Invitation to non-Abelian gauge theory (Yang-Mills)

$\begin{cases} \text{lattice YM} \\ \text{Continuum YM.} \end{cases}$

Unlike Φ^4 where lattice model is defined by simple "finite difference" YM has more subtle discretization, to preserve gauge symmetry

Setup:

$G = U(N)$ unitary matrices: $gg^* = g^*g = \mathrm{id}$
where $g^* = \bar{g}^T$

$\mathfrak{g} = u(N)$ skew-Hermitian matrices. $\bar{A}^T = -A$

Recall $\exp: \mathfrak{g} \to G$
$A \mapsto e^A$
$0 \mapsto \mathrm{id}$

Recall: $A, B \in \mathfrak{g}$
$[A, B] = AB - BA \in \mathfrak{g}$

Recall: G action on $\mathfrak{g}$ by: $A \mapsto gAg^{-1}$

Inner product:

$\langle A, B \rangle = \mathrm{Re}\,\mathrm{tr}(AB^*)$.

If $A, B \in \mathfrak{g}$ then $\langle A, B \rangle = -\mathrm{tr}(AB)$

G-action invariance: $\langle gAg^{-1}, gBg^{-1} \rangle = \langle A, B \rangle$

[In general, G = Lie group. $\mathfrak{g}$ = Lie algebra.]

Continuum YM:

Def: $A = (A_1, \dots, A_d)$. $A_i : \mathbb{R}^d \to \mathfrak{g}$

$$F^A_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \qquad F_{ij} = -F_{ji}$$

$$S(A) = \int_{\mathbb{R}^d} |F^A|^2 \, dx$$

$$\left(= -\sum_{ij} \int_{\mathbb{R}^d} tr(F^A_{ij})^2 \right)$$

(If $\mathfrak{g}$ replaced by $\mathbb{R}$. $S(A)$ is quadratic)

Gauge transformation: $\forall g : \mathbb{R}^d \to G$

$$A \mapsto g A g^{-1} + g \, dg^{-1} =: A^g$$
$$= g A g^{-1} - dg \cdot g^{-1}$$

Here:

- $dg = (\partial_1 g, \dots, \partial_d g)$ note: $dg^{-1} = -g^{-1} \, dg \, g^{-1}$
  (since $\partial_i (g \cdot g^{-1}) = 0$)
- $g \partial_i g^{-1}, \ \partial_i g \, g^{-1} : \mathbb{R}^d \to \mathfrak{g}$ (exercise)

Gauge invariance: $S(A^g) = S(A)$

proof :

$$\tilde{A} = gAg^{-1} + g\,dg^{-1}$$

$$\tilde{F}_{ij} = (d\tilde{A})_{ij} + [\tilde{A}_i, \tilde{A}_j]$$

$$= \partial_i(gA_jg^{-1} + g\partial_jg^{-1}) - \partial_j(gA_ig^{-1} + g\partial_ig^{-1})$$

$$+ [gA_ig^{-1} + \underline{g\partial_ig^{-1}}, \underline{gA_jg^{-1}} + g\partial_jg^{-1}]$$

Cancelations :

$$(\partial_i g)A_jg^{-1} + g\underbrace{\partial_ig^{-1}}_{= -g^{-1}\partial_ig\cdot g^{-1}}gA_jg^{-1} = 0$$

$$(\partial_i g)\partial_jg^{-1} + g\partial_ig^{-1}g\partial_jg^{-1} = 0 \qquad \text{etc.}$$

$$\tilde{F}_{ij} = g(\partial_iA_j - \partial_jA_i + [A_i, A_j])g^{-1} = gF_{ij}g^{-1}$$

(exercise)

inner product is G-invariant. $\blacksquare$

measure : $e^{-S(A)}DA$ formal!

In general, YM-Higgs model: $\Phi : \mathbb{R}^d \to \mathbb{R}^N$

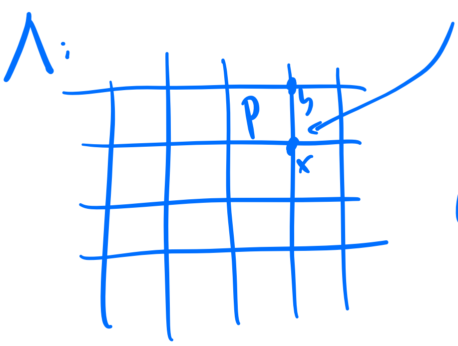
$$S(A, \Phi) = \int_{\mathbb{R}^d} |F_A|^2 + |D_A\Phi|^2 + |\Phi|^4 \, dx$$

$$D_A^i\Phi \overset{\text{def}}{=} \partial_i\Phi + A_i\Phi \qquad \Phi^g \overset{\text{def}}{=} g\Phi$$

$$S(A^g, \Phi^g) = S(A, g)$$

Lattice YM:

Λ:



$Q_{xy} \in G$

$Q_{yx} = Q_{xy}^{-1}$

$p = x_1 x_2 x_3 x_4$ "plaquette"

$Q_p \stackrel{def}{=} Q_{x_1 x_2} \cdots Q_{x_4 x_1}$

$$S(Q) = \sum_p \mathrm{Re}\,\mathrm{tr}(Q_p)$$

Gauge transf:

$g : \Lambda \longrightarrow G$, $Q^g_{xy} = g_x Q_{xy} g_y^{-1}$

Gauge inv:

$$S(Q^g) = S(Q)$$

- base point / orientation of p not important:
  $\sum$ over all p or fix some lexicographic order
- make sense on $\forall$ graph.
- $\mathrm{Re}\,\mathrm{tr}$ can be replaced by $f : G \to \mathbb{R}$
  s.t. $f(aba^{-1}) = f(b)$ $\quad f(a^{-1}) = f(a)$
- $\sum_p \mathrm{Re}\,\mathrm{tr}(Q_p) + \sum_{(xy)} |Q_{xy} \Phi_y - \Phi_x|^2 \qquad \Phi_x \in \mathbb{R}^N$

Haar measure:

μ on $G = U(N)$

- $\mu(gS) = \mu(S) \qquad \forall g \in G. \quad S \subseteq G$
  where $gS = \{ g \cdot s : s \in S \}$
- $\mu(Sg) = \mu(S)$

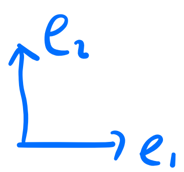
YM measure

$$\frac{1}{Z} e^{S(Q)} d\mu \qquad \text{well-defined}$$

$S(Q)$ as a discretization of $S(A)$:

Let's redefine $\quad S(Q) = \sum_p \operatorname{Re} \operatorname{Tr}(Q_p - Id)$

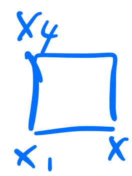
(just amounts to change of Z)



$e_1 \cdots e_d$ ON basis of R^d

Let $Q(x, x+\varepsilon e_j) = e^{\varepsilon A_j(x)}$

Then for $p = x_1 x_2 x_3 x_4$,



$x_3 = x_2 + \varepsilon e_k$

$x_2 = x_1 + \varepsilon e_j$

$$Q_p = e^{\varepsilon A_j(x_1)} e^{\varepsilon A_k(x_2)} e^{-\varepsilon A_j(x_4)} e^{-\varepsilon A_k(x_1)}$$

Baker-Campbell-Hausdorff :

$$e^{B_1} \cdots e^{B_m} = e^{\sum_{a=1}^m B_a + \frac{1}{2}\sum_{1 \le a < b \le m} [B_a, B_b] + \text{higher comm}}$$

$$\begin{cases} \text{If: } B_1 \cdots B_4 \text{ skew-Hermitian, order } \varepsilon \\ \text{and } B_1 + \cdots + B_4 \sim \varepsilon^2. \end{cases}$$

Then. By Taylor:

$$\operatorname{Re}\operatorname{Tr}\left(e^{B_1} \cdots e^{B_4} - I\right)$$

$$= \operatorname{Re}\operatorname{Tr}\Big(e^{\underbrace{\sum_i B_i + \frac{1}{2}\sum_{i<j}[B_i\, B_j]}_{\sim \varepsilon^2} + O(\varepsilon^3)} - I \Big)$$

$$= \operatorname{Re}\operatorname{Tr}\Big(\underbrace{\sum_i B_i + \frac{1}{2}\sum_{i<j}[B_i, B_j] + O(\varepsilon^3)}_{\in \mathfrak{g}\ \Rightarrow\ \operatorname{Re}\operatorname{Tr} = 0}$$

$$+ \frac{1}{2}\Big(\sum_i B_i + \frac{1}{2}\sum_{i<j}[B_i, B_j]\Big)^2 + O(\varepsilon^5) \Big)$$

$$= \frac{1}{2} \cancel{\operatorname{Re}} \operatorname{Tr}\Big[\Big(\sum_i B_i + \frac{1}{2}\sum_{i<j}[B_i\ B_j]\Big)^2\Big] + O(\varepsilon^5)$$

Now $B_1 = \varepsilon A_j(x_1)$, $B_2 = \cdots\cdots$

$$\varepsilon A_j(x_1) + \varepsilon \underbrace{A_k(x_2)} - \varepsilon A_j(x_4) - \varepsilon \underbrace{A_k(x_1)}$$

$$= \varepsilon^2 (\underbrace{\partial_j A_k} - \partial_k A_j) + O(\varepsilon^3)$$

$$[B_1, B_2] = \varepsilon^2 [A_j(x_1),\ A_k(x_2)] \simeq [A_j, A_k] + O(\varepsilon^3) = [B_3, B_4]$$

$$[B_1, B_3] = \varepsilon^2 [A_j(x_1),\ -A_j(x_4)] = O(\varepsilon^3) = [B_2, B_4]$$

$$[B_1\ B_4] = -[A_j, A_k] + O(\varepsilon^3)$$

$$[B_2, B_3] = -[A_k, A_j] + O(\varepsilon^3) = [A_j, A_k] + O(\varepsilon^3)$$

So we get $\frac{\varepsilon^4}{2} \mathrm{Tr}\left(\partial_j A_k - \partial_k A_j + [A_j, A_k] \right) + O(\varepsilon^5)$

$$e^{\beta \sum_p \mathrm{Tr}(Q_p - I)} \qquad \varepsilon^d \sum_p \longrightarrow \int \cdots dx$$

scale $\beta = \varepsilon^{d-4}$

SPDE in continuum.

$\xi = (\xi_1, \cdots \xi_d)$ $\quad \xi_i$: $\mathfrak{g}$-valued white noise

$$E\left[\langle \xi_i(f), A \rangle \langle \xi_j(g), B \rangle\right] \qquad A, B \in \mathfrak{g}$$
$$= \delta_{ij} \langle A, B \rangle \langle f, g \rangle_{L^2}.$$

$\partial_t A = - D_A^* F_A + \xi$ $\quad$ (D_A^* : adjoint of D_A)