Abelian Higgs model

Fix $\mathbb{T}^2$. $\quad A: T^2 \to \mathbb{R}^2 \quad \phi: T^2 \to \mathbb{C}$

$\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{(A_1, A_2)}$

In $d=2$, only $F_A^{12} = -F_A^{21}$

Define curvature $F_A^{jk} = \partial^j A^k - \partial^k A^j$

covariant derivative: $D_A^j \phi = \partial^j \phi + i A^j \phi$

Energy functional:

$$E(A, \phi) = \int_{T^2} \frac{1}{4}|F_A|^2 + \frac{1}{2}|D_A\phi|^2 + \frac{1}{4}|\phi|^4 \, dx$$

where: $\quad |F_A|^2 = \sum_{j,k} (F_A^{jk})^2$

$\quad\quad\quad |D_A\phi|^2 = \sum_j |D_A^j \phi|^2$

Rmk: Drop A $(A=0) \Rightarrow \int \frac{1}{2}|\partial\phi|^2 + \frac{1}{4}|\phi|^4 dx \quad \Phi^4$-model

Drop ϕ $(\phi = 0) \Rightarrow$ quadratic ("2D version of Maxwell)

gauge invariance/symmetry

gauge transformation: $\mathcal{G} = \{ g : \mathbb{R}^2 \to \mathbb{R} \text{ and } \nabla g, e^{-ig} \text{ periodic} \}$

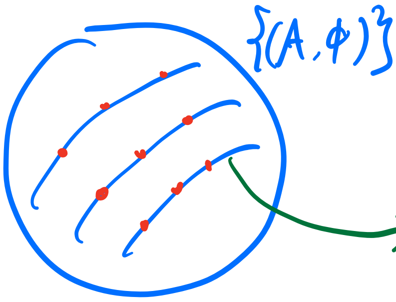
$$(A, \phi) \mapsto (A^g, \phi^g) := (A + \nabla g, e^{-ig}\phi)$$

(This is an Abelian group action)

$$\begin{cases} F_{A^g} = F_A \\ D_{A^g}\phi^g = \nabla\phi^g + i A^g \phi^g = \nabla(e^{-ig}\phi) + i(A + \nabla g) e^{-ig}\phi \\ \quad\quad = e^{-ig} D_A \phi = (D_A\phi)^g \end{cases}$$

(exercise)

Therefore: $E(A^g, \phi^g) = E(A, \phi)$



$\rightarrow$ a gauge orbit (i.e. a gauge equivalent class)

Rmk: $\int |\nabla \phi|^2 dx$ also has symmetry under $\phi \to \phi + c$

But gauge symmetry is infinite dim symmetry.

The word "gauge" is from Maxwell theory (Electro-magnetism)

<u>gauge fixing</u> ("eliminate" gauge freedom by imposing a condition)

We impose Coulomb gauge condition

$$\operatorname{div} A := \sum_j \partial_j A^j = 0$$

Rmk: Helmholtz / Hodge decomposition (on $\mathbb{T}^2$):

$$A = \nabla f + B \quad \text{where } \operatorname{div} B = 0$$

$$(B = \operatorname{curl}^* G + (n_1, n_2). \quad (n_1, n_2) \in \mathbb{R}^2)$$

So $\forall A.\ \exists g$, s.t. A^g is div 0.

$\searrow$ not unique: If $\operatorname{div} A = 0$, $\operatorname{div}(A + (n_1, n_2)) = 0$

for (n_1, n_2) to be ∇g for some $g \in G$,

i.e. $g_{(x_1, x_2)} = n_1 x_1 + n_2 x_2$. n_1, n_2 must be in $\mathbb{Z}$

Stochastic PDE:

$$\begin{cases}\partial_t A = \Delta A - P_\perp \,\mathrm{Im}(\bar{\phi} D_A \phi) + P_\perp \xi \\ \partial_t \phi = \sum_j D_A^j D_A^j \phi - |\phi|^2 \phi + \zeta \end{cases}$$

where $P_\perp$ is "Leray" Projection to div=0 part.

$$\left(P_\perp A = A - \nabla(\Delta^{-1} \operatorname{div} A) \right)$$

Local: Shen '19 global: Bringmann – Cao '24 (hard: no term like $-A^3$)

measure: [Brydges - Fröhlich - Seiler ~'80] [King 86]

$\boxed{\text{Local theory:}}$

$A, \phi \in C^{0-}$. ϕ^3, $A\phi^2$, $A^2\phi$ are ill-defined

$\phi\partial\phi$, $A\partial\phi$ are worse! $\rightarrow$ more singular than ϕ^4_2.

Da Prato - Debussche won't work:

$$\begin{cases} A = \mathring{\mathsf{I}} + C^{1-}?\ C^{1+}? \\ \phi = \dot{\mathsf{I}} + C^{1-}?\ C^{1+}? \end{cases} \qquad \underbrace{A}_{C^{1+}}\underbrace{\partial\phi}_{C^{-1-}}$$

C^{0-}

ansatz:

$$A = \mathring{\mathsf{I}} + \mathsf{Y} + X$$

assume Einstein sum rule below.

$$\phi = \dot{\mathsf{I}} + 2i\,\mathsf{V} + \eta + \psi$$

where the paracontrolled component η solves

$$(\partial_t - \Delta)\eta = 2i\,(A - \mathring{\mathsf{I}})^j \varolessthan \partial_j \dot{\mathsf{I}}$$

We want to solve $(X, \eta, \psi) \in C^{2-} \times C^{1-} \times C^{2-}$

Toy model:

$$\partial_t A = \Delta A + A \partial A + \xi$$

$$A = \dot{\mathsf{I}} + \underbrace{\dot{\mathsf{Y}} + Z}_{B} = \dot{\mathsf{I}} + B$$

$$(\partial_t - \Delta) Z = \dot{\mathsf{I}}\,\partial B + B\,\partial \dot{\mathsf{I}} + B \partial B$$

This decomposition is not enough. since: $Z \in C^{1-}$ at most.

$Z \partial \dot{\mathsf{I}}$ is problematic.

However, for this toy model, we can write

$$(\partial_t - \Delta) Z = \partial(\underbrace{\dot{\mathsf{I}} B}_{C^{0-}}) + \underbrace{B}_{C^{1-}}\underbrace{\partial B}_{C^{0-}} \in C^{-1-}$$

$\boxed{\text{Can solve in } C^{1-}}$

In fact DaPrato-Debussche did this for 2D NS + white noise

Since AH doesn't have this nice feature, we should not use this total derivative trick here.

So Let's instead consider the toy problem

$$\boxed{(\partial_t - \Delta) Z = B \partial \dot{\mathsf{I}}}$$

paracontrolled ansatz:

$$Z = X + Y. \quad \Big(\text{i.e. } A = \dot{\mathsf{I}} + \overbrace{\dot{\mathsf{Y}} + X + Y}^{B}\Big)$$

where X is paracontrolled component

$$(\partial_t - \Delta) X = \underbrace{B \olessthan \partial \dot{\mathsf{I}}}$$

Want to solve $(X, Y) \in C^{1-2\kappa} \times C^{2-5\kappa}$

Then $(\partial_t - \Delta) Y$

$$= \underbrace{B \odot \partial \mathsf{I}} + \cancel{\mathsf{I} \odot \partial B} + \underbrace{B \varogreaterthan \partial \mathsf{I}} + \cancel{\mathsf{I} \varogreaterthan \partial B} + \cancel{B \partial B}$$

$$= \curlyvee \odot \partial \mathsf{I} + X \odot \partial \mathsf{I} + Y \odot \partial \mathsf{I} + B \varogreaterthan \partial \mathsf{I}$$

$$X \uparrow \; P * (B \varolessthan \partial \mathsf{I})$$

① $\curlyvee\!\curlyvee - C\, \mathsf{I}$ converges in C^{0-}

with $C = \curlyvee$

similar with "2nd renormalization" of ϕ^4_3

② $[P * (B \varolessthan \partial \mathsf{I})] \odot \partial \mathsf{I}$

$\approx [B \varolessthan (P * \partial \mathsf{I})] \odot \partial \mathsf{I}$ error $\in C^{0-}$

$\approx B \varolessthan [(P * \partial \mathsf{I}) \odot \partial \mathsf{I}]$ error $\in C^{1-}$

$\hookrightarrow C^{2-}$ OK

1^- $\quad -1^-$ $\quad -1^-$

1^- $\quad 1^-$ $\quad -1^-$

Lemma 1: $\alpha < 1 \quad \alpha + \beta + \gamma > 0 \quad \beta + \gamma < 0$

$$\| (f \varolessthan g) \odot h - f (g \odot h) \|_{C^{\alpha+\beta+\gamma}} \lesssim \|f\|_{C^\alpha} \|g\|_{C^\beta} \|h\|_{C^\gamma}$$

Lemma 2:

$$\| (P * (f \varolessthan g) - f \varolessthan P g) \odot h \|_{C^{0-}} \lesssim \|f\|_{C^{1-}} \|g\|_{C^{-1-}} \|h\|_{C^{-1-}}$$

So we <u>essentially</u> have $B \circledcirc$ "$\underbrace{\Psi}$"

$\downarrow$

C^{0-} by renormalization

The point is that we turn things into an explicit object Ψ which we can use stochastic analysis

Recall $\|f \circledcirc g\|_{C^{\alpha\wedge 0+\beta}} \le \|f\|_{C^\alpha}\|g\|_{C^\beta}$

$$B \circledcirc (\Psi - c) \in C^{0-} \xrightarrow{\text{Schauder}} C^{2-} \quad \boxed{\text{OK}}$$

③ $\underset{2^-}{Y} \ominus \underset{-1^-}{\partial \tau} \in C^{1-} \Rightarrow$ ok

Recall $\|f \ominus g\|_{C^{\alpha+\beta}} \le \|f\|_{C^\alpha}\|g\|_{C^\beta}$ if $\alpha+\beta>0$

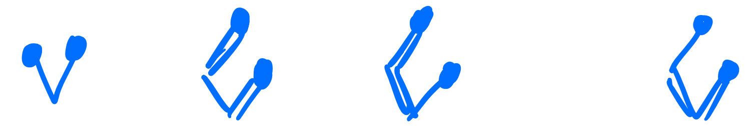
④ $\underset{1^-}{B} \circledcirc \underset{-1^-}{\partial \tau} \in C^{0-} \Rightarrow$ ok

Recall $\|f \circledcirc g\|_{C^{\alpha+(0\wedge\beta)}} \le \|f\|_{C^\alpha}\|g\|_{C^\beta}$

Renormalized equation:

$$\begin{cases} \partial_t A = \Delta A - P_\perp \operatorname{Im}(\bar{\phi} D_A \phi) + P_\perp \xi + CA \\ \partial_t \phi = \sum_j D_A^j D_A^j \phi - :|\phi|^2\phi: + \zeta + \sigma\phi \end{cases}$$

Interestingly:

① 

$\phi A \phi$ $\quad$ $P(A\,\partial\phi)\cdot\partial\phi$ $\quad$ $\partial P(A\cdot\partial\phi)\cdot\phi$

actually add up to a finite C $\qquad$ ($\sigma \sim \log$ div)

② "diagonal": only CA in A equ. $\sigma\phi$ in ϕ equ.

③ [BC] showed that:

If $C = \frac{1}{8\pi}$. solution is "gauge covariant":

Denote $S(A_0, \phi_0, \xi, \zeta)$ the solution (A, ϕ)

$$S(A_0, \phi_0, \xi, \zeta)^n \overset{\text{law}}{=} S(A_0^n, \phi_0^n, \xi, \zeta)$$

$$\forall\, n \in \mathbb{Z}^2$$

④ "$A =$  $+$  $+ X$"

null form: