2D Abelian Higgs global theory

Ref: Bringmann-Cao "Global well-posedness ..."

DISCLAIMER.

Many of the statements in this note are not literally true. The intent is to touch on the key ideas, and sometimes rigor is sacrificed as a result. In the talk, I will try to mention which statements are only approximately true.

CONCEPTUAL POINT.

Understand how geometry and singular SPDEs interact. Still much left to figure out.

Stochastic Abelian Higgs

$$(\partial_t - \Delta)A = -P_\perp \mathrm{Im}(\bar\phi D_A \phi) + P_\perp \xi$$

$$(\partial_t - D_A^j D_{A,j})\phi = -|\phi|^{q-1}\phi + \zeta$$

(SAH).

It's not necessary, but for convenience, we will work with the initial data:

$$A_0 = \Psi_0 + A_0^\dagger$$

$$\phi_0 = \psi_0 + \phi_0^\dagger,$$

$A_0^\dagger \in \mathcal{C}_x^\eta$, $\phi_0^\dagger \in L_x^r$, $0 < \eta \ll 1$, $r \gg 1$ fixed.

Local solution ansatz:

$$A = \overset{0-}{\overset{\circ}{\mid}} + \overset{1-}{\mathsf{Y}} + \overset{2-}{X}$$

$$\Phi = \overset{0-}{\overset{\bullet}{\mid}} + 2i\, \overset{1-}{{}^{\circ}\mathsf{Y}} + \overset{1-}{\eta} + \overset{2-}{\chi}$$

$\uparrow$
para-controlled term

Here,

$$(\partial_t - \Delta)\overset{\circ}{\mid} = P_\perp \xi, \quad (\partial_t - \Delta)\overset{\bullet}{\mid} = \zeta$$

$$(\partial_t - \Delta)\mathsf{Y} = -P_\perp \operatorname{Im}(\overset{\bullet}{\mid}\, d\, \overset{\bullet}{\mid})$$

$$(\partial_t - \Delta)\,{}^{\circ}\mathsf{Y} = \partial_j (\overset{\circ}{\mid}^j\, \overset{\bullet}{\mid})$$

$$(\partial_t - \Delta)\eta = 2i\,\partial_j \left((A - \overset{\circ}{\mid})^j \prec \overset{\bullet}{\mid} \right)$$

First attempt at global theory:
let
$$\tilde{\eta} = 2i\phi + \eta + \chi,$$
and try to obtain energy estimates on $\tilde{\eta}$. Equation for $\tilde{\eta}$:

$$(\partial_t - D_A^j D_{A_j})\tilde{\eta} = (D_A^j D_{A_j} - \Delta)\phi - |\phi + \tilde{\eta}|^{q-1}(\phi + \tilde{\eta})$$

$$= 2i\partial_j(A^j\phi) - |A|^2\phi - |\phi + \tilde{\eta}|^{q-1}(\phi + \tilde{\eta})$$

Obtain

$$\frac{1}{2}(\partial_t - \Delta)(|\tilde{\eta}|^2) = -|D_A\tilde{\eta}|^2 - \cdots$$

$$- |A|^2 \operatorname{Re}(\phi\bar{\tilde{\eta}})$$

$$- \cdots$$

OBSTACLE:

Can not absorb .

Reflection: the $|A|^2$ arises b/c we defined

$$(\partial_t - \Delta)\Psi = \xi ,$$

as one always does for local theory.

Idea: why not define modified linear object by:

$$(\partial_t - D_A^j D_{A,j})\Psi_A = \xi \qquad \text{(CSHE)}.$$

Forget about SAH for the moment. Take A deterministic. What bounds on (CSHE) can we hope for?

DO NOT write as

$$(\partial_t - \Delta)\mathring{\imath}_A = 2i\partial_j(A^j \mathring{\imath}_A) - |A|^2 \mathring{\imath}_A + \xi.$$

Once you do this, you just LOSE.

Instead, write as

$$\mathring{\imath}_A(t,x) = \int_0^t ds \int_{\mathbb{T}^2} dy\, P_A(s,y;t,x)\,\xi(s,y)$$

where P_A is the covariant heat kernel.

Estimate 2nd moment:

$$\mathbb{E}[|\mathring{\imath}_A(t,x)|^2] = \|P_A\|^2_{L^2_s L^2_y}$$

$$= \int_0^t ds \int_{\mathbb{T}^2} dy\, |P_A(s,y;t,x)|^2$$

Diamagnetic inequality for heat kernel:

$$|p_A(s,y;t,x)| \leq p(s,y;t,x) \;!!$$

Follows by Feynman-Kac-Itô formula:

$$p_A(s,y;t,x) =$$

$$p(s,y;t,x)\, \mathbb{E}_{\substack{s\to t \\ y\to x}}\left[\exp\left(-i\int_s^t du\, A(u,W_u)\circ dW_u\right)\right]$$

Here, W is a rate-2 BB $(s,y) \to (t,x)$.

<u>Remark</u>

The term has a beautiful interpretation as the expectation of stochastic parallel transport of the connection A along the Brownian path W.

Thus, obtain

$$\mathbb{E}\left[|\Phi_A(t,x)|^2\right] \leq \int_0^t ds \int dy\, p(s,y;t,x)^2$$

$$= \mathbb{E}\left[|\Phi(t,x)|^2\right]$$

(of course, this is ∞, so need to add in some smoothing).

From this, one can get bounds on Φ_A which are essentially independent of A:

$$\mathbb{E}\left[\|\Phi_A\|^p_{C_t^0 C_x^{-\kappa}([0,1]\times\mathbb{T}^2)}\right]^{\frac{1}{p}} \lesssim p^{\frac{1}{2}}$$

(in practice, give up a power of A).

Back to SAH. We would like to take φ_A, but problem is that A and ξ are dependent.

Workaround: take $B = e^{t\Delta} A_0$, and work with

$$\varphi_B .$$

This leads to ansatz:

$$A = \varphi + B + Z$$

$$\phi = \varphi_B + \psi, \qquad \psi(0) = \phi_0^+$$

where

$$(\partial_t - \Delta) z = -P_\perp \operatorname{Im}(\bar{\phi} D_A \phi)$$

$$(\partial_t - D_A^j D_{A_{|j}}) \psi = (D_A^j D_{A_{|j}} - D_B^j D_{B_{|j}}) \dot{\iota}_B - |\dot{\iota}_B + \psi|^{q-1} (\dot{\iota}_B + \psi)$$

$$= 2i(\dot{\iota} + z) D_A^j \dot{\iota}_B + |\dot{\iota} + z|^2 \dot{\iota}_B - |\dot{\iota}_B + \psi|^{q-1} (\dot{\iota}_B + \psi)$$

$$z(0) = 0, \quad \psi(0) = \phi_0 .$$

By applying various energy estimates to the evolution of

$$(z, \psi),$$

and estimating various functionals of $\dot{\iota}_B$, can eventually show, for $\tau \in [0,1]$ and r very large; q very small,

$$\frac{1}{r}\partial_t\left(\|\psi(t)\|_{L^r_x}^{r}\right) + \|\psi(t)\|_{L_x^{r+q-1}}^{r+q-1} \lesssim \|z\|_{C^0_t \ell^q_x}^{\frac{2}{q}(r+q-1)} + \text{l.o.t.},$$

$$\|z\|_{C^0_t \ell^q_x([0,\tau])} \lesssim \tau^{\frac{3}{4}}\|A_0\|_{\ell^q_x}^{\frac{1}{2}} + \tau^{\frac{1}{2}}\|\psi\|_{C^0_t L^r_x([0,\tau])}^{2}$$

$$+ \tau\|\psi\|_{C^0_t L^r_x([0,\tau])}\|z\|_{C^0_t \ell^q_x([0,\tau])}^{\frac{q+1}{q}} + \text{l.o.t.}$$

The point is to reduce the dependence on A_0 as much as possible.

Remark

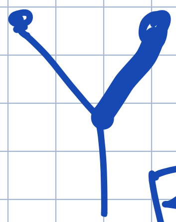
Technically, ϕ only has 1-regularity, so cannot directly apply energy estimates to ϕ (b/c a term like

$$-|D_A \phi|^2$$

appears, and this will be infinite).

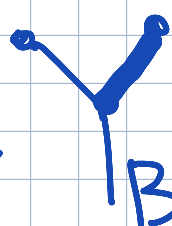
Thus, we have to truncate at frequency L, where L is large relative to the initial data A_0^+, ϕ_0^+, and apply energy estimates to the low frequency portion of ϕ. The high frequency portion turns out to be amenable to perturbative estimates, essentially by the same arguments as in the local theory.

<u>Remark.</u> The $\tau^{\frac{3}{4}} \|A_0\|_{\mathcal{C}_x^{\eta}}^{\frac{1}{2}}$ term in $\mathfrak{T}$ arises from estimating the covariant quadratic object



$$_B = \mathrm{Duh}\big(\mathrm{Im}(\overline{\phi}_B D_B \phi_B)\big).$$

Unlike for the usual quadratic object, this object actually has a nonzero resonant term, and in the end we bound

$$\Big\| \mathbb{E}\Big[$$  $$_B \Big]\Big\|_{C_t^0 \mathcal{C}_x^{\eta}([0,\tau])}$$

$$\lesssim \tau^{\frac{3}{4}} \|A_0\|_{\mathcal{C}_x^{\eta}}^{\frac{1}{2}}$$

The non-resonant piece satisfies the better bound

$$\mathbb{E}\left[\| :Y_B: \|^p_{C^0_t C^{\eta}_x([0,\tau])} \right]^{\frac{1}{p}} \lesssim 1,$$

in that there is no A_0-dependence (Actually, give up ε power in practice).

The proofs of these bounds on

$$Y_B$$

rely on covariant energy estimates which lead to operator bounds for integral operators such as

$$(KA)(w) = \int dz \, D_{A(z)} P_A(w;z) A(w).$$

It turns out that the estimates

[redacted] are enough to obtain global existence for all odd $q \geq 3$ (though this is far from obvious, at least to us).

In the following, we sketch the argument. Fix parameters

$$\alpha = \frac{3}{4}, \quad \beta = \frac{5}{2}, \quad \gamma = \frac{2}{7}$$

These parameters are chosen to satisfy all the inequalities that will appear.

Take time scale

$$T \sim C_3^{-1} \max\left(\|A_0^{+}\|^{\alpha}_{\ell_x^{\gamma}}, \|\phi_0^{+}\|^{\beta}_{L_x^{\eta}}, C\right)^{-1}.$$

Here, $C \ll C_1 \ll C_2 \ll C_3$.

We first prove some crude estimates of ϕ and Z using [redacted].

As in Session 3, we may bound

$$\partial_t\left(\|\psi\|_{L^r_x}\right) = \|\psi\|_{L^r_x}^{1-r}\,\frac{1}{r}\,\partial_t\left(\|\psi\|_{L^r_x}^{r}\right)$$

$$\leq \|\psi\|_{L^r_x}^{1-r}\Big(-\|\psi\|_{L^r_x}^{r+q-1} + C\|z\|_{C^0_t C^\gamma_x}^{\frac{2}{\delta}(r+q-1)} + C\Big)$$

$$\leq -\|\psi\|_{L^r_x}^{q} + \frac{C\|z\|_{C^0_t C^\gamma_x}^{\frac{2}{\delta}(r+q-1)}}{\|\psi\|_{L^r_x}^{r-1}} + \frac{C}{\|\psi\|_{L^r_x}^{r-1}}.$$

If $\|\psi(t)\|_{L^r_x} \geq C_1 \max\left(\|Z\|^{\frac{2}{q}}_{C^0_t \ell^{\gamma}_x}, 1\right)$
where

$$C_1 >> C,$$

for all $t \in [0, \tau]$, then

$$-\frac{1}{q-1} \partial_t \left(\|\psi\|^{-(q-1)}_{L^r_x} \right)$$

$$= \|\psi\|^{-q}_{L^r_x} \, \partial_t \|\psi\|_{L^r_x}$$

$$\leq -\frac{1}{2},$$

$$\Rightarrow \quad \|\psi(t)\|^{-(q-1)}_{L^r_x} - \|\psi(0)\|^{-(q-1)}_{L^r_x} \geq \frac{q-1}{2} t$$

$$\Rightarrow \quad \|\psi(t)\|_{L^r_x} \leq \left(\|\psi_0\|^{-(q-1)}_{L^r_x} + \frac{q-1}{2} t \right)^{-\frac{1}{q-1}}$$

$$= \frac{1}{\left(\|\psi_0\|_{L^r_x}^{-(q-1)} + \frac{q-1}{2} t \right)^{\frac{1}{q-1}}}$$

$$= \|\psi_0\|_{L^r_x} \left(\frac{1}{1 + \frac{q-1}{2} \|\psi_0\|_{L^r_x}^{q-1} t} \right)^{\frac{1}{q-1}} .$$

Thus $\|\psi(t)\|_{L^r_x} \leq \|\psi_0\|_{L^r_x}$, and moreover if

$$\|\psi_0\|_{L^r_x}^{q} t \geq C_1,$$

have that

$$\|\psi(t)\|_{L^r_x} \leq \|\psi_0\|_{L^r_x} \left(\frac{1}{1 + \frac{q-1}{2} C_1 \|\psi_0\|^{-1}} \right)^{\frac{1}{q-1}}$$

$$\leq \|\psi_0\|_{L^r_x} \left(1 - C_1 \|\psi_0\|^{-1} \right)$$

$$\leq \|\psi_0\|_{L^r_x} - C_1 .$$

Note: this extra C_1 decay is important for the global iteration later.

Otherwise, suppose there exists $t \in [0, \tau]$ st

$$\|\psi(t)\|_{L^r_x} \leq C_1 \max\left(\|z\|^{\frac{2}{\delta}}_{C^0_t \mathcal{C}^{\eta}_x}, 1\right).$$

Then by similar arguments as in Session 3, can show that

$$\|\psi(\tau)\|_{L^r_x} \leq C_1 \max\left(\|z\|^{\frac{2}{\delta}}_{C^0_t \mathcal{C}^{\eta}_x}, 1\right).$$

Thus,

$$\|\psi\|_{C^0_t L^r_x} \leq \max\Big(\|\psi_0\|_{L^r_x},$$
$$C_1 \|z\|^{\frac{2}{\delta}}_{C^0_t \mathcal{C}^{\eta}_x}, C_1\Big).$$

Suppose we guess that in fact

$$\|\psi\|_{C^0_t L^r_x} \le \|\psi_0\|_{L^r_x} + C_1 .$$

Then the bound for z gives

$$\|z\|_{C^0_t \ell^q_x} \le \tau^{\frac{1}{2}} \|\psi\|^2_{C^0_t L^r_x([0,\tau])} + \tau \|\psi\|_{C^0_t L^r_x([0,\tau])} \|z\|^{\frac{q+1}{q}}_{C^0_t \ell^q_x([0,\tau])} + C .$$

$$\le \|\psi_0\|^{2-\frac{\beta}{2}}_{L^r_x} + C_1^2 + C_3^{-1} \|\psi_0\|^{1-\beta}_{L^r_x} \|z\|^{\frac{q+1}{q}}_{C^0_t \ell^q_x} + C$$

(the term $\tau^{\frac{3}{4}} \|A_0\|^{\frac{1}{2}} \lesssim 1$

since $\tau \le \|A_0\|^{-\alpha} = \|A_0\|^{-\frac{3}{4}}$)

Thus, at best, we can hope for

$$\|z\|_{C_t^0 \ell_x^q} \le \|\psi_0\|^{2-\frac{\beta}{2}} + C_2)$$

where $C_2 \gg C_1$.

By applying the <u>continuity method</u>, can show that indeed

$$\|\psi\|_{C_t^0 L_x^r} \le \|\psi_0\|_{L_x^r} + C_1$$

$$\|z\|_{C_t^0 \ell_x^q} \le \|\psi_0\|_{L_x^r}^{2-\frac{\beta}{2}+\nu} + C_2,$$

where $0 < \nu \ll 1$ fixed.

The continuity method is essentially continuous induction: first, verify that the bound holds at $t=0$ (trivial, since $z(0) = 0$), then show that if bound holds on an interval $[0,t] \subseteq [0,\tau]$, then in fact have the stronger bounds on the same interval:

$$\|\psi\|_{C_t^0 L_x^r} \leq \|\psi_0\|_{L_x^r} + \frac{C_1}{2}$$

$$\|z\|_{C_t^0 \ell_x^q} \leq \|\psi_0\|_{L_x^r}^{2-\frac{\beta}{2}+\nu} + \frac{C_2}{2}$$

(by plugging in the assumed bounds into and arguing as before).

Since ψ, z are continuous in time, this implies that the assumed bounds in fact hold on some slightly larger interval $[0,t']$, where $t < t' \leq \tau$ (assuming that $t < \tau$, otherwise we are done).

For instance, plugging the assumed bound into , obtain

$$\|z\| \leq \|\psi_0\|^{2-\frac{\beta}{2}} + C_1^2 + C + C_3^{-1} \|\psi_0\|^{1-\beta} \left(\|\psi_0\|^{2-\frac{\beta}{2}+\nu} + C_2 \right)^{\frac{q+1}{q}}$$

Can absorb

$$C_1^2 + C + C_3^{-1} C_2^{\frac{q+1}{q}} \ll \frac{C_2}{100}.$$

Next,

$$\|\psi_0\|^{1-\beta+(2-\frac{\beta}{2}+\nu)\left(\frac{q+1}{q}\right)} \ll \|\psi_0\|^{2-\frac{\beta}{2}+\nu},$$

provided (assuming also $\|\psi_0\| \gg 1$)

$$1-\beta + \left(2-\frac{\beta}{2}\right)\left(\frac{q+1}{q}\right) < 2 - \frac{\beta}{2}$$

(recall $\beta = \frac{5}{2}$, and take $q = 3$ say).

Can also absorb

$$\|\psi_0\|^{2-\frac{\beta}{2}} \ll \|\psi_0\|^{2-\frac{\beta}{2}+\nu}$$

(also assuming $\|\psi_0\| \gg 1$).

Similarly, to bound ψ,

$$\|\psi\|_{C_t^0 L_x^r} \le \max\Big(\|\psi_0\|_{L_x^r},$$
$$C_1 \|z\|_{C_t^0 L_x^q}^{\frac{2}{q}}, C_1\Big).$$

$$\le \max\Big(\|\psi_0\|, C_1\big(\|\psi_0\|^{2-\frac{\beta}{2}+\nu} + C_2\big)^{\frac{2}{q}}\Big).$$

If

$$\Big(2-\frac{\beta}{2}+\nu\Big)\frac{2}{q} < 1,$$

then

$$\|\psi_0\|^{\left(2-\frac{\beta}{2}+\nu\right)\frac{2}{q}} \ll \|\psi_0\|,$$

and so obtain

$$\|\psi\|_{C_t^0 L_x^r} \leq \|\psi_0\| + \frac{C_1}{2} .$$

Finally, we use the continuity bound and to prove refined estimates which allow us to iterate in time.

Claim:

$$\max\left(\|A(\tau) - \ell(\tau)\|^{\gamma}_{C_x^{\gamma}}, \|\phi(\tau) - \ell(\tau)\|_{L_x^r}, C_4\right)$$

$$\leq e^{c\tau} \max\left(\|A_0^+\|^{\gamma}_{C_x^{\gamma}}, \|\phi_0^+\|_{L_x^r}, C_4\right).$$

Clearly, once this is shown, we may iterate in time to obtain a soft global existence statement. With more effort, can prove uniform-in-time bounds in the relevant norms.

Have that

$$A - \ell = B + Z$$

$$\Phi - \ell = \ell_B - \ell + \psi .$$

By probalistic estimates, can show

$$\| \ell_B - \ell \|_{C_t^0 L_x^r([0,1])} \lesssim 1 .$$

First, suppose $\max(\|A_0^\dagger\|^\alpha, \|\Phi_0^\dagger\|^\beta) \le C_4$.
Then we simply use the continuity bound

$$\|\psi(\tau)\| \le \|\psi_0\| + C_1$$

$$\ll C_4 ,$$

$$\|Z(\tau)\| \le \|\psi_0\|^{2-\frac{\beta}{2}} + C_2$$

$$<< C_4 ,$$

and so

$$\|A(\tau) - \phi(\tau)\| \le \|A_0^+\| + \|Z(\tau)\|$$

$$<< C_4 .$$

Thus, we may assume

$$\max\left(\|A_0^+\|^\alpha, \|\phi_0^+\|^\beta\right) > C_4 .$$

Next, assume that

$$\|\phi_0^+\|^\beta >> \|A_0^+\|^\alpha .$$

Then

$$\tau \sim C_3^{-1} \|\phi_0^+\|^{-\beta} .$$

From the bound with $t = \tau$,

[note $\|\phi_0^+\|^q \tau \sim C_3^{-1} \|\phi_0^+\|^{q-\beta} > C_1$ since $\beta = \frac{5}{2} < q$], obtain that either

$$\|\phi(\tau) - j(\tau)\| \le \|j_B(\tau) - j(\tau)\| + \|\phi(\tau)\|$$
$$\le C + \|\phi_0\| - C_1$$
$$\le \|\phi_0^+\|,$$

or

$$C + \|\phi(\tau)\| \le C + C_1 \max\left(\|z\|^{\frac{2}{q}}_{C_t^0 C_x^4}, 1\right)$$

[using]

$$\le C + C_1 \left(\|\phi_0^+\|^{\frac{2}{q}\left(2 - \frac{\beta}{2} + \nu\right)} + C_2^{\frac{2}{q}} \right)$$

Since

$$\frac{2}{q}\left(2 - \frac{\beta}{2} + \nu\right) < 1,$$

can bound whole thing by

$$\|\phi_0^+\|.$$

This shows the bound for

$$\|\phi(\tau) - q(\tau)\| .$$

To bound $\|A(\tau) - q(\tau)\|$, apply the continuity bound to obtain

$$\|A(\tau) - q(\tau)\| \leq \|A_0^+\| + \|Z\|$$

$$\leq \|\phi_0^+\|^{\frac{\beta}{\alpha}} + \|\phi_0^+\|^{2-\frac{\beta}{2}+\nu} + C_2 .$$

Thus,

$$\|A(\tau) - q(\tau)\|^{\gamma} \leq \|\phi_0^+\|^{\frac{\beta}{\alpha}\gamma} + \|\phi_0^+\|^{(2-\frac{\beta}{2}+\nu)\gamma} + C_2,$$

and using that

$$\frac{\beta}{\alpha}\gamma = \frac{\left(\frac{5}{2}\right)}{\left(\frac{3}{4}\right)} \times \left(\frac{2}{7}\right) < 1,$$

$$\left(2 - \frac{\beta}{2} + \nu\right)\gamma = \left(2 - \frac{5}{4} + \nu\right)\frac{2}{7} < 1,$$

can bound by

$$\leq \|\phi_0^+\| .$$

This shows the case $\|\phi_0^+\|^\beta \geq \|A_0^+\|^\alpha$.

Finally, suppose $\|A_0^+\|^\alpha > \|\phi_0^+\|^\beta$, so

$$\tau \sim C_3^{-1} \|A_0^+\|^{-\alpha} .$$

Then by the continuity bound,

$$\|\varphi(\tau) - \eta(\tau)\| \leq C + \|\psi(\tau)\|$$

$$\leq C + \|\psi_0^+\| + C_1$$

$$C + \|A_0^+\|^{\frac{\alpha}{\beta}} + C_1.$$

Since $\frac{\alpha}{\beta} = \frac{\frac{3}{4}}{\left(\frac{5}{2}\right)} < 1$, can

bound by

$$\leq \|A_0^+\| .$$

Next,

$$\|A(\tau) - \eta(\tau)\| \leq \|A_0^+\| + \|z(\tau)\|$$

$$\leq \|A_0^+\| + \|\phi_0^+\|^{2-\frac{\beta}{2}+\nu} + C_2$$

$$\leq \|A_0^+\| + \|A_0^+\|^{\frac{\alpha}{\beta}(2-\frac{\beta}{2}+\nu)} + C_2 .$$

We claim that

$$\|A_0^+\|^{\frac{\alpha}{\beta}(2-\frac{\beta}{2}+\nu)} + C_2 \leq C\tau \|A_0^+\| .$$

To see this, recall $\tau \sim C_3^{-1} \|A_0^+\|^{-\alpha}$, thus we want to show

$$C_3\left(\|A_0^+\|^{\frac{\alpha}{\beta}(2-\frac{\beta}{2}+\nu)} + C_2\right) \lesssim \|A_0^+\|^{1-\alpha} .$$

Since

$$\frac{\alpha}{\beta}\left(2-\frac{\beta}{2}+\nu\right) < 1-\alpha$$

$$\left[\quad \frac{\left(\frac{3}{4}\right)}{\left(\frac{5}{2}\right)}\left(2-\frac{5}{4}+\nu\right) < \frac{1}{4} \quad \right],$$

this inequality holds, and thus

$$\|A(\tau) - \varphi(\tau)\| \le \|A_0^\dagger\| (1 + c\tau)$$
$$\le e^{c\tau} \|A_0^\dagger\|.$$

It follows that

$$\|A(\tau) - \varphi(\tau)\|^\gamma \le e^{\gamma c\tau} \|A_0^\dagger\|^\gamma,$$

finishing the proof of the claim.