\|_{L^2}^2 + \gamma\lambda \|Y\|_{L^{2p+2}}^{2p+2} + C p^{3(p+1)} \lambda^{p+2} \|Z\|_{B^{-K}_{\infty,\infty}}^{4(p+1)}.$$

Putting the estimates for and together,

obtain

$$\frac{\lambda}{2}\left|\int z\, \psi^{2p+1}\right| \leq \gamma p \|\psi^{p-1}\nabla\psi\|_{L^2}^2$$

$$+ \gamma p \lambda \|\psi\|_{L^{2p+2}}^{2p+2}$$

$$+ C\left(\lambda + p^{3(p+1)}\lambda^{p+2}\right)\|z\|_{B^{-k}_{\infty,\infty}}^{4(p+1)} .$$

Arguing similarly, can obtain

$$\frac{\lambda}{2}\left|\int :z^2:\, \psi^{2p}\right| \leq \gamma p \|\psi^{p-1}\nabla\psi\|_{L^2}^2$$

$$+ \gamma \lambda \|\psi\|_{L^{2p+2}}^{2p+2}$$

$$+ C\left(\lambda + p^{3(p+1)}\lambda^{1+\frac{p+1}{3}}\right) \| :z^2: \|_{B^{-k}_{\infty p \infty}}^{\frac{4(p+1)}{3}},$$

$$\frac{\lambda}{2} \left| \int :z^3: \, y^{2p-1} \right| \le \gamma p \| y^{p-1} \nabla y \|_{L^2}^2$$

$$+ \gamma \lambda \| y \|_{L^{2p+2}}^{2p+2}$$

$$+ C\left(\lambda + p^{3(p+1)}\lambda^{1+\frac{p+1}{5}}\right) \| :z^3: \|_{B^{-k}_{\infty p \infty}}^{\frac{4(p+1)}{5}}$$

In the end, obtain

$$\frac{1}{2p}\partial_t \| y \|_{L^{2p}}^{2p} + m \| y \|_{L^{2p}}^{2p} + \frac{p}{2} \| y^{p-1} \nabla y \|_{L^2}^2$$
$$+ \frac{\lambda}{4} \| y \|_{L^{2(p+1)}}^{2(p+1)}$$

$$\leq C\left(\lambda + p^{3(p+1)}\lambda^{p+2}\right)$$

$$\left(\|Z\|_{B^{-\kappa}_{\infty,\infty}}^{4(p+1)} + \|{:}Z^2{:}\|_{B^{-\kappa}_{\infty,\infty}}^{\frac{4(p+1)}{3}} + \|{:}Z^3{:}\|_{B^{-\kappa}_{\infty,\infty}}^{\frac{4(p+1)}{5}} \right).$$

From this, can obtain moment bounds for Y. In particular, by integrating over $t \in [0,T]$, obtain

$$\frac{1}{2p}\mathbb{E}\left[\|Y(T)\|_{L^{2p}}^{2p}\right]$$

$$+ \int_0^T m\,\mathbb{E}\left[\|Y(t)\|_{L^{2p}}^{2p}\right] dt$$

$$+ \int_0^T \mathbb{E}\left[\|Y(t)^{p-1}\nabla Y(t)\|_{L^2}^2\right] dt$$

$$+ \frac{\lambda}{4} \int_0^T \mathbb{E}\left[\| Y(t) \|_{L^{2p+2}}^{2p+2} \right] dt$$

$$\leq \frac{1}{2p} \mathbb{E}\left[\| Y(0) \|_{L^{2p}}^{2p} \right]$$

$$+ C_p T (\lambda + p^{3(p+1)} \lambda^{2+p})$$

If Y is stationary in time, then by dividing by T and sending $T \to \infty$, we obtain

$$m \mathbb{E}\left[\| Y(0) \|_{L^{2p}}^{2p} \right] + \mathbb{E}\left[\| Y(0)^{p-1} \nabla Y(0) \|_{L^2}^{2} \right]$$

$$+ \frac{\lambda}{4} \mathbb{E}\left[\| Y(0) \|_{L^{2p+2}}^{2p+2} \right]$$

$$\leq C_p (\lambda + p^{3(p+1)} \lambda^{2+p}).$$