SIMONS LAUFER MATHEMATICAL SCIENCES INSTITUTE

SLMath Summer Graduate School

Stochastic quantisation

Week 2

[website of the school]

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[These notes are based on a graduate course given by M. Gubinelli at the University of Milan in 2021]

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9 UV limit in three dimensions

What happens in $d=3$. Let us go back to the a-priori estimates: test the equation (?) where $V'(\varphi)=\lambda\varphi^3+\beta\varphi$ (and as in two dimensions we take $\beta=-3\lambda c_\varepsilon+\beta'$) with Z and integrate in space $\mathbb{T}_\varepsilon^3$:

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3} Z_t^2+\int_{\mathbb{T}_\varepsilon^3}\left[|\nabla_\varepsilon Z_t|^2+m^2|Z_t|^2+\frac{\lambda}{2}|Z_t|^4\right]$$

$$=-\frac{1}{2}\int_{\mathbb{T}_\varepsilon^3}[\lambda\mathbb{Y}^3 Z+3\lambda\mathbb{Y}^2Z^2+3\lambda\mathbb{Y}^1Z^3+\beta'\mathbb{Y}^1Z+\beta'Z^2].$$

But now $\mathbb{Y}^2$ has regularity $-1-2\kappa$ and $\mathbb{Y}^3$ even worser than $-3/2-3\kappa$. For Z we can hope only for H^1 regularity from these estimates. *Big problem!!*

The term $\mathbb{Y}^1Z^3$, $\mathbb{Y}^1Z$ are still ok because $\mathbb{Y}^1$ has regularity $-1/2-\kappa$.

We go back to the equation (?) and write it more explicitly

$$\frac{\partial}{\partial t} Z_t + (m^2 - \Delta_\varepsilon) Z_t = \underbrace{-\frac{1}{2} \lambda \mathbb{Y}^3}_{\mathscr{C}^{-3/2-3\kappa}} - \frac{3}{2} \lambda \mathbb{Y}^2 Z + \cdots$$

From the theory of parabolic equations one sees that Z cannot have better regularity that $2 + -3/2 - 3\kappa = 1/2 - 3\kappa > 0$ surely it cannot be H^1. Moreover in this case we even have a worser problem for the term $\mathbb{Y}^2 Z$ which is a prod. of something of reg. $-1 - 2\kappa$ and something of reg. $1/2 - 3\kappa$ which do not sum up to a positive quantity. The first step is to separated the problems in the product $\mathbb{Y}^2 Z$ via a decomposition, we write

$$\mathbb{Y}^2 Z = \mathbb{Y}^2 \succ Z + \mathbb{Y}^2 \preccurlyeq Z,$$

where $\mathbb{Y}^2 \preccurlyeq Z = \mathbb{Y}^2 \prec Z + \mathbb{Y}^2 \circ Z$. By paraproducts estimates one has that $\mathbb{Y}^2 \succ Z$ has regularity of $\mathbb{Y}^2$ that is $-1 - 2\kappa$ and it is well-defined. The term containing the resonant product $\mathbb{Y}^2 \preccurlyeq Z$ is however not well defined.

Define a new stochastic object $\mathbb{Y}^{[3],\varepsilon}$ to be the solution of the equation

$$\frac{\partial}{\partial t} \mathbb{Y}_t^{[3],\varepsilon} + (m^2 - \Delta_\varepsilon) \mathbb{Y}_t^{[3],\varepsilon} = -\frac{1}{2} \lambda \mathbb{Y}_t^{3,\varepsilon},$$

(for example, take the stationary solution). Again this is a very explicit functional of the Gaussian process Y^ε and will be easy to analyze, in particular one can show that uniformly in ε it belongs to

$$\mathbb{Y}^{[3],\varepsilon} \in C(\mathbb{R}, \mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3)),$$

in the sense that, for example,

$$\sup_\varepsilon \mathbb{E}\left[\sup_{t \in [0,T]} \| \mathbb{Y}_t^{[3],\varepsilon} \|_{\mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3)}^K \right] < \infty,$$

for any K, T.

Now define $\mathbb{H}$ as the solution to

$$\boxed{\frac{\partial}{\partial t} \mathbb{H}_t + (m^2 - \Delta_\varepsilon) \mathbb{H}_t = -\frac{1}{2} \lambda \mathbb{Y}_t^3 - \frac{3}{2} \lambda \mathbb{Y}_t^2 \succ \mathbb{H}_t, \qquad (1)}$$

this is a linear equation which can be easily solved and analyzed and its solution $\mathbb{H}$ does not looks much different than $\mathbb{Y}^{[3],\varepsilon}$ and lives also in $C(\mathbb{R}, \mathscr{C}^{1/2-3\kappa}(\mathbb{T}^3))$.

Define Φ as

$$Z =: \mathbb{H} + \Phi$$

which solves

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_t &= (\Delta_\varepsilon - m^2)\,\Phi_t - \frac{\lambda}{2}[-3\,\lambda\,\mathbb{Y}^2 \succ \mathbb{H} + 3\,\mathbb{Y}^2 \succ Z + 3\,\mathbb{Y}^2 \circ Z + 3\,\mathbb{Y}^2 \prec Z + 3\,\mathbb{Y} Z^2 + Z^3] \\ &= (\Delta_\varepsilon - m^2)\,\Phi_t - \frac{\lambda}{2}\left[3\,\mathbb{Y}^2 \succ \Phi + \underbrace{3\,\mathbb{Y}^2 \circ \Phi + 3\,\mathbb{Y}^2 \circ \mathbb{H}}_{\text{dangerous terms!!!}} + 3\,\mathbb{Y}^2 \prec Z + 3\,\mathbb{Y} Z^2 + Z^3\right]\end{aligned} \tag{2}$$

This is the right equation to get a-priori estimates for (almost since Φ cannot be expected to be in H^1 exactly due to this equation). Let's test it with Φ to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}^3_\varepsilon}\Phi^2 + \int_{\mathbb{T}^3_\varepsilon}\Phi\,(m^2 - \Delta_\varepsilon)\,\Phi + \frac{\lambda}{2}\int_{\mathbb{T}^3_\varepsilon}\Phi^4$$

$$= \int_{\mathbb{T}^3_\varepsilon}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2 \succ \Phi - \frac{3}{2}\lambda\,\mathbb{Y}^2 \circ \Phi - \frac{3}{2}\lambda\,\mathbb{Y}^2 \circ \mathbb{H} - \frac{1}{2}\beta'(\mathbb{Y}^1 + Z)\right]$$

$$+ \int_{\mathbb{T}^3_\varepsilon}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2 \prec Z_t - \frac{3}{2}\lambda\,\mathbb{Y}^1 Z^2\right] - \frac{\lambda}{2}\int_{\mathbb{T}^3_\varepsilon}\Phi((\mathbb{H} + \Phi)^3 - \Phi^3)$$

We have now to cross fingers and check that all the terms in the r.h.s. can be controlled with the l.h.s.

The term

$$-\frac{\lambda}{2}\int_{\mathbb{T}^3_\varepsilon}\Phi_t((\mathbb{H}_t + \Phi_t)^3 - \Phi_t^3),$$

is not scary at all since $\mathbb{H}$ is a nice function and it contains only powers less than 4 of Φ so it can be controlled via the $\int \Phi^4$ is the l.h.s. (like in the infinite vol estimates of last week). The term

$$-\frac{3}{2}\lambda\int_{\mathbb{T}^3_\varepsilon}\Phi_t\,\mathbb{Y}^1 Z_t^2 = -\frac{3}{2}\lambda\int_{\mathbb{T}^3_\varepsilon}\Phi_t\,\mathbb{Y}^1\,(\mathbb{H} + \Phi_t)^2$$

is also fine since $\mathbb{Y}^1$ is only $-1/2 - \kappa$ irregular and we have the H^1 norm of Φ and it is at most cubic in Φ^3. With some work one can get a nice estimate. Note however that this term will contain products

$$\mathbb{Y}^1\,\mathbb{H}, \quad \mathbb{Y}^1\,\mathbb{H}^2,$$

which are not well defined because $\mathbb{H}$ is only of regularity $1/2 - 2\kappa$ so the reg. do not sum up to positive number. However these terms can be analyzed with probabilistic estimates and shown to be well defined and not needing renormalization. We will assume in the following that they have uniform estimates as $\varepsilon \to 0$ in

$$\mathbb{Y}^1\,\mathbb{H}, \mathbb{Y}^1\,\mathbb{H}^2 \in C(\mathbb{R}; \mathscr{C}^{-1/2-\kappa}(\mathbb{T}^3)).$$

We are worried about the terms:

$$-\frac{3}{2}\lambda\int_{\mathbb{T}^3_\varepsilon}\Phi_t[\mathbb{Y}^2 \succ \Phi_t + \mathbb{Y}^2 \preccurlyeq \Phi_t]$$

since Φ is not regular enough for $\mathbb{Y}^2$. Here we use the following fact.

Lemma 1. *We have*

$$D(f,g,h) := \int_{\mathbb{T}_\varepsilon^3} f(g \succ h) - \int_{\mathbb{T}_\varepsilon^3} (g \circ f) h$$

is well defined and continuous when the sum of the regularities of f,g,h is positive. For example

$$|D(f,g,h)| \leqslant \|f\|_{H^\alpha} \|h\|_{H^\gamma} \|g\|_{\mathscr{C}^\beta}$$

whenever $\alpha+\beta+\gamma>0$.

Using this lemma we have

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t + \mathbb{Y}^2 \circ \Phi_t] = \int_{\mathbb{T}_\varepsilon^3} \Phi_t[2\,\mathbb{Y}^2 \succ \Phi_t] + D(\Phi_t, \mathbb{Y}_t^2, \Phi_t)$$

We got rid of the resonant product but the term

$$\langle \text{danger} | \int_{\mathbb{T}_\varepsilon^3} \Phi_t[2\,\mathbb{Y}^2 \succ \Phi_t] \rangle$$

is still dangerous.

Going back to the a-priori estimate we focus on two terms

$$\cdots + \langle \text{danger} | \int_{\mathbb{T}_\varepsilon^3} \Phi_t (m^2 - \Delta_\varepsilon)\, \Phi_t \rangle = \langle \text{danger} | -3\,\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t] \rangle + \cdots$$

and try to cancel the one in r.h.s. using that in the l.h.s. This is possible by defining

$$\Psi_t := \Phi_t + \frac{3\,\lambda}{2} \mathcal{Q}^{-1}[\mathbb{Y}^2 \succ \Phi_t]$$

where

$$\mathcal{Q} := (m^2 - \Delta_\varepsilon).$$

Substituting the estimate (i.e. we are completing the above square). One get

$$\int_{\mathbb{T}_\varepsilon^3} \Phi_t\, \mathcal{Q}\, \Phi_t + 3\,\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \left[\Psi_t - \frac{3\,\lambda}{2} \mathcal{Q}^{-1}[\mathbb{Y}^2 \succ \Phi_t]\right] \mathcal{Q} \left[\Psi_t - \frac{3\,\lambda}{2} \mathcal{Q}^{-1}[\mathbb{Y}^2 \succ \Phi_t]\right] + 3\,\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Psi_t\, \mathcal{Q}\, \Psi_t - 3\,\lambda \int_{\mathbb{T}_\varepsilon^3} \Psi_t[\mathbb{Y}^2 \succ \Phi_t] + \frac{9\,\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)\, \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t) + 3\,\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi_t[\mathbb{Y}^2 \succ \Phi_t]$$

$$= \int_{\mathbb{T}_\varepsilon^3} \Psi_t\, \mathcal{Q}\, \Psi_t - \frac{9\,\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)\, \mathcal{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

10 The paracontrolled a-priori estimate

To recap, we defined $\mathbb{H}$ to be

$$\frac{\partial}{\partial t}\mathbb{H}_t+(m^2-\Delta_\varepsilon)\,\mathbb{H}_t=-\frac{\lambda}{2}\mathbb{Y}_t^3-\frac{3\lambda}{2}\mathbb{Y}_t^2\succ\mathbb{H}_t,$$

(solve this equation by a fix-point) and defined $\Phi:=Z-\mathbb{H}$ which satisfies

$$\frac{\partial}{\partial t}\Phi_t=(\Delta_\varepsilon-m^2)\,\Phi_t-\frac{\lambda}{2}[-3\,\lambda\,\mathbb{Y}^2\succ\mathbb{H}+3\,\mathbb{Y}^2\succ Z+3\,\mathbb{Y}^2\circ Z+3\,\mathbb{Y}^2\prec Z+3\,\mathbb{Y}Z^2+Z^3]$$

$$=(\Delta_\varepsilon-m^2)\,\Phi_t-\frac{\lambda}{2}\left[3\,\mathbb{Y}^2\succ\Phi+\underbrace{3\,\mathbb{Y}^2\circ\Phi+3\,\mathbb{Y}^2\circ\mathbb{H}}_{\text{dangerous terms!!!}}+3\,\mathbb{Y}^2\prec Z+3\,\mathbb{Y}Z^2+Z^3\right].$$

Recall the various regularities (we use κ for an arbitrary small >0 which can be different from line to line)

term	reg
$\mathbb{Y}^1$	$-1/2-\kappa$
$\mathbb{Y}^2$	$-1-\kappa$
$\mathbb{Y}^3$	"$-3/2-\kappa$"(as space-time distribution)
$\mathbb{H}$	$1/2-\kappa$
$\mathbb{Y}^2\succ\Phi$	$-1-\kappa$
Φ	$1-\kappa$

Then we tested with Φ to get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}_\varepsilon^3}\Phi_t^2+\int_{\mathbb{T}_\varepsilon^3}\Phi_t(m^2-\Delta_\varepsilon)\,\Phi_t+\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi_t^4$$

$$=\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2\succ\Phi-\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\Phi\right]$$

$$+\int_{\mathbb{T}_\varepsilon^3}\Phi\left[-\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\mathbb{H}-\frac{1}{2}\beta'(\mathbb{Y}^1+Z)\right]$$

$$-\frac{3}{2}\lambda\int_{\mathbb{T}_\varepsilon^3}\Phi(\mathbb{Y}^2\prec Z+\mathbb{Y}^1Z^2)-\frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3}\Phi((\mathbb{H}+\Phi)^3-\Phi^3)$$

and we did a transformation to the combination (in which all the terms are "ill defined", i.e. I cannot hope to control them separately in the limit)

$$A:=\int_{\mathbb{T}_\varepsilon^3}\Phi(m^2-\Delta_\varepsilon)\,\Phi+\int_{\mathbb{T}_\varepsilon^3}\Phi\left[\frac{3}{2}\lambda\,\mathbb{Y}^2\succ\Phi+\frac{3}{2}\lambda\,\mathbb{Y}^2\circ\Phi\right]$$

we use a "commutator lemma" to replace $\int \Phi(\mathbb{Y}^2 \circ \Phi)$ with $\int (\mathbb{Y}^2 \succ \Phi)\,\Phi$ modulo nice error term:

$$A = \int_{\mathbb{T}_\varepsilon^3} \Phi (m^2 - \Delta_\varepsilon)\, \Phi + \int_{\mathbb{T}_\varepsilon^3} \Phi[3\,\lambda\, \mathbb{Y}^2 \succ \Phi] + \lambda D(\Phi, \mathbb{Y}^2, \Phi)$$

Then we defined Ψ so that

$$\Phi = -\frac{3\,\lambda}{2} (m^2 - \Delta_\varepsilon)^{-1} [\mathbb{Y}^2 \succ \Phi] + \Psi,$$

$$A = \int_{\mathbb{T}_\varepsilon^3} \Psi_t (m^2 - \Delta_\varepsilon)\, \Psi_t + \frac{9\,\lambda}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1} (\mathbb{Y}^2 \succ \Phi_t) + \lambda D(\Phi, \mathbb{Y}^2, \Phi)$$

At this point we have decomposed X as

$$X = \mathbb{Y}^1 + \mathbb{H} - \frac{3\,\lambda}{2} (m^2 - \Delta_\varepsilon)^{-1} [\mathbb{Y}^2 \succ \Phi] + \Psi \tag{3}$$

where

$$\Phi = X - \mathbb{Y}^1 - \mathbb{H} = -\frac{3\,\lambda}{2} (m^2 - \Delta_\varepsilon)^{-1} [\mathbb{Y}^2 \succ \Phi] + \Psi$$

with these different functions satisfying the a-priori equation

$$\boxed{\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t} \int_{\mathbb{T}_\varepsilon^3} \Phi^2 + \int_{\mathbb{T}_\varepsilon^3} \Psi (m^2 - \Delta_\varepsilon)\, \Psi + \frac{\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi^4 \\ &= \int_{\mathbb{T}_\varepsilon^3} \left[-\frac{9\,\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1} (\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2}\,\lambda\, \Phi\, \mathbb{Y}^2 \circ \mathbb{H} - \frac{1}{2}\,\beta'\, \Phi(\mathbb{Y}^1 + Z) \right] \\ &+ \lambda D(\Phi, \mathbb{Y}^2, \Phi) - \frac{3}{2}\,\lambda \int_{\mathbb{T}_\varepsilon^3} \Phi (\mathbb{Y}^2 \prec Z + \mathbb{Y}^1 Z^2) - \frac{\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi ((\mathbb{H} + \Phi)^3 - \Phi^3) \end{aligned}}$$

The good guys are on the l.h.s and the bad guys on the r.h.s., with the ugly guys in orange.

The terms in orange are still out of control, in particular they contain products which are not well defined (because the regularities do not sum up to positive).

Let us pause a moment and try to understand the meaning of the decomposition (3): this is the key point of these new approaches to singular SPDEs (i.e. regularity structures or paracontrolled distributions). The message is that we cannot just look at generic functions in a given vector space (like in classical PDE theory) but we need to specify the solution as an "expansion" in terms (explicit or implicit) of different character. In the paracontrolled approach this involves the regularity of the various terms

$$X = \underbrace{\mathbb{Y}^1}_{-1/2-\kappa} + \underbrace{\mathbb{H}}_{1/2-\kappa} \underbrace{- \frac{3\,\lambda}{2} (m^2 - \Delta_\varepsilon)^{-1} [\mathbb{Y}^2 \succ \Phi]}_{1-\kappa} + \underbrace{\Psi}_{H^1},$$

(actually Ψ is even better than H^1, if I remember correctly it has regularity $3/2$).

For example one could see from this that for the LP blocks one has

$$
\begin{array}{ll}
\Delta_i X & \sim (2^i)^{1/2+\kappa}, \\
\Delta_i X - \Delta_i \mathbb{Y}^1 & \sim (2^i)^{-1/2-\kappa} \\
\Delta_i X - \Delta_i \mathbb{Y}^1 - \Delta_i \mathbb{H} & \sim (2^i)^{-1-\kappa} \\
\Delta_i X - \Delta_i \mathbb{Y}^1 - \Delta_i \mathbb{H} + \frac{3\lambda}{2} \Delta_i \{(m^2 - \Delta_\varepsilon)^{-1} [\mathbb{Y}^2 \succ \Phi]\} & \sim (2^i)^{-1}
\end{array}
$$

which can be interpreted by saying that my solution lives in a very particular subspace of the space of Besov functions of regularity $-1/2$ (we could take for example $H^{-1/2-\kappa}$).

In particular the stochastic objects $\mathbb{Y}^1, \mathbb{H}, \mathbb{Y}^2$ do not have better regularity as those stated (i.e. they are almost surely not in $\mathscr{C}^{-1/2}, \mathscr{C}^{1/2}, \mathscr{C}^1$ (think about Hölder regularity of BM).

11 The second renormalization

We need to understand what is going on with the red term

$$\int_{\mathbb{T}_\varepsilon^3} \left[-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1} (\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2} \lambda \Phi \mathbb{Y}^2 \circ \mathbb{H} - \frac{1}{2} \beta' \Phi (\mathbb{Y}^1 + Z) \right]$$

which contains not-well defined products.

Start with $\mathbb{Y}^2 \circ \mathbb{H}$: use the definition of $\mathbb{H}$ (where $\mathscr{L} = \partial_t + (m^2 - \Delta_\varepsilon)$)

$$\mathbb{H} = -\frac{\lambda}{2} \mathbb{Y}^{[3]} - \frac{3\lambda}{2} \mathscr{L}^{-1} (\mathbb{Y}^2 \succ \mathbb{H}),$$

recall also that $\mathbb{Y}^{[3]} = \mathscr{L}^{-1} \mathbb{Y}^3$ (with reg. $1/2 - \kappa$), and write it as

$$\mathbb{Y}^2 \circ \mathbb{H} = -\frac{\lambda}{2} \mathbb{Y}^2 \circ \mathbb{Y}^{[3]} - \frac{3\lambda}{2} \mathbb{Y}^2 \circ \mathscr{L}^{-1} (\mathbb{Y}^2 \succ \mathbb{H}).$$

For $\mathbb{Y}^2 \circ \mathbb{Y}^{[3]}$ we can show by probabilistic arguments involving Wick products (i.e. explicit formulas for polynomials of Gaussian) that one can define other polynomials $\mathbb{Y}^{2\circ[3]}$ and $\mathbb{Y}^{2\circ[2]}$

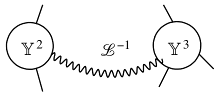
$$\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} = \mathbb{Y}^2 \circ \mathscr{L}^{-1} \mathbb{Y}^3 = [\![Y^2]\!] \circ \mathscr{L}^{-1} [\![Y^3]\!] = \mathbb{Y}^{2\circ[3]} + 3 d_\varepsilon \mathbb{Y}^1,$$

$$\mathbb{Y}^2 \circ \mathscr{L}^{-1} \mathbb{Y}^2 = \mathbb{Y}^{2\circ[2]} + d_\varepsilon$$

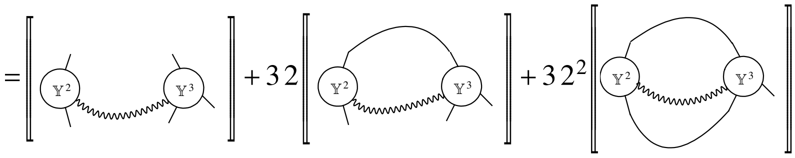
where d_ε is a constant which diverges logarithmically with ε. This is not much different from what we did in $d = 2$ and in $d = 3$ for the products Y^3, Y^2. The random field $\mathbb{Y}^{2\circ[3]}$ and $\mathbb{Y}^{2\circ[2]}$ converge as $\varepsilon \to 0$ to well defined random field such that

$$\boxed{\begin{array}{c} \mathbb{Y}^{2\circ[3]} := \mathbb{Y}^2 \circ \mathbb{Y}^{[3]} - 3 d_\varepsilon \mathbb{Y}^1 \in C(\mathbb{R}_+, \mathscr{C}^{1/2-\kappa}) \\[1em] \mathbb{Y}^{2\circ[2]} := \mathbb{Y}^2 \circ \mathscr{L}^{-1} \mathbb{Y}^2 - d_\varepsilon \in C(\mathbb{R}_+, \mathscr{C}^{-\kappa}) \end{array}}$$

In terms of Feynman graphs one could write

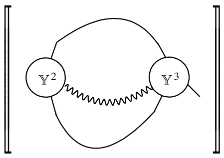
$$\mathbb{Y}^2 \circ \mathbb{Y}^{[3]} =$$ 

which can be decomposed in orthogonal terms



$$= [\![Y^2]\!] \diamond_0 \mathscr{L}^{-1} [\![Y^3]\!] + 3\,2 [\![Y^2]\!] \diamond_1 \mathscr{L}^{-1} [\![Y^3]\!] + 3\,2^2 [\![Y^2]\!] \diamond_2 \mathscr{L}^{-1} [\![Y^3]\!]$$

and one has that the last one is diverging while the other two are well defined



$$\approx \int_{\mathbb{T}_\varepsilon^3} \mathrm{d}x \underbrace{P(x-y)}_{\mathscr{L}^{-1}} \underbrace{G(x-y)^2}_{\text{two contraction lines}} Y(y) \approx \underbrace{\int_{\mathbb{T}_\varepsilon^3} \mathrm{d}x \frac{1}{|x|^3}}_{\propto d_\varepsilon} Y(y)$$

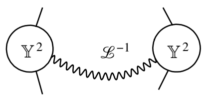
since the correlation function

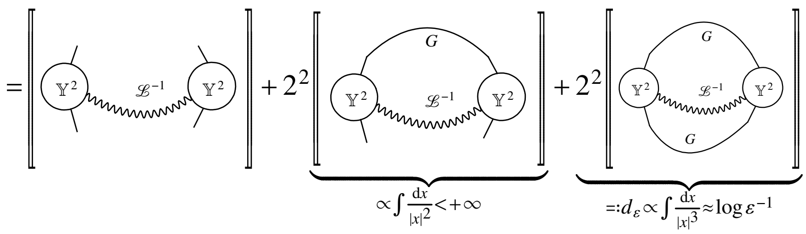
$$G(x-y) = \mathbb{E}[Y(x)Y(y)] \approx \sum_{k \in \mathbb{Z}^3 \cap [-\varepsilon^{-1}, \varepsilon^{-1}]^3} \frac{e^{ik(x-y)}}{k^2+m^2} \approx \frac{1}{|x-y|}$$

and the kernel P of $\mathscr{L}^{-1}$ behaves in the same way

$$P(x-y) \approx |x-y|^{-1}.$$

For $\mathbb{Y}^2 \circ \mathscr{L}^{-1}\, \mathbb{Y}^2$ one can do the same:

$$\mathbb{Y}^2 \circ \mathbb{Y}^{[2]} =$$ 



$$= \mathbb{Y}^{2\circ[2]} + d_\varepsilon$$

where $\mathbb{Y}^{2\circ[2]}$ denotes the sum of the first two graphs.

Let us go back to $\mathbb{Y}^2 \circ \mathbb{H}$. The first step is to use commutator lemmas for paraproducts and resonant products:

Commutator lemmas roughly say that one can usually write

$$f \circ (g \succ h) \approx (f \circ g) h$$

modulo "nice terms". Similar statements can be made when there are other nice linear operations in between, e.g.

$$f \circ \mathscr{L}^{-1}(g \succ h) \approx (f \circ \mathscr{L}^{-1} g) h, \qquad f \circ (m^2 - \Delta)^{-1}(g \succ h) \approx (f \circ (m^2 - \Delta)^{-1} g) h$$

This is enough to show that

$$\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}) = \underbrace{[\mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2)]}_{\text{ugly guy!}} \mathbb{H} + \underbrace{C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})}_{\text{nice commutator}}$$

$$= (\mathbb{Y}^{2 \circ [2]} + d_\varepsilon) \mathbb{H} + C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})$$

Therefore we can handle the full term $\mathbb{Y}^2 \circ \mathbb{H}$ as

$$\mathbb{Y}^2 \circ \mathbb{H} = -\frac{\lambda}{2} \underbrace{\mathbb{Y}^2 \circ \mathbb{Y}^{[3]}}_{\text{ugly guy!}} - \frac{3\lambda}{2} \mathbb{Y}^2 \circ \mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H})$$

$$= -\frac{\lambda}{2} (\mathbb{Y}^{2 \circ [3]} + 3 d_\varepsilon \mathbb{Y}^1) - \frac{3\lambda}{2} (\mathbb{Y}^{2 \circ [2]} + d_\varepsilon) \mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H})$$

$$= -\frac{\lambda}{2} \mathbb{Y}^{2 \circ [3]} - \frac{3\lambda}{2} \mathbb{Y}^{2 \circ [2]} \mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) - \frac{3\lambda}{2} d_\varepsilon (\mathbb{Y}^1 + \mathbb{H})$$

and we see precisely how $\mathbb{Y}^2 \circ \mathbb{H}$ diverges as $\varepsilon \to 0$, due to the presence of d_ε.

Now our task is to handle the other dangerous term (highlighted in red):

$$\int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \mathscr{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

with $\mathscr{Q} = (m^2 - \Delta_\varepsilon)$. We can decompose it with paraproducts and some commutator lemma as

$$B = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \mathscr{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t) = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t) \circ \mathscr{Q}^{-1} (\mathbb{Y}^2 \succ \Phi_t)$$

(only the resonant term counts in integrals)

$$B = \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \circ \mathscr{Q}^{-1} \mathbb{Y}^2) \Phi^2 + \underbrace{C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi)}_{\text{nice commutator}}$$

The same considerations as above apply to the explicit polynomial $\mathbb{Y}^2 \circ \mathscr{Q}^{-1} \mathbb{Y}^2$ and one defines

$$\mathbb{Y}^{2 \circ \{2\}} := \mathbb{Y}^2 \circ \mathscr{Q}^{-1} \mathbb{Y}^2 - d_\varepsilon$$

with the same constant as above. It is very similar to $\mathbb{Y}^2 \circ \mathscr{L}^{-1} \mathbb{Y}^2$, in particular the divergent part is the same! (very important). So the analysis of B gives

$$B = \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2\circ\{2\}} \Phi^2 + C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi) + \boxed{\int_{\mathbb{T}_\varepsilon^3} d_\varepsilon \Phi^2}.$$

Putting all together we have

$$\boxed{\int_{\mathbb{T}_\varepsilon^3} \left[-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} (\mathbb{Y}^2 \succ \Phi_t)(m^2 - \Delta_\varepsilon)^{-1} (\mathbb{Y}^2 \succ \Phi_t) - \frac{3}{2} \lambda \Phi(\mathbb{Y}^2 \circ \mathbb{H}) - \frac{1}{2} \beta' \Phi(\mathbb{Y}^1 + Z) \right]}$$

$$= -\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2\circ\{2\}} \Phi^2 - \frac{9\lambda}{4} C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi)$$

$$-\frac{3\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi \left[-\frac{\lambda}{2} \mathbb{Y}^{2\circ[3]} - \frac{3\lambda}{2} \mathbb{Y}^{2\circ[2]} \mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) \right]$$

$$\boxed{-\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \Phi[d_\varepsilon(\mathbb{Y}^1 + \mathbb{H} + \Phi)]} - \frac{1}{2} \beta' \Phi(\mathbb{Y}^1 + Z),$$

and now the remarkable fact is that we can choose $\beta' = -9\lambda^2 d_\varepsilon / 2$ in order to cancel the divergences coming from d_ε. This means by choosing appropriately β we can remove all the divergences coming from ill-defined products of irregular Gaussian polynomials.

This is possible because this model is "superrenormalizable", or also called "subcritical", i.e. the linear part of the equation dominates the irregular terms in small scales, or said otherwise the non-linear irregular terms can be treated as a perturbation of the linear part.

We are at the point where in our a-priori estimate we do not have any more ugly term, all the products are well defined with the available regularity and the only step remaining is to check that we can close the a-priori estimates, i.e. estimate every term in the l.h.s. with the good terms in the r.h.s.

Let's summarize the discussion of this morning by writing down the final equation which will give rise to our a-priori estimates.

$$\boxed{\begin{gathered}
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{T}_\varepsilon^3} \Phi^2 + \int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi|^2 + m^2 \Psi^2] + \frac{\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi^4 = \mathscr{A} \\
\mathscr{A} := -\frac{9\lambda^2}{4} \int_{\mathbb{T}_\varepsilon^3} \mathbb{Y}^{2\circ\{2\}} \Phi^2 - \frac{9\lambda}{4} C'(\mathbb{Y}^2, \mathbb{Y}^2, \Phi, \Phi) \\
-\frac{3\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi \left[-\frac{\lambda}{2} \mathbb{Y}^{2\circ[3]} - \frac{3\lambda}{2} \mathbb{Y}^{2\circ[2]} \mathbb{H} - \frac{3\lambda}{2} C(\mathbb{Y}^2, \mathbb{Y}^2, \mathbb{H}) \right] \\
+\lambda D(\Phi, \mathbb{Y}^2, \Phi) - \frac{3}{2} \lambda \int_{\mathbb{T}_\varepsilon^3} \Phi(\mathbb{Y}^2 \prec Z + \mathbb{Y}^1 Z^2) - \frac{\lambda}{2} \int_{\mathbb{T}_\varepsilon^3} \Phi((\mathbb{H} + \Phi)^3 - \Phi^3) \\
\boxed{-\underbrace{\left[\frac{9\lambda^2}{4} + \frac{1}{2} \beta' \right]}_{=0} \int_{\mathbb{T}_\varepsilon^3} \Phi[d_\varepsilon(\mathbb{Y}^1 + \mathbb{H} + \Phi)]}
\end{gathered} \qquad (4)}$$

where we have a series of explicit probabilistic objects

$$
\begin{array}{ll|ll}
\mathbb{Y}^1 = Y & \in \mathscr{C}^{-1/2-\kappa} & \mathbb{Y}^{2\circ\{2\}} := \mathbb{Y}^2 \circ \mathscr{Q}^{-1}\mathbb{Y}^2 - d_\varepsilon & \in \mathscr{C}^{-\kappa} \\
\mathbb{Y}^2 = Y^2 - c_\varepsilon & \in \mathscr{C}^{-1-\kappa} & \mathbb{Y}^{2\circ[2]} := \mathbb{Y}^2 \circ \mathscr{L}^{-1}\mathbb{Y}^2 - d_\varepsilon & \in \mathscr{C}^{-\kappa} \\
\mathbb{Y}^{[3]} = \mathscr{L}^{-1}(Y^3 - 3c_\varepsilon Y) & \in \mathscr{C}^{1/2-\kappa} & \mathbb{Y}^{2\circ[3]} := \mathbb{Y}^2 \circ \mathscr{L}^{-1}\mathbb{Y}^3 - 3d_\varepsilon Y & \in \mathscr{C}^{-1/2-\kappa} \\
\mathbb{H} = -\frac{\lambda}{2}\mathbb{Y}^{[3]} - \frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}), & \in \mathscr{C}^{1/2-\kappa} & &
\end{array}
$$

(meaning that we can have uniform estimates in the corresponding spaces which do not blow up as $\varepsilon \to 0$). The unknowns $X \in H^{-1/2-\kappa}$, $Z \in H^{1/2-\kappa}$, $\Phi \in H^{1-\kappa}$, $\Psi \in H^1$, satisfying the decomposition

$$X = \mathbb{Y}^1 + \mathbb{H} - \frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi] + \Psi \tag{5}$$

where

$$Z := X - \mathbb{Y}^1,$$

$$\Phi := X - \mathbb{Y}^1 - \mathbb{H} = \underbrace{-\frac{3\lambda}{2}(m^2 - \Delta_\varepsilon)^{-1}[\mathbb{Y}^2 \succ \Phi]}_{\in \mathscr{C}^{1-\kappa} = B^{1-\kappa}_{\infty,\infty}} + \underbrace{\Psi}_{\in H^1 = B^1_{2,2}}.$$

With this decomposition one is able to prove that for small $\delta > 0$ there exist an explicit function $Q(\mathbb{Y})$ such that

$$|\mathscr{A}| \leqslant Q(\mathbb{Y}) + \delta\left[\int_{\mathbb{T}^3_\varepsilon} [|\nabla_\varepsilon \Psi|^2 + m^2\Psi^2] + \frac{\lambda}{2}\int_{\mathbb{T}^3_\varepsilon} \Phi^4\right] \tag{6}$$

(at this point this kind of argument proceed as in $d = 2$, i.e. via functional analytic estimates). The function $Q(\mathbb{Y})$ depends only on

$$\mathbb{Y} := (\mathbb{Y}^1, \mathbb{Y}^2, \mathbb{Y}^{[3]}, \mathbb{Y}^{2\circ\{2\}}, \mathbb{Y}^{2\circ[2]}, \mathbb{Y}^{2\circ[3]})$$

via norms of the kind

$$Q(\mathbb{Y}) = Q(\|\mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa})}, \|\mathbb{Y}^2\|_{C([0,T],\mathscr{C}^{-1-\kappa})}, \ldots),$$

in particular

$$\sup_{\varepsilon>0} \mathbb{E}[Q(\mathbb{Y}_\varepsilon)^K] < \infty$$

for any power $K \geqslant 1$.

Using (6) in (4) we get that for δ small enough

Theorem 2. *We have*

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{T}^3_\varepsilon} \Phi^2 + (1-\delta)\left[\int_{\mathbb{T}^3_\varepsilon} [|\nabla_\varepsilon \Psi|^2 + m^2\Psi^2] + \frac{\lambda}{2}\int_{\mathbb{T}^3_\varepsilon} \Phi^4\right] \leqslant Q(\mathbb{Y}_\varepsilon)$$

provided β is chosen depending on ε in a precise divergent way

$$\beta = C_1\varepsilon^{-1} + C_2\lambda\log(\varepsilon^{-1}),$$

with constants C_1, C_2 which we computed above.

As in $d=2$ this can be now used to obtain a-priori estimates for the measure by taking the average and use "stationarity".

Remark. I'm ignoring some technical problem which need to be addressed, in particular one cannot construct stationary solutions to the equation for $\mathbb{H}$, as a consequence both Φ and Ψ are not stationary and the argument to get the appropriate estimates in average has to be modified. But the changes are minor.

Anyway one obtain at the end that for any $t \in [0,T]$,

$$\mathbb{E}\left[\int_{\mathbb{T}_\varepsilon^3} [|\nabla_\varepsilon \Psi_t|^2 + m^2 \Psi_t^2] + \frac{\lambda}{2}\int_{\mathbb{T}_\varepsilon^3} \Phi_t^4\right] \leqslant \mathbb{E}\, Q(\mathbb{Y}_\varepsilon),$$

which given the relation of Ψ, Φ with X allows to obtain tightness, i.e. one can prove that

$$\sup_\varepsilon \mathbb{E}\left[\|X_0^\varepsilon\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^p\right] < \infty,$$

for any $p>1$. And even better, with some more care one can prove that

$$\sup_\varepsilon \mathbb{E}[\exp(\beta \|X_0^\varepsilon\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^{1-\kappa})] = \sup_\varepsilon \int \exp(\beta \|\varphi\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}^{1-\kappa})\, \nu^\varepsilon(\mathrm{d}\varphi) < \infty,$$

for small $\kappa>0$ and $\beta>0$. So the measure ν^ε allows uniform exponential integrability for some power less than 1 of the norm $\|\varphi\|_{H^{-1/2}(\mathbb{T}_\varepsilon^3)}$. This is more than enough to obtain tightness.

12 The Φ_3^4 measure without cutoffs

We must now combine the $\varepsilon \to 0$ proof with the $M \to \infty$ proof. This is not difficult but one needs to pay attention to some subtle detail.

Let us start by noting that the above a-priori estimates works also with weights, i.e. instead of testing the elaborated equation with Φ one tests with $\rho^2 \Phi$ for some polynomial weight ρ. This make appear weighted Besov norms of the type

$$\|\rho^\sigma \mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa}(\Lambda_{\varepsilon,M}))}, \|\rho^\sigma \mathbb{Y}^2\|_{C([0,T],\mathscr{C}^{-1-\kappa}(\Lambda_{\varepsilon,M}))}, \ldots$$

for some $\sigma>0$, and also norms like

$$\|\rho^{1/2} \Phi\|_{L^4(\Lambda_{\varepsilon,M})}, \quad \|\rho \nabla_\varepsilon \Phi\|_{L^2(\Lambda_{\varepsilon,M})}, \quad \|\rho\, \Phi\|_{L^2(\Lambda_{\varepsilon,M})}$$

for the solution. The first point is to make sure that norms like

$$\|\rho^\sigma \mathbb{Y}^1\|_{C([0,T],\mathscr{C}^{-1/2-\kappa}(\Lambda_{\varepsilon,M}))}$$

are uniformly bounded in M as $M \to \infty$. The idea is that all the processes $(\mathbb{Y}^\tau)_\tau$ growth at infinity at most polynomially with a small power, e.g. one can prove

$$|\Delta_i \mathbb{Y}_t^1(x)| \leqslant C(1+|x|)^\delta (1+|t|)^\delta, \qquad t \in \mathbb{R}, x \in \Lambda_{\varepsilon,M}$$

uniformly in M and ε for some finite random constant C. It is somehow clear that one cannot get better estimates, in particular this kind of stochastic processes cannot be bounded in the full space without weight.

Example. A discrete model. Let $(G_n)_{n\geqslant 1}$ a family of i.i.d $\mathscr{N}(0,1)$, then one can prove that there exists a random constant $C<\infty$ almost surely such that

$$|G_n(\omega)| \leqslant (C(\omega) + c \log^{1/2} n), \qquad n \geqslant 1$$

almost surely for some deterministic constant c. To prove this one shows that

$$Q(\omega) := \sum_{n\geqslant 1} \frac{1}{n^2} e^{\beta |G_n(\omega)|^2}$$

is integrable for small β. This implies that it is finite a.s. and then of course that

$$e^{\beta |G_n(\omega)|^2} \leqslant n^2 Q(\omega), \qquad \Rightarrow \qquad |G_n(\omega)| \leqslant \left(\frac{2}{\beta}\log n + \frac{1}{\beta}\log Q(\omega)\right)^{1/2}$$

for all $n \geqslant 1$.

However the biggest problem come from the equation of $\mathbb{H}$:

$$\mathbb{H} = -\frac{\lambda}{2}\mathbb{Y}^{[3]} - \frac{3\lambda}{2}\mathscr{L}^{-1}(\mathbb{Y}^2 \succ \mathbb{H}),$$

since it cannot be solved in weighted spaces: indeed there is a loss of weight in the estimate of the second term. One can then use a trick to solve this problem. See the paper.

At the end one obtains estimates of the form

$$\boxed{\sup_{\varepsilon,M} \mathbb{E}^{\varepsilon,M}[\exp(\beta \|\rho X_0^\varepsilon\|_{H^{-1/2}(\Lambda_\varepsilon)}^{1-\kappa})] = \sup_{\varepsilon,M} \int \exp(\beta \|\rho\varphi\|_{H^{-1/2}(\Lambda_\varepsilon)}^{1-\kappa})\, \nu^{\varepsilon,M}(\mathrm{d}\varphi) < \infty.} \tag{7}$$

It is well enough to have full tightness both in the $\varepsilon \to 0$ and in the $M \to \infty$ limit, irrespective of the size of λ (can be arbitrary large). This is a fully non-perturbative technique.

In conclusion one obtain accumulation points of the family $(\nu^{\varepsilon,M})$ and then it is easy to prove that any of these acc. points is translation invariant and RP, moreover the estimate (7) allow to prove the technical condition required by the OS reconstruction. Any limit point ν give rise to a translation invariant (no rotation so far), RP and "nice" measure and then to a QFT by OS reconstruction.

We do not have uniqueness, nor rotation invariance. If one can prove uniqueness it should be "easy" to prove rotation invariance.

13 Some properties of Φ_3^4

Let's now prove some properties of this measure. First of all that for any $\lambda > 0$ any accumulation point is non-Gaussian.

Remark. Ideally one would like to have non-triviality, i.e. that the corresponding QFT describes interacting particles. A Gaussian measure is trivial.

Let us call ν a arbitrary accumulation point. The first remark is that one can actually prove tightness for the triple of stationary processes $(X_{\varepsilon,M}, Y_{\varepsilon,M}, \mathbb{Y}^{[3]}_{\varepsilon,M})_{\varepsilon,M}$. Let us call

$$(X, Y, \mathbb{Y}^{[3]}),$$

a limit in law of the family. Of course $X_0 \sim \nu$ (i.e. is an accumulation point for $(\nu^{\varepsilon,M})_{\varepsilon,M}$). But we have also the dynamics and a coupling of X, Y, in particular this coupling satisfy the same estimates as we have before the limit, that is

$$\zeta := X - \mathbb{Y}^1 + \frac{\lambda}{2} \mathbb{Y}^{[3]} \in H^{1-\kappa}$$

in particular we have that

$$\left| \Delta_i X - \Delta_i \mathbb{Y}^1 + \frac{\lambda}{2} \Delta_i \mathbb{Y}^{[3]} \right| \approx (2^i)^{-1+\kappa}.$$

This estimate tells us that there exists a coupling between the interacting field X and the free field Y so that X is in "first approximation" given as above. In this very precise sense that if I look at the interacting field $\Delta_i X$ at high-momenta, then I see essentially the free field $\Delta_i Y$ and then a correction $\frac{\lambda}{2} \Delta_i \mathbb{Y}^{[3]}$ coming from the interaction (first-order in perturbation theory) and the something else whose size is well controlled. This is not perturbation theory because λ can be very large.

This is an expression of asymptotic freedom in the UV of the theory.

In order to show that X_0 is not gaussian one can show that the 4-th moment is not given by the usual formula for Gaussians. So we look at four-point function

$$U_4^i(X, X, X, X) := \mathbb{E}[\Delta_i X_0(x)\, \Delta_i X_0(x)\, \Delta_i X_0(x)\, \Delta_i X_0(x)] - 3(\mathbb{E}[\Delta_i X_0(x)\, \Delta_i X_0(x)])^2$$

and we want to prove that

$$U_4(X, X, X, X) \neq 0$$

because this implies that X is non-Gaussian. Note that

$$U_4(Y, Y, Y, Y) = 0$$

since Y is Gaussian. Moreover U_4 is a multilinear function, so we can use our decomposition of X to write

$$X = Y - \frac{\lambda}{2}\mathbb{Y}^{[3]} + \zeta,$$

and

$$U_4(X,X,X,X) = \underbrace{U_4(Y,Y,Y,Y)}_{=0} - 2\lambda U_4(Y,Y,Y,\mathbb{Y}^{[3]})$$

$$+ c\,U_4(Y,Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]}) + c\,U_4(Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]},\mathbb{Y}^{[3]}) + c\,U_4(Y,Y,Y,\zeta) + \cdots$$

An explicit computation shows that (both upper and lower bounds)

$$U_4(Y,Y,Y,\mathbb{Y}^{[3]}) \approx \mathbb{E}\underbrace{|\Delta_i Y|^3}_{(2^i)^{3/2}}\underbrace{|\Delta_i \mathbb{Y}^{[3]}|}_{(2^i)^{-1/2}} \approx (2^i)^{3/2-1/2} \approx 2^i,$$

and rough bounds show that

$$|U_4(Y,Y,\mathbb{Y}^{[3]},\mathbb{Y}^{[3]})| \approx [(2^i)^{1/2+\kappa}]^2[(2^i)^{-1/2+\kappa}]^2 \approx (2^i)^{4\kappa}$$

and all the other terms are as small and cannot compensate for $U_4(Y,Y,Y,\mathbb{Y}^{[3]})$ so finally one deduce that

$$U_4(X,X,X,X) = -2\lambda U_4(Y,Y,Y,\mathbb{Y}^{[3]}) + O((2^i)^{(1/2+5\kappa)}) \neq 0$$

for i large enough and $\lambda \neq 0$. This proves non-Gaussianity of X_0. And actually shows that the correlation functions in the UV are given by first order perturbation theory.

14 Conclusion

The lectures end here. Our main goal was to present several topics:

- The basic conceptual structure of QM and link with probability theory via the Euclidean approach
- The meaning of RP as the bridge between QM and EQM or QFT and EQFT.
- How stochastic quantization via a Langevin equation allows to study certain "difficult" measures as push-forward of "easy" Gaussian measures.
- How to control the infinite volume limit via PDE arguments involving weighted spaces.
- How to control the UV limit via paraproducts and decompositions involving paraproducts.
- How divergences can be extracted and matched with local counter-terms in the "bare" interaction.
- How the resulting measure can be analyzed via the decompositions obtained and the coupling with the free theory (despite the fact that Φ^4_3 is not absolutely continuous wrt. the free field).