Schauder estimates.

Ref: Bahouri, Chemin, Danchin "Fourier analysis and nonlinear PDE"

Simplification: work on $\mathbb{T}^d$, since this avoids weights (which are needed in infinite volume) and lattice effects.

Recall given $f : \mathbb{T}^d \to \mathbb{C}$,

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(x) \underbrace{e^{-i2\pi n \cdot x}}_{\overline{e_n}} dx ,$$

$$f = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e_n$$

Since $\Delta e_n = -4\pi^2 |n|^2 e_n$, have that

$$(\Delta - m)f = -\sum_n \hat{f}(n)\,(4\pi^2|n|^2+m)\,e_n)$$

and

$$e^{t(\Delta-m)}f = \sum_n e^{-t(4\pi^2|n|^2+m)} \times \hat{f}(n)\, e_n .$$

Recall also Besov norm

$$\|f\|_{B^{\alpha}_{p,q}} = \Big(\sum_{j\geq -1} \big(2^{j\alpha}\|\Delta_j f\|_{L^p}\big)^q\Big)^{\frac{1}{q}} .$$

Prop. Let $\mathcal{A} \subseteq \mathbb{R}^d$ be an annulus. For $\lambda > 0$, have that if

$$\operatorname{supp}(\hat{f}) \subseteq \lambda \mathcal{A},$$

then for $1 \leq p \leq \infty$,

$$\|e^{t\Delta} f\|_{L^p} \leq C e^{-c\lambda^2 t} \|f\|_{L^p},$$

where C, c do not depend on λ, t, p.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ be supported on an annulus containing A, and moreover $\psi \equiv 1$ on A. Then if $\mathrm{supp}(\hat{f}) \subseteq \lambda A$, we have that

$$e^{t\Delta} f = \sum_{n \in \mathbb{Z}^d} e^{-4\pi^2 |n|^2 t} \psi(\lambda^{-1} n) \hat{f}(n) e_n$$

$$= \phi_{\lambda,t} * f,$$

where

$$\phi_{\lambda,t} = \sum_n e^{-4\pi^2 |n|^2} \psi(\lambda^{-1} n) e_n.$$

By Young's convolution inequality, we have that

$$\|\phi_{\lambda,t} * f\|_{L^p} \le \|\phi_{\lambda,t}\|_{L^1} \|f\|_{L^p},$$

and so it remains to show that

$$\|\phi_{\lambda,t}\|_{L^1} \le C e^{-\lambda^2 t} .$$

First, note that

$$\psi(\lambda^{-1} n) \neq 0 \iff |n| \sim \lambda.$$

In particular, we can assume that

$$\lambda \ge c_0 > 0.$$

Then, we have that

$$\Big\| \sum_n e^{-4\pi^2 |n|^2 t} \psi(\lambda^{-1} n) e_n \Big\|_{L^1}$$

$$\lesssim \sum_{|n| \sim \lambda} e^{-c\lambda^2 t}$$

$$\lesssim \lambda^d e^{-c\lambda^2 t}.$$

Let p_t be the heat kernel on $\mathbb{T}^d$. Then note that

$$\sum_n e^{-4\pi^2 |n|^2 t} \psi(\lambda^{-1} n) e_n$$

$$= p_t * \phi_\lambda,$$

where

$$\phi_\lambda = \sum_n \psi(\lambda^{-1} n) e_n .$$

We claim that $\|\phi_\lambda\|_{L^1} \lesssim 1$, in which case (also by Young)

$$\|p_t * \phi_\lambda\|_{L^1} \leq \|\phi_\lambda\|_{L^1} \lesssim 1.$$

Now if $\lambda^2 t \leq 1$, we bound

$$\|p_t * \phi_\lambda\|_{L^1} \leq C e^{-c\lambda^2 t}$$

for appropriate choices of C, c, and if $\lambda^2 t > 1$, then

$$t^{-1} < \lambda^{-2} \leq c_0^{-2} \lesssim 1,$$

and thus

$$\begin{aligned}\|p_t * \phi_\lambda\|_{L^1} &\lesssim \lambda^d e^{-c\lambda^2 t} \\ &\lesssim t^{-\frac{d}{2}} (\lambda^2 t)^{\frac{d}{2}} e^{-\frac{c}{2}\lambda^2 t} e^{-\frac{c}{2}\lambda^2 t} \\ &\lesssim e^{-\frac{c}{2}\lambda^2 t},\end{aligned}$$

where we used that

$$\sup_{x \geq 1} x^{\frac{d}{2}} e^{-\frac{c}{2}x} \lesssim 1.$$

It remains to show the claim that

$$\|\phi_\lambda\|_{L^1} \lesssim 1 .$$

First, note that the Poisson summation formula gives that for $x \in \mathbb{T}^d \cong [0,1)^d$,

$$\sum_{n \in \mathbb{Z}^d} \eta_\lambda(x+n) = \sum_{n \in \mathbb{Z}^d} \psi(\lambda^{-1} n)\, e_n ,$$

where $\eta_\lambda \in \mathcal{S}$ is st $\hat{\eta}_\lambda = \psi(\lambda^{-1}\cdot)$.
Indeed, compute

$$\int \sum_{n \in \mathbb{Z}^d} \eta_\lambda(x+n)\, \overline{e_{n_0}(x)}\, dx$$

$$= \hat{\eta}_\lambda(n_0) = \psi(\lambda^{-1} n_0) .$$

This implies

$$\Big\| \sum_{n\in\mathbb{Z}^d} \psi(\lambda^{-1} n)\, e_n \Big\|_{L^1}$$

$$= \int_{[0,1)^d} \Big| \sum_{n\in\mathbb{Z}^d} \eta_\lambda(x+n) \Big|\, dx$$

$$\leq \int_{\mathbb{R}^d} |\eta_\lambda(x)|\, dx = \|\eta_\lambda\|_{L^1}$$

Now, an explicit computation shows that

$$\eta_\lambda = \lambda^d \eta(\lambda \,\cdot\,),$$

where η is st $\hat{\eta} = \psi$. Thus,

$$\|\eta_\lambda\|_{L^1} = \|\eta\|_{L^1} \lesssim 1. \quad \square$$

Given the proposition, we may prove the following Schauder estimate.

Prop. For $t>0$, $\beta \geq \alpha$, $p, q \in [1, \infty]$, we have that

$$\|e^{t\Delta} f\|_{B^{\beta}_{p,q}} \lesssim (\beta-\alpha)^{\frac{\beta-\alpha}{2}} t^{-\frac{(\beta-\alpha)}{2}} \|f\|_{B^{\alpha}_{p,q}}.$$

Pf. For $j \geq 0$, we have by the Proposition that

$$\|\Delta_j e^{t\Delta} f\|_{L^p} = \|e^{t\Delta} \Delta_j f\|_{L^p}$$
$$\leq C e^{-c 2^{2j} t} \|\Delta_j f\|_{L^p}.$$

Thus,

$$2^{j\beta} \|\Delta_j e^{t\Delta} f\|_{L^p} \lesssim 2^{j\alpha} \|\Delta_j f\|_{L^p} \times 2^{j(\beta-\alpha)} e^{-c 2^{2j} t}$$

$$\lesssim (\beta-\alpha) t^{-\frac{(\beta-\alpha)}{2}} 2^{j\alpha} \|\Delta_j f\|_{L^p},$$

where we used that

$$\sup_{x \geq 1} x^{\beta-\alpha} e^{-c x^2 t} \lesssim (\beta-\alpha)^{\frac{\beta-\alpha}{2}} t^{-\frac{(\beta-\alpha)}{2}}.$$

The desired result now follows. $\square$

Now, given $f : [0,T] \to B^{\alpha}_{p,q}$, define

$$\|f\|_{L^r_t B^{\alpha}_{p,q}} = \left(\int_0^T dt\, \|f(t)\|^r_{B^{\alpha}_{p,q}} \right)^{\frac{1}{r}}.$$

Define

$$\mathrm{Duh}(f)(t) := \int_0^t e^{(t-s)(\Delta - m)} f(s)\,ds.$$

Then we have the following Schauder estimate.

Prop. Let $\alpha \le \beta < 2$, $p, q \in [1, \infty]$. Then

$$\|\mathrm{Duh}(f)\|_{L_t^\infty B_{p,q}^\beta} \lesssim_{\beta-\alpha} T^{1-\frac{(\beta-\alpha)}{2}} \|f\|_{L_t^\infty B_{p,q}^\alpha}.$$

Pf. For $t \in [0, T]$, we bound

$$\|\mathrm{Duh}(f)(t)\|_{B_{p,q}^\beta}$$

$$\le \int_0^t \|e^{(t-s)(\Delta - m)} f(s)\|_{B^{\beta}_{p,q}}\, ds$$

$$\lesssim \int_0^t (t-s)^{-\frac{(\beta-\alpha)}{2}} \|f(s)\|_{B^{\alpha}_{p,q}}\, ds$$

$$\lesssim t^{1-\frac{(\beta-\alpha)}{2}} \|f\|_{L^\infty_t B^{\alpha}_{p,q}},$$

where we used that $\beta < \alpha + 2$ to integrate the singularity. Taking sup over $t \in [0, T]$, the desired result now follows. $\square$