$\boxed{\Phi^4_2 \text{ measure has Sub-Gaussian tail.}}$

Consider Φ^4_2 measure:

$$\mu \sim \text{"} e^{-\int_{T^2} \frac{1}{2}|\nabla\Phi|^2 + \frac{1}{4}\Phi^4 - \infty \Phi^2 \, dx} \prod_x d\Phi(x) / Z \text{"}$$

$$\Phi \in S'(\mathbb{T}^2)$$

Formally, it's non-Gaussian. i.e "non-trivial".

We show it's indeed non-Gaussian. [Hairer-Steele, 3D]

<u>Theorem</u>: $\forall \varphi \in C^\infty$ for $\beta > 0$ small enough (only dep on φ)

$$\mathbb{E}_{\Phi \sim \mu}\left[e^{\beta \langle \Phi, \varphi \rangle^4}\right] < \infty$$

(1) known [Fröhlich 1976 in 2D]

(2) μ not Gaussian. i.e. $\langle \Phi, \varphi \rangle$ not Gaussian

(3) Similar proof also apply to $e^{\beta |\Phi|^4_{C^{-\kappa}}}$

(4) bound is indep of size of torus

(5) Regularity axiom in Osterwalder-Schrader.

In fact, Gubinelli - Hofmanova, Moinat - Weber proved

$$\mathbb{E}\left[e^{\beta\langle\Phi,\psi\rangle^{1-}}\right] < \infty \quad \text{in 3D}$$

By stochastic quantization argument (Weber-Moinat)

$$\Phi = Y + Z$$

$$\sup Z \lesssim \max\left(\frac{1}{R},\ \|Y\|_{C^{-\kappa}}^{\frac{2}{1-\kappa}},\ \|\Psi^2\|_{C^{-2\kappa}}^{\frac{2}{2-2\kappa}},\ \|\Psi^3\|_{C^{-3\kappa}}^{\frac{2}{3-3\kappa}}\right)$$

$$= Q$$

By Wiener chaos:

$$\mathbb{E}\, e^{\lambda \|\Psi^n\|_{C^{-n\kappa}}^{2/n}} < \infty \qquad n = 1, 2, 3.$$

i.e. $\mathbb{E}\, e^{Q^{1-\kappa}} < \infty$

at most get $\mathbb{E}_{\Phi\sim\mu}\, e^{\beta\langle\Phi,\psi\rangle^{1-\kappa}} < \infty$

OS Regularity axiom:

$$\left|\mathbb{E}\left[\Phi(f_1)\cdots\Phi(f_n)\right]\right| \le C_n \prod_{i=1}^{n} \|f_i\|$$

Suppose $d=2$.

$$\left|\mathbb{E}\left[\Phi(f_1)\cdots\Phi(f_n)\right]\right| \le \mathbb{E}\,\|\Phi\|_{C^{-\kappa}}^{n} \prod_{i=1}^{n} \|f_i\|_{C^{\kappa}}$$

- [S22] has bound on $\mathbb{E}\,\|\Phi\|_{C^{-\kappa}}^{n}$
- [GH]:

$$\|\Phi\|^n \le \|\Phi\|^{\alpha\cdot\lceil\frac{n}{\alpha}\rceil} \le \beta^{-\lceil n/\alpha\rceil} \left\lceil\frac{n}{\alpha}\right\rceil!\; e^{\beta\|\Phi\|^{\alpha}}$$

$$\overset{\text{Stirling}\cdots}{\le} K^n\cdot(n!)^{1/\alpha}\, e^{\beta\|\Phi\|^{\alpha}}$$

[GlimmJaffe] stronger axiom: $\left|\mathbb{E}\, e^{(\Phi,f)}\right| \le e^{c\left(\|f\|_{L^1} + \|f\|_{L^p}^{p}\right)}$

$1 \le p \le 2.$

[Hairer-Steele]:

Tilt measure: $d\nu = e^{\frac{\beta}{4}\langle \Phi, \psi\rangle^4} d\mu$

Now goal: prove ν is finite measure

Formally: ν "should" be invariant for

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \infty \Phi + \beta \psi \langle \Phi, \psi\rangle^3 + \xi$$

if $\beta > 0$ small, $-\Phi^3$ may still be dominant.

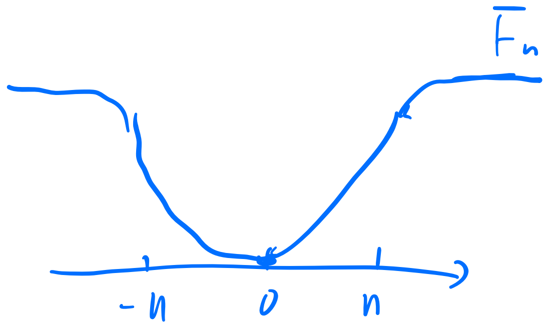
so we may still get a priori bound.

To prove invariance, we modify by n, as follows:

$$d\nu_n = \frac{1}{Z_n} e^{\beta F_n(\langle \Phi, \psi \rangle)} d\mu$$

$$F_n(x) = \begin{cases} \dfrac{x^4}{4} & |x| \leq n \\ \dfrac{n^4}{4} + 1 & |x| \geq n+1 \end{cases}$$

$$F_n' \in [0, n^3]$$



$$Z_n = \mathbb{E}_{\Phi \sim \mu}\left(e^{\beta F_n}\right)$$ finite for each n

By Fatou: $$\mathbb{E}_{\Phi \sim \mu} e^{\beta \langle \Phi, \psi \rangle^4} \leq \underbrace{\liminf_{n \to \infty} Z_n}_{}$$

goal: bound this.

Lemma 1 For each n, ν_n is invari meas of

$$\partial_t \Phi^{(n)} = \Delta \Phi^{(n)} - \Phi^{(n)3} + \text{"}\infty\text{"}\, \Phi^{(n)} + \beta \psi \langle \Phi^{(n)}, \psi \rangle^3 + \xi$$

Proof: Discretize, pass to limit.

Lemma 2 $$\sup_{t \in (R^2, T)} \left| (\Phi^{(n)} - Y)(t,x) \right|$$

$$\leq C \max\left(R^{-1},\ \|Y\|_{C^{-\kappa}}^{\frac{2}{1-\kappa}},\ \|Y^2\|_{C^{-\kappa}}^{\frac{2}{2-2\kappa}},\ \|Y^3\|^{\frac{2}{3-3\kappa}} \right)$$

Proof of main result:

- Start $\Psi^{(n)}$ from its invariant measure ν_n
- RHS of Lemma 2 is a.s. finite.

  so $\exists\, K>0$ s.t

$$\mathbb{P}(\|\Psi^{(n)}\|_{C^{-\kappa}} \le K)$$

$$\geq \mathbb{P}(\|\Psi^{(n)} - Y\|_{C^{-\kappa}} + \|Y\|_{C^{-\kappa}} \le K)$$

$$\geq \mathbb{P}(\|\Psi^{(n)} - Y\|_{L^\infty} + \|Y\|_{C^{-\kappa}} \le K)$$

$$\geq \frac{1}{2}$$

Ball of radius K in $C^{-\kappa}$.

$$LHS = \nu_n(B_K) = \int_{B_K} \nu_n$$

$$= \int \frac{1}{Z_n} e^{\beta F_n(\langle \Phi, \psi\rangle)} d\mu \quad \left(\geq \frac{1}{2}\right)$$

But for $\Phi \in B_K$.

$$F_n(\langle \Phi, \psi\rangle) \le \frac{1}{4}\langle \Phi, \psi\rangle^4 \le C K^4. \quad \text{unif in } n$$

So,

$$Z_n \le 2\int e^{\beta F_n} d\mu \le 2 e^{\beta C K^4} \quad \text{unif in } n$$

Maximum principle: ($\mathbb{T}^d$ case. $\mathbb{R}^d$ case same with weight)

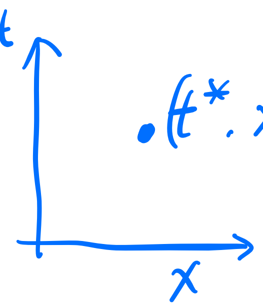
see [GH] Lem 2.12

Lem: suppose u is C^1 in t, C^2 in x

$$\begin{cases} Lu + u^3 = g \\ u(0) = u_0 \end{cases} \qquad (L = \partial_t - \Delta + m, \quad m \in \mathbb{R})$$

Then $\|u\|_{L^\infty_{tx}} \lesssim 1 + \|u_0\|_{L^\infty_x} + \|g\|_{L^\infty_{tx}}^{1/3}$

pf:



$\cdot(t^*, x^*)$ max

$u(t^*, x^*) = M$

Assume $M > 0$ (If $M \le 0$ look at minimum ...)

If $t_* = 0$, then $u(t,x) \le u(0, x^*) = \|u_0\|_{L^\infty}$ ✓

If $t_* > 0$:

$$\underbrace{\partial_t u}_{0} \underbrace{- \Delta u}_{\ge 0} + m u + u^3 = g \qquad (1)$$

$$\begin{cases} \partial_t u(t^*, x^*) = 0 \\ \nabla u(t^*, x^*) = 0 \\ \Delta u(t^*, x^*) \le 0 \end{cases}$$

$$(1) \Longrightarrow M^3 \le g(t^*, x^*) - mM$$

$$\le \|g\|_{L^\infty} + |m| \cdot \|u\|_{L^\infty}$$

So $u \le \|u_0\|_{L^\infty_x} + \|g\|_{L^\infty_{tx}}^{1/3} + c\|u\|_{L^\infty_{tx}}^{1/3}$

Same argument to $-u$:

$$-u \lesssim \|u_0\|_{L^\infty_x} + \|g\|^{1/3}_{L^\infty_{tx}} + c\|u\|^{1/3}_{L^\infty_{tx}}$$

So $$\|u\|_{L^\infty_{tx}} \lesssim \|u_0\|_{L^\infty_x} + \|g\|^{1/3}_{L^\infty_{tx}} + c\|u\|^{1/3}_{L^\infty_{tx}}$$

$$a^{1/3}\cdot 1 \leq a + 1^{3/2} \qquad \left(\tfrac{1}{3}+\tfrac{2}{3}=1\right)$$

$\Rightarrow$

$$\|u\|_{L^\infty_{tx}} \lesssim \|u_0\|_{L^\infty_x} + \|g\|^{1/3}_{L^\infty_{tx}} + 1$$

To be indep of u_0:

Consider $\eta\cdot u$ $\quad$ $\eta>0$ bulk. $\eta=0$ on boundary.

$(\eta\cdot u)(t^*, x^*)$ max

$\nabla(\eta\cdot u)(t^*\ x^*)=0$. $\Rightarrow$ $\nabla u = -\frac{\nabla\eta}{\eta}u$

and $\Delta(\eta\cdot u)\leq 0$

At (t^*, x^*):

$$0\leq(\partial_t-\Delta)(u\eta) = \eta(\partial_t-\Delta)u + u(\partial_t-\Delta)\eta - 2\nabla u\cdot\nabla\eta$$

$$= -\eta(u^3-g) + u\left((\partial_t-\Delta)\eta + 2\cdot\frac{(\nabla\eta)^2}{\eta}\right)$$

Assume $\dfrac{(\partial_t-\Delta)\eta}{\eta} + 2\dfrac{(\nabla\eta)^2}{\eta^2} \leq \dfrac{1}{2\eta^2}$

Then $u^2 \leq \dfrac{\eta^{-2}}{2} + \dfrac{\|g\|}{u} \leq 2\max\left(\dfrac{\eta^{-2}}{2}, \dfrac{\|g\|}{u}\right)$

If $\frac{\eta^{-2}}{2}$ is larger, $u\eta \le 1$

If $\frac{\|g\|}{u}$ is larger, $u\eta \le 2\eta^3\|g\| \le 2$

if $\eta \le \|g\|^{-\frac{1}{3}}$

Choose $\eta(t,x) = \dfrac{1}{\|g\|^{1/3} + \frac{c}{\sqrt{t}}}$ for suitable c.

See [Moinat-Weber 18 CPAM]