LECTURES ON GEOMETRIC SINGULAR ANALYSIS, WITH APPLICATIONS TO ELLIPTIC AND HYPERBOLIC PDE

PETER HINTZ

Abstract. These lectures provide an introduction to *geometric singular analysis*, a systematic framework for the study of partial differential equations (PDE) on noncompact or singular spaces, and of families of PDE depending on a parameter which become singular as the parameter tends to 0. We illustrate this framework in a number of examples ranging from the Laplace operator on Euclidean space, possibly with a singular potential or with a family of potentials which become singular at a point, and the Dirichlet problem on polygonal domains to hyperbolic problems such as the wave equation in the interior of a Schwarzschild black hole or on Minkowski and de Sitter spacetimes. Various problems invite the reader to work out details and further examples.

Contents

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1. Introduction and motivation

The goal of these lectures[1] is to introduce a general-purpose perspective, called[2] *geometric singular analysis*, for the analysis of partial differential equations (PDE) in the following contexts.

(1) PDE on *noncompact spaces* such as $\mathbb{R}^n$. A typical example is the Laplacian[3] $\Delta = -\sum_{j=1}^n \partial_{x^j}^2$, where we write $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.
(2) PDE on *singular spaces*, or PDE with *singular* (but *structured*) *coefficients*. Examples include the Laplacian with a Coulomb potential $\Delta - \frac{1}{|x|}$, where we are interested in the behavior near $|x| = 0$, or the Laplacian on polygonal domains in $\mathbb{R}^2$.
(3) *Families of PDEs* depending on a parameter $\epsilon > 0$ which *degenerate or become singular* as $\epsilon \searrow 0$. An example is $\Delta_x + W(x) + \epsilon^{-2} V(x/\epsilon)$ on $\mathbb{R}^n_x$, where $W, V \in \mathcal{C}^\infty_c(\mathbb{R}^n)$.

As we shall see, item (1) is really a (very important) special case of item (2) via the process of *compactification* of the noncompact space. Moreover, in the study of singular family of PDEs, the key work often lies in analyzing individual PDE on suitable singular spaces. We discuss this in the example of item (3) in §7.1.

[1]The author produced these notes for his lectures at the *PDE mini-school* at the University of North Carolina, Chapel Hill, on March 24–26, 2023. Many thanks to the organizers, Yaiza Canzani and Jian Wang, for their kind invitation, and to their NSF RTG grant "Partial Differential Equations on Manifolds" (DMS-2135998) for supporting the mini-school.

[2]The term 'geometric microlocal analysis' is also often used, especially in the context of constructing (approximate) solutions for PDE on singular spaces. In these lectures, we shall not employ any microlocal tools however, as the PDE we shall study are not particularly complicated.

[3]The sign convention used in these notes makes Δ a non-negative operator, i.e. $\langle \Delta u, u\rangle_{L^2(\mathbb{R}^n)} \geq 0$ for all $u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$.

1.1. **The Laplacian on $\mathbb{R}^3$.** As our first example of a partial differential operator on a non-compact space, consider the operator

$$\Delta = -\sum_{j=1}^{3} \partial_{x^j}^2 \tag{1.1}$$

on $\mathbb{R}^3$. A good functional analytic setup for the global analysis P is not entirely trivial to find. Ideally, we would like to find function spaces $\mathcal{X}, \mathcal{Y}$ so that $P \in L(\mathcal{X}, \mathcal{Y})$ is a bounded linear map with

- finite-dimensional nullspace ('almost' uniqueness),
- closed range (so that solutions of $Pu = f$, if they exist, can be found with norm control $\|u\|_{\mathcal{X}} \leq C\|f\|_{\mathcal{Y}}$),
- finite-dimensional cokernel ('almost' solvability, i.e. $Pu = f \in \mathcal{Y}$ has a solution if and only if f satisfies finitely many linearly independent constraints).

That is, we would like P to be a *Fredholm operator* from $\mathcal{X}$ to $\mathcal{Y}$. Hölder spaces $\mathcal{C}^{k,\alpha}(\mathbb{R}^3)$ would be acceptable if one only cares about elliptic PDE; but hyperbolic equations (like the wave equation) demand L^2-based function spaces. In these notes, we will only work with L^2 spaces. A natural first attempt for the operator (1.1) is then to take $\mathcal{X} = H^2(\mathbb{R}^3)$, $\mathcal{Y} = L^2(\mathbb{R}^3)$. (Further credibility to this choice comes from the fact that P is self-adjoint on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$.) *However*, $\Delta\colon H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ does not have closed range. This remains true on all weighted Sobolev spaces too. See Problem 1.3. Thus, standard weighted Sobolev spaces are not appropriate for our aims.

By contrast, since the Laplacian Δ_g on a compact Riemannian manifold (M, g) without boundary is Fredholm $H^2(M) \to L^2(M)$ (with nullspace spanned by constants, and range equal to the orthocomplement of the space of constant functions). Thus, the failure of the closed range property of $\Delta\colon H^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ must be due to the non-compactness of $\mathbb{R}^3$.

A weak hint as to what is going on is given by the observation that if $f \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, then the solution

$$u(x) = \Big(\frac{1}{4\pi|\cdot|} * f\Big)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,\mathrm{d}y$$

of $\Delta u = f$ has a full asymptotic expansion as $|x| \to \infty$,

$$u(x) = |x|^{-1} f_1\Big(\frac{x}{|x|}\Big) + |x|^{-2} f_2\Big(\frac{x}{|x|}\Big) + \ldots,\ \ |x| \to \infty.$$

(See Problem 1.4.) If one (formally) differentiates this expansion along ∂_{x^j}, one gains a factor of $|x|^{-1}$. Thus, a more appropriate notion of regularity in $|x| > 1$ should be regularity under application of the vector fields $|x|\partial_{x^j}$, $j = 1, \ldots, 3$.

In order to focus on a 'neighborhood of infinity', we introduce inverse polar coordinates

$$\rho := r^{-1} = |x|^{-1}, \quad \omega = \frac{x}{|x|} \in \mathbb{S}^2.$$

The Laplace operator is

$$\begin{aligned}\Delta &= -\partial_r^2 - \frac{2}{r}\partial_r + r^{-2}\Delta_{\mathbb{S}^2} \\ &= -(-\rho^2\partial_\rho)^2 - 2\rho(-\rho^2\partial_\rho) + \rho^2\Delta_{\mathbb{S}^2} \\ &= \rho^2\big(-(\rho\partial_\rho)^2 + \rho\partial_\rho + \Delta_{\mathbb{S}^2}\big),\end{aligned} \tag{1.2}$$

where $\Delta_{\mathbb{S}^2} = -\frac{1}{\sin\theta}\partial_\theta \sin\theta\,\partial_\theta - (\sin\theta)^{-2}\partial_\phi^2$ in standard coordinates $\theta \in (0,\pi)$, $\phi \in (0,2\pi)$ on the sphere $\mathbb{S}^2$. Thus, $\rho^{-2}\Delta$ is constructed from $\rho\partial_\rho$ and spherical derivatives, and it is an elliptic operator of this type (i.e. its leading order part is a positive definite quadratic form in $\rho\partial_\rho$, ∂_θ, ∂_ϕ). We say that Δ is a (weighted) *b-differential operator*. A natural notion of regularity for solutions of $\Delta u = f$ is thus regularity under precisely these derivatives (called *b-regularity*). One can check that this is the same as regularity under the vector fields $|x|\partial_{x^j}$.

We are thus led to define *weighted b-Sobolev spaces* $H_{\mathrm{b}}^{s,\alpha}$ for $s \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ by

$$H_{\mathrm{b}}^{s,\alpha} = \bigl\{u \colon (\langle x\rangle\partial_x)^\beta(\langle x\rangle^\alpha u) \in L^2\ \forall\,\beta \in \mathbb{N}_0^n,\ |\beta| \le s\bigr\}. \tag{1.3}$$

(Usage of $\langle x\rangle\partial_x$ ensures that we are recovering the standard Sobolev spaces over compact subsets of $\mathbb{R}^3$, in the sense that if $u \in H_{\mathrm{b}}^{s,\alpha}$, then $\chi u \in H^s(\mathbb{R}^3)$ for all $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^3)$.) This is a Hilbert space with squared norm

$$\|u\|_{H_{\mathrm{b}}^{s,\alpha}}^2 = \sum_{|\beta|\le s} \bigl\|(\langle x\rangle\partial_x)^\beta(\langle x\rangle^\alpha u)\bigr\|_{L^2}^2.$$

We have $\langle x\rangle^2\Delta \colon H_{\mathrm{b}}^{s,\alpha} \to H_{\mathrm{b}}^{s-2,\alpha}$ for $s \ge 2$, $\alpha \in \mathbb{R}$. (See Problem 1.5.) We might then hope that

$$\Delta \colon H_{\mathrm{b}}^{s,\alpha} \to H_{\mathrm{b}}^{s-2,\alpha+2}$$

is Fredholm. This is *almost* true: the weight α needs to be chosen carefully (namely, it must avoid the discrete set $\frac12 + \mathbb{Z}$). One can moreover show that for $\alpha \in (-\frac32, -\frac12)$, this operator is invertible.

One can see why a condition on α is necessary by considering the action of $\rho^{-2}\Delta$ on separated functions of the form $\rho^\lambda v(\omega)$, which is

$$\rho^{-2}\Delta(\rho^\lambda v) = \rho^\lambda N(\rho^{-2}\Delta,\lambda)v, \qquad N(\rho^{-2}\Delta,\lambda) := -\lambda^2 + \lambda + \Delta_{\mathbb{S}^2} \in \mathrm{Diff}^2(\mathbb{S}^2). \tag{1.4}$$

Restricting v further to be a spherical harmonic $Y_{lm}(\omega)$ of degree $l \in \mathbb{N}_0$, $N(\rho^{-2}\Delta,\lambda)$ becomes multiplication by $-\lambda^2 + \lambda + l(l+1) = -(\lambda+l)(\lambda-l-1)$. This vanishes for $\lambda = -l, l+1$, corresponding to the fact that $u(\rho,\omega) := \rho^\lambda Y_{lm}(\omega) \in \ker\rho^{-2}\Delta$. This function should be regarded as relevant only near $\rho = 0$; let thus $\chi \in \mathcal{C}_{\mathrm{c}}^\infty([0,1)_\rho)$ be identically 1 near 0. For $\alpha = -\frac32 + \lambda$, the function $\chi(\rho)u(\rho,\omega)$ just barely fails to lie in $H_{\mathrm{b}}^{2,\alpha}$, and then $u_\epsilon = \chi(\rho)\rho^\epsilon u(\rho,\omega)$ blows up in $H_{\mathrm{b}}^{2,\alpha}$ as $\epsilon \searrow 0$, while Δu_ϵ remains bounded in $H_{\mathrm{b}}^{0,\alpha+2}$. (See Problem 1.5.)

The computation (1.4) also tells us what to expect about the asymptotic behavior of solutions of $\Delta u = 0$ (or $\Delta u = f$ where $f \in \mathscr{S}(\mathbb{R}^3)$, say): they have asymptotic expansions as $|x| \to \infty$, or equivalently $\rho \searrow 0$, into terms of the form $\rho^\lambda Y_{lm}(\omega)$ where $\lambda = -l, l+1$. Note, finally, that the operator family $N(\rho^{-2}\Delta,\lambda)$, insofar as it controls the asymptotic behavior of solutions of $\Delta u = f$ at infinity, should be regarded as living 'at (the sphere at) infinity' in $\mathbb{R}^3$, i.e. at '$\rho = 0$'. Thus, we wish to add to $\mathbb{R}^3$ this sphere at infinity; this is accomplished via the *radial compactification* $\overline{\mathbb{R}^3}$, defined more generally for $\mathbb{R}^n$ as

$$\begin{aligned} \overline{\mathbb{R}^n} &:= \Bigl(\mathbb{R}^n \sqcup \bigl([0,\infty)_\rho \times \mathbb{S}^{n-1}\bigr)\Bigr)/\sim, \\ &\qquad \mathbb{R}^n \setminus \{0\} \ni x = r\omega \sim (r^{-1},\omega). \end{aligned} \tag{1.5}$$

This is a compact manifold with boundary $\rho^{-1}(0)$; it is diffeomorphic to the closed unit ball. See Figure 1.1.

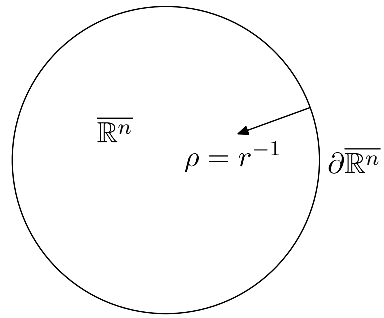

FIGURE 1.1. The *radial compactification* $\overline{\mathbb{R}^3}$ of $\mathbb{R}^3$, obtained by attaching a sphere $\mathbb{S}^2 = \rho^{-1}(0)$ 'at infinity' to $\mathbb{R}^3$, where $\rho = r^{-1}$.

Theme I.1. *Infinity is closer than you might think.* Working 'locally near infinity' ($\rho = 0$) in a systematic fashion allows one to see the structure of the PDE clearly, understand its mapping properties, and read off the expected asymptotic behavior of solutions.

Theme I.2. Infinity is a *singular place*: PDEs typically degenerate in some (controlled) fashion there. (Cf. the appearance of $\rho\partial_\rho$, not ∂_ρ, in (1.2).)

As hyperbolic examples, we shall discuss the wave equation near a Schwarzschild singularity in §3.2 and the wave equation on de Sitter space in §3.3.

1.2. **Singular PDE.** We already encountered a singular PDE that did not look like one at first sight, namely the Laplace operator on $\mathbb{R}^3$ which is singular at infinity $\rho = 0$, cf. (1.2). Other types of singularities are less hidden. One example is the following operator near the origin of $\mathbb{R}^3$ (with $\mathsf{Z} \in \mathbb{R}$):

$$P := \Delta - \frac{\mathsf{Z}}{|x|} = -\partial_r^2 - \frac{2}{r}\partial_r + r^{-2}\Delta_{\mathbb{S}^2} - \frac{1}{r} = r^{-2}\bigl(-(r\partial_r)^2 - r\partial_r + \Delta_{\mathbb{S}^2} - \mathsf{Z}r\bigr). \tag{1.6}$$

If we regard this as an operator on $(0,\infty)_r \times \mathbb{S}^2$, this looks quite similar to (1.2), except for the presence of the term r in parenthesis. If one seeks approximate elements in the nullspace of this operator of the form $r^\lambda v(\omega)$, one is led to the operator

$$N(r^2 P, \lambda) = -\lambda^2 - \lambda + \Delta_{\mathbb{S}^2} \in \mathrm{Diff}^2(\mathbb{S}^2)$$

which is the same as (1.4) except for a sign change in λ. This operator, on the 2-sphere, wants to live at $r = 0$. To make this happen, one regards polar coordinates as valid *down to* $r = 0$, and thus considers P as a (weighted) b-differential operator on

$$[\mathbb{R}^3; \{0\}] := [0,\infty)_r \times \mathbb{S}^2.$$

This manifold is called the *blow-up* of $\mathbb{R}^3$ at $\{0\}$. The smooth map $[\mathbb{R}^3;\{0\}] \ni (r,\omega) \mapsto r\omega \in \mathbb{R}^3$ is a diffeomorphism on $\{r > 0\}$, but the preimage of $\{0\}$ is a 2-sphere. This perspective has the following benefits.

(1) The operator $r^2 P$, regarded as a second order b-differential operator (constructed from $r\partial_r$ and angular derivatives) on $[\mathbb{R}^3;\{0\}]$, has *smooth coefficients*. Thus, we have exchanged analytic complexity (singular potential) on a simple space ($\mathbb{R}^3$) for analytic simplicity (smooth coefficients) on a more complicated space ($[\mathbb{R}^3;\{0\}]$).

(2) On the more complicated space $[\mathbb{R}^3; \{0\}]$, the analysis of P naturally takes place in the already familiar b-Sobolev spaces.
(3) Solutions of $Pu = 0$ have asymptotic expansions as $r \searrow 0$ into terms of the form $r^\lambda Y_{lm}(\omega)$ where $\lambda = -l-1, l$ with $l \in \mathbb{N}_0$ (and possibly involving $\log r$ terms, too). This is as simple a behavior as one can hope for at a boundary hypersurface when studying a degenerate PDE.

Other examples of singular PDEs, such as the Laplacian on manifolds with conic singularities, or the Laplacian on polygonal domains in $\mathbb{R}^2$, will be discussed in these notes as well.

Theme II.[4] One should trade analytic complexity (singularities) for geometric complexity (e.g. via blowing up submanifolds). This cleanly exhibits the analytic structure of the singularity, and the behavior of solutions near the singularity will be simple(r) on the more complicated manifold.

As a hyperbolic example, we will discuss the wave equation on Minkowski space in §5.3.

Some readers might want to argue that compactifications and blow-ups do nothing but introduce more appropriate (singular) local coordinates near infinity or some other type of singularity; so one might as well forget about the blow-up and work with the singular local coordinates. But then again: nobody really questions the utility of the notion of a smooth manifold, which one could dispense of if one formulated everything in local coordinates and described how estimates and notions of regularity etc. in different local coordinates patch together. The 'compactification/blow-up/manifolds with boundary or corners' perspective provides the analogue of the smooth manifold notion for singular analysis. Even a skeptic may thus consider these simply as highly efficient organizing tools. (In more complicated settings, such as spaces whose singularities have an iterated structure, such as stratified spaces, one would get rather completely lost without these tools.)

1.3. **Singular limits.** Let $W, V \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^3)$, and consider for $\epsilon > 0$ the operator

$$P_\epsilon = \Delta + W + \epsilon^{-2} V(\cdot/\epsilon)$$

on $\mathbb{R}^3$. As $\epsilon \searrow 0$ and for $x \neq 0$, this (naively) converges to $P_0 := \Delta + W$. On the other hand, if we set $\hat{x} = \frac{x}{\epsilon}$, then for fixed $\hat{x}$ and as $\epsilon \searrow 0$,

$$\begin{aligned} \epsilon^2 P_\epsilon &= -\sum_{j=1}^3 \epsilon^2 \partial_{x^j}^2 + \epsilon^2 W(x) + V(x/\epsilon) = -\sum_{j=1}^3 \partial_{\hat{x}^j}^2 + V(\hat{x}) + \epsilon^2 W \\ &\to \hat{P} := \hat{\Delta} + V(\hat{x}), \end{aligned}$$

where $\hat{\Delta}$ is the Laplacian in the $\hat{x}$-coordinates. Thus, $\hat{P}$ lives on $\mathbb{R}^3_{\hat{x}}$, but also somehow at $x = \epsilon = 0$. Moreover, the points $x(\epsilon) = \epsilon^{1/2}\omega$ (where $\omega \in \mathbb{S}^2$ is fixed) converge to 0 in the x-coordinates as $\epsilon \searrow 0$, but $\hat{x}(\epsilon) = \frac{x(\epsilon)}{\epsilon} = \epsilon^{-1/2}\omega$ converges to infinity in $\mathbb{R}^3_{\hat{x}}$ in the direction ω. So somehow infinity of $\mathbb{R}^3_{\hat{x}}$ is attached to $r = 0$ in $\mathbb{R}^3_x$. The way to make these

[4] The distinction between Themes I.2 and II is typically not very clear-cut. Sometimes the first guess for the compactification of the noncompact manifold on which some PDE might be easily analyzable turns out to not be good enough yet (e.g. solutions are still highly singular on it), and the correct manifold is really a blow-up of it. But then one could ask why one picked the wrong compactification in the first place.

ideas rigorous is to consider P_ϵ as a *single* differential operator on $[0,1)_\epsilon \times \mathbb{R}^3_x$ and blow up $\epsilon = x = 0$. See Figure 1.2 (albeit in two dimension less, for artistic reasons). In this manner, the limit of P_ϵ as $\epsilon \searrow 0$ is described by one operator ($\hat{P}$) on a compactification of $\mathbb{R}^3_{\hat{x}}$ and another operator (P_0) on $[\mathbb{R}^3_x; \{0\}]$.

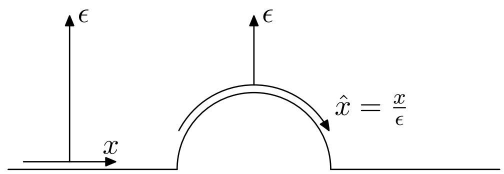

FIGURE 1.2. Blow-up of $[0,1)_\epsilon \times \mathbb{R}_x$ at $\epsilon = x = 0$, and some local coordinates.

For example then, if both $\hat{P}$ and P_0 are invertible, on function spaces which are related in an appropriate manner at the corner at the intersection of the two limiting regimes, then one can hope to prove the invertibility of P_ϵ for small $\epsilon > 0$. We will carry this out in detail in §7.1.

Theme III. Analyze families of PDE which depend on a parameter in a singular manner by studying all operators at once on a *total space* which incorporates both the underlying manifold and the parameter, and which is resolved (blown up) so as to exhibit all different asymptotic regimes.

As an example from general relativity, we sketch in §7.2, following [HX22], how to study the quasinormal mode spectrum of Schwarzschild–de Sitter black holes in the limit that the black hole mass is small.

1.4. **Plan of these lectures.** In §2, we describe the notions of b-geometry and b-analysis on general manifolds with boundary, including b-vector fields and b-differential operators and their normal operators and indicial families, b-Sobolev spaces, and asymptotic expansions ('polyhomogeneity'). We discuss applications of this theory in §3.

The general procedure of blow-ups of suitable submanifolds of manifolds with corners is discussed in §4. We apply it to a variety of settings (mentioned in §1.2 above) in §5.

In §6 finally, we describe, following the motivation in §1.3, how to study (a certain class of) singular limits of families of PDE using techniques from geometric singular analysis, with examples gives in §7.

1.5. **Further reading.** Grieser's notes on the b-calculus [Gri01] provide a detailed introduction to and motivation for many notions discussed here, including manifolds with corners, blow-ups, conormality, and polyhomogeneity. The focus of later parts of those notes is on distributions and pseudodifferential operators on manifolds with boundary. (By contrast, in these notes, we do not use any microlocal techniques.) For a textbook account of (microlocal) b-analysis, with applications to index theory, see Melrose's book [Mel93]; see also [Hör07, §18.3]. A systematic, extensive, and general treatment of analysis on manifolds with corners is Melrose's book (in progress) [Mel96].

On average, applications of geometric singular analysis techniques in the literature are skewed towards elliptic theory (spectral theory, Fredholm analysis, index theory) and related topics (such as heat kernel asymptotics) on noncompact and/or singular spaces, often via the construction of parametrices (approximate inverses) and their integral ('Schwartz') kernels. Important examples concern the spectral theory on asymptotically hyperbolic manifolds [MM87] and low energy resolvent asymptotics on asymptotically conic spaces [GH08]. Spectral theory on asymptotically Euclidean spaces has a non-elliptic character when viewed through the lens of singular ('scattering') analysis on the radial compactification [Mel94]. Vasy's lecture notes [Vas18] provide a detailed account of microlocal analysis in this setting.

Early non-elliptic applications include studies of local interactions of singularities for nonlinear wave equations [MR85]. Applications to the global theory of (non)linear wave have only emerged more recently [Vas10, MSBV14, Vas13, HV15], with applications to nonlinear stability problems in general relativity [HV18, Hin18]. In these lectures, we briefly describe two such examples using only 'hands-on'[5] techniques on de Sitter space in §3.3 (following [Vas10, HX22]) and Minkowski space in §5.3 (following [HV20, Hin23]). We also re-interpret some results in the literature in geometric singular analysis terms, e.g. results by Fournodavlos–Sbierski [FS20] on the behavior of waves near the Schwarzschild singularity in §3.2.

Geometric singular analysis has featured implicitly in gluing constructions and singular perturbation theory; for explicit examples (and references to earlier gluing results), we refer the reader to [KS22, SS21, HX22, Hin21, Hin22]. We briefly sketch one such example in §7.2.

1.6. Problems.

Problem 1.1 (Utility of closed range)**.** Let $A\colon X \to Y$ be an operator between two Banach spaces $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$. Show that A has closed range if and only if there exists a constant $C < \infty$ so that for all $y \in \operatorname{ran} A$ there exists $x \in X$ with $Ax = y$ and $\|x\|_X \leq C\|y\|_Y$.

Problem 1.2 (Derivative on $\mathbb{R}$)**.** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\colon H^1(\mathbb{R}) \to L^2(\mathbb{R})$ has trivial nullspace and dense but not closed range. Make the final statement explicit by finding a sequence $f_j \in L^2(\mathbb{R})$ in the range so that $f_j \to f \in L^2(\mathbb{R})$ does not lie in the range.

Problem 1.3 (Laplacian on $\mathbb{R}^n$)**.** Let $n \geq 1$. Show that $\Delta\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ has trivial nullspace and dense but not closed range. Show that for all $s, \alpha \in \mathbb{R}$,

$$\Delta\colon H^{s,\alpha}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n)\colon \langle x\rangle^\alpha u \in H^s(\mathbb{R}^n)\} \to H^{s-2,\alpha}(\mathbb{R})$$

has finite-dimensional kernel, but its range is not closed (but the closure of the range may be positive-dimensional). Here $\langle x\rangle = (1 + |x|^2)^{1/2}$. *Hint.* First prove this for $\alpha = 0$. Deduce the general statement from this by showing that $\langle x\rangle^{-\alpha}\Delta\langle x\rangle^\alpha = \Delta + R_\alpha$ where $R_\alpha\colon H^s \to H^{s-2}$ is a compact operator (use the Rellich compactness theorem).

[5] occasionally termed 'physical space', and in any case not microlocal

Problem 1.4 (Asymptotic expansion)**.** Let $f \in \mathcal{C}^\infty_c(\mathbb{R}^3)$ and $u = \frac{1}{4\pi|\cdot|} * f$. Show that there exist $f_j \in \mathcal{C}^\infty(\mathbb{S}^2)$, $j = 1, 2, \ldots$, so that

$$u \sim \sum_{j=1}^\infty |x|^{-j} f_j\Big(\frac{x}{|x|}\Big), \qquad |x| \to \infty,$$

in the sense that $r_N(x) := u - \sum_{j=1}^{N-1} |x|^{-j} f_j(\frac{x}{|x|})$ satisfies $|\partial_x^\alpha r_N(x)| \le C_{N,\alpha}|x|^{-N-|\alpha|}$ for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$. Compute f_1.

Problem 1.5 (Laplacian on weighted b-Sobolev spaces)**.** Recall the definition of the function space $H_b^{s,\alpha}$ on $\mathbb{R}^3$ from (1.3). Use polar coordinates $x = r\omega$ on $\mathbb{R}^3$.

(1) Let $s \ge 2$, $\alpha \in \mathbb{R}$. Show that $\Delta\colon H_b^{s,\alpha} \to H_b^{s-2,\alpha+2}$ is a bounded linear operator.
(2) Let $s \in \mathbb{N}_0$. Show that $\langle x\rangle^{-\lambda} \in H_b^{s,\alpha}$ if and only if $\alpha < -\frac{3}{2} + \lambda$. (*Hint.* Do this first for $s = 0$.)
(3) Let $\lambda \in \{l, -l-1\}$. Fix $\chi \in \mathcal{C}^\infty_c([0,1))$ to be identically 1 near 0. Write $\rho = |x|^{-1}$. For $\epsilon > 0$, let $u_\epsilon(x) = \chi(\rho)\rho^{\lambda+\epsilon}Y_{lm}(\omega)$, and let $\alpha = -\frac{3}{2} + \lambda$. Show that $\|u_\epsilon\|_{H_b^{2,\alpha}} \nearrow \infty$ as $\epsilon \searrow 0$, while $\limsup_{\epsilon\searrow 0} \|\Delta u_\epsilon\|_{H_b^{0,\alpha+2}} < \infty$.
(4) Deduce that for $\alpha \in \frac{1}{2} + \mathbb{Z}$, the operator $\Delta\colon H_b^{2,\alpha} \to H_b^{0,\alpha+2}$ does not have closed range.
(5) Compute, for every $\alpha \in \mathbb{R}$, the nullspace of Δ on $H_b^{2,\alpha}$.

2. Manifolds with boundary

Throughout this section, M denotes an n-dimensional manifold with boundary. Thus, points in M come in two flavors: those in the *manifold interior* M° which have an open neighborhood diffeomorphic to $\mathbb{R}^n$, and those in the *boundary* ∂M which have an open neighborhood diffeomorphic to $\mathbb{R}^n_1 := [0,\infty) \times \mathbb{R}^{n-1}$. See Figure 2.1. Examples to keep mind are the radial compactification $M = \overline{\mathbb{R}^n}$ of $\mathbb{R}^n$ defined in (1.5), in which case $M^\circ = \mathbb{R}^n$ and ∂M is the 'sphere at infinity'; or $M = [\mathbb{R}^3; \{0\}] = [0,\infty)_r \times \mathbb{S}^2$, in which case $M^\circ = \mathbb{R}^3 \setminus \{0\}$ and $\partial M = \{0\} \times \mathbb{S}^2$. Standard function spaces on M are

$$\mathcal{C}^\infty(M), \qquad \dot{\mathcal{C}}^\infty(M) := \{u \in \mathcal{C}^\infty(M)\colon u \text{ vanishes to infinite order at } \partial M\}.$$

Let us write $\mathcal{V}(M) = \mathcal{C}^\infty(M; TM)$ for the space of smooth vector fields on M. In local coordinates $x \ge 0$, $y \in \mathbb{R}^{n-1}$ near a boundary point, these are linear combinations of ∂_x, ∂_{y^j} ($j = 1, \ldots, n-1$) with smooth coefficients. For $m \in \mathbb{N}_0$, we write $\mathrm{Diff}^m(M)$ for the space of m-th order differential operators: these are locally finite sums of up to m-fold compositions of elements of $\mathcal{V}(M)$. (For $m = 0$, $\mathrm{Diff}^0(M)$ is the space of multiplication operators by elements of $\mathcal{C}^\infty(M)$.)

Already at this point, we note how working on *compact* manifolds with boundary captures uniform behavior very cleanly: in the case of $\overline{\mathbb{R}^n}$, smooth functions $u \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$ are not only automatically bounded as $|x| \to \infty$, but in fact they are smooth functions of $\frac{1}{|x|}$ and $\frac{x}{|x|}$ down to $\frac{1}{|x|} = 0$. That is, they have an asymptotic expansion

$$u \sim \sum_{j=0}^\infty |x|^{-j} u_j\Big(\frac{x}{|x|}\Big), \quad u_j \in \mathcal{C}^\infty(\mathbb{S}^{n-1}) \qquad (|x| \to \infty). \tag{2.1}$$

See Problem 2.1.

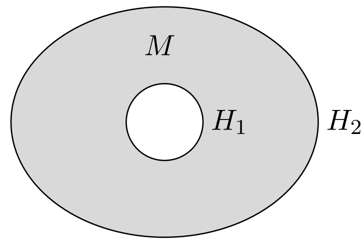

FIGURE 2.1. A compact manifold M whose boundary ∂M has two connected components H_1, H_2.

2.1. **b-vector fields and b-differential operators.** As we saw in (1.2) and (1.6), PDEs on M° viewed from the perspective of M degenerate at ∂M, and their solutions are typically not smooth, but rather have behavior like $x^z(\log x)^k v(y)$ in the best of cases. What are the vector fields with respect to which such functions are regular? The derivatives transversal to ∂M featuring in (1.2) and (1.6) are weaker ($x\partial_x$ instead of ∂_x). We phrase this invariantly:

Definition 2.1 (b-vector fields and b-differential operators)**.** The space $\mathcal{V}_{\mathrm{b}}(M) \subset \mathcal{V}(M)$ of *b-vector fields* consists of all smooth vector fields which are tangent to ∂M. For $m \in \mathbb{N}_0$, the space $\mathrm{Diff}_{\mathrm{b}}^m(M)$ of *b-differential operators* consists of all locally finite sums of up to m-fold compositions of elements of $\mathcal{V}_{\mathrm{b}}(M)$.

These are sometimes called *totally characteristic* vector fields/operators. A b-differential operator defines continuous linear maps $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ and $\dot{\mathcal{C}}^\infty(M) \to \dot{\mathcal{C}}^\infty(M)$; thus, two operators $P_i \in \mathrm{Diff}_{\mathrm{b}}^{m_i}(M)$, $i = 1, 2$, can be composed, and we have $P_1 \circ P_2 \in \mathrm{Diff}_{\mathrm{b}}^{m_1+m_2}(M)$.

Concretely, $\mathcal{V}_{\mathrm{b}}(M)$ consists of all $V \in \mathcal{V}(M)$ which in local coordinates $x \geq 0$, $y \in \mathbb{R}^{n-1}$ near a boundary point are of the form

$$a(x,y)x\partial_x + \sum_{j=1}^{n-1} b_j(x,y)\partial_{y^j}, \qquad a, b_j \in \mathcal{C}^\infty([0,\infty) \times \mathbb{R}^{n-1}),$$

and b-differential operators are of the form

$$P = \sum_{j+|\alpha| \leq m} a_{j\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha, \qquad a_{j\alpha} \in \mathcal{C}^\infty. \tag{2.2}$$

Near an interior point, there is no difference between differential and b-differential operators. We then have the notion of *ellipticity*, which in M° is the standard one[6] and which near ∂M and in terms of (2.2) requires $\sum_{j+|\alpha|\leq m} a_{j\alpha}(x,y)\xi^j\eta^\alpha \neq 0$ for all $(\xi,\eta) \in (\mathbb{R} \times \mathbb{R}^{n-1}) \setminus \{(0,0)\}$.

In order to study b-differential operators globally near the boundary ∂M, and also to measure growth and decay at the boundary, we introduce:

Definition 2.2 (Boundary defining functions)**.** A *boundary defining function* on M is a nonnegative function $\rho \in \mathcal{C}^\infty(M)$ so that $\partial M = \rho^{-1}(0)$ and $\mathrm{d}\rho(p) \neq 0$ for all $p \in \partial M$.

Lemma 2.3 (Conjugation by weights)**.** *Let $P \in \mathrm{Diff}_{\mathrm{b}}^m(M)$ and $\alpha \in \mathbb{R}$. Then $\rho^\alpha P \rho^{-\alpha} \colon u \mapsto \rho^\alpha P(\rho^{-\alpha}u)$ defines an element of $\mathrm{Diff}_{\mathrm{b}}^m(M)$.*

[6] Namely, if $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial_x^\alpha$, then the polynomial $\sum_{|\alpha|\leq m} \xi^\alpha$ does not vanish for $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof. This is ultimately due to the fact that $x^\alpha x\partial_x x^{-\alpha} = x\partial_x - \alpha$. See Problem 2.7. $\square$

This result allows us to define spaces of weighted b-differential operators,

$$\rho^{-\alpha}\mathrm{Diff}_\mathrm{b}^m(M) = \{\rho^{-\alpha}P \colon P \in \mathrm{Diff}_\mathrm{b}^m(M)\}.$$

Operators of this class still map $\dot{\mathcal{C}}^\infty(M) \to \dot{\mathcal{C}}^\infty(M)$ continuously, but of course they are not continuous anymore as operators on $\mathcal{C}^\infty(M)$.

Example 2.4 (Regular singular ODEs). On $M = [0,\infty)$, b-differential operators are of the form $P = \sum_{j=0}^m a_j(x)(x\frac{\mathrm{d}}{\mathrm{d}x})^j$ where $a_j \in \mathcal{C}^\infty([0,\infty))$. If $a_m(x) \neq 0$, the equation $Pu = f$ (or $Pu = 0$ with specified initial data) is a regular singular ODE of order m, and the standard method for analyzing and solving it involves the characteristic polynomial $N(P,\lambda) = \sum_{j=0}^m a_j(0)\lambda^j$. For example, a root λ of $N(P,\lambda)$ gives rise to x^λ asymptotics of u.

Example 2.5 (Laplace operator). The function[7] $\langle x\rangle^{-1}$ on $\mathbb{R}^n$ is a boundary defining function on $\overline{\mathbb{R}^n}$, and $\Delta \in \langle x\rangle^{-2}\mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^n})$. Indeed, for bounded x this says nothing more than that Δ is a second order differential operator with smooth coefficients, whereas near $\partial\overline{\mathbb{R}^n}$ it follows from (1.2). The analogue of the characteristic polynomial of $\langle x\rangle^2\Delta$ at $\rho = r^{-1} = 0$ is $N(\langle x\rangle^2\Delta, \lambda) = -\lambda^2 + \lambda + \Delta_{\mathbb{S}^2}$ from (1.4). Moreover, $\Delta \in \langle x\rangle^{-2}\mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^n})$ is elliptic (in the sense that $\langle x\rangle^2\Delta \in \mathrm{Diff}_\mathrm{b}^2$ is elliptic).

We proceed to define the 'characteristic polynomial' in general. First, recall that ∂M has a *collar neighborhood*, i.e. an open neighborhood of ∂M is diffeomorphic to

$$[0,\infty)_\rho \times \partial M.$$

In such a collar neighborhood, one can write $P \in \mathrm{Diff}_\mathrm{b}^m(M)$ as

$$P = \sum_{j=0}^m P_j(\rho)(\rho\partial_\rho)^j, \qquad P_j \in \mathcal{C}^\infty\big([0,\infty)_\rho; \mathrm{Diff}^{m-j}(\partial M)\big).$$

Freezing coefficients at $\rho = 0$ gives the *normal operator*[8]

$$N(P) := \sum_{j=0}^m P_j(0)(\rho\partial_\rho)^j \in \mathrm{Diff}_\mathrm{b}^m([0,\infty) \times \partial M). \tag{2.3}$$

Unlike P, which does not need to have any symmetries, the operator $N(P)$ is invariant under dilations in ρ. Just as the principal symbol of P captures P modulo operators of lower differential order, i.e. modulo $\mathrm{Diff}_\mathrm{b}^{m-1}$ (i.e. the principal symbol of $P \in \mathrm{Diff}_\mathrm{b}^m(M)$ vanishes if and only if $P \in \mathrm{Diff}_\mathrm{b}^{m-1}(M)$), the normal operator captures P modulo operators with additional *decay* at ∂M: if $P \in \mathrm{Diff}_\mathrm{b}^m(M)$, then $N(P) = 0$ if and only if $P \in \rho\mathrm{Diff}_\mathrm{b}^m(M)$. It is useful to express this more quantitatively as follows: if $\chi \in \mathcal{C}_\mathrm{c}^\infty([0,\infty) \times \partial M)$ is equal to 1 near $\{0\} \times \partial M$, then $\chi N(P)\chi$ is a well-defined operator on M, and

$$P - \chi N(P)\chi \in \rho\mathrm{Diff}_\mathrm{b}^m(M). \tag{2.4}$$

[7] or more precisely its continuous extension to $\overline{\mathbb{R}^n}$, which is then a smooth function on $\overline{\mathbb{R}^n}$

[8] One can define this in a manner that is independent of the choice of collar neighborhood, namely as a dilation-invariant m-th order b-differential operator on the inward pointing normal bundle ${}^+N\partial M$.

Next, we define the *indicial family*[9] of P by

$$N(P,\lambda) := \sum_{j=0}^m P_j(0)\lambda^j \in \mathrm{Diff}^m(\partial M), \qquad \lambda \in \mathbb{C}. \tag{2.5}$$

(Generalizing the situation encountered in Example 2.5, we observe that if $P \in \mathrm{Diff}_\mathrm{b}^m(M)$ is elliptic, then $N(P,\lambda)$ is elliptic for all $\lambda \in \mathbb{C}$; see Problem 2.9.) We then define the *boundary spectrum* of P to be the generalization of the set of indicial roots:[10]

$$\mathrm{spec}_\mathrm{b}(P) := \{\lambda \in \mathbb{C} \colon N(P,\lambda) \colon \mathcal{C}^\infty(\partial M) \to \mathcal{C}^\infty(\partial M) \text{ is not invertible}\}. \tag{2.6}$$

Example 2.6 (Boundary spectrum of the Laplacian). For $\Delta \in \langle x\rangle^{-2}\mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^3})$, we have $\mathrm{spec}_\mathrm{b}(\langle x\rangle^2\Delta) = \{-l-1, l \colon l \in \mathbb{N}_0\} = \mathbb{Z}$. See Problem 2.10 for a generalization.

Remark 2.7 (Multiplicity). When $N(P,\lambda)^{-1}$ is meromorphic, one can define the finer set

$$\mathrm{Spec}_\mathrm{b}(P) = \{(\lambda,k) \in \mathbb{C}\times\mathbb{N}_0 \colon N(P,z)^{-1} \text{ has a pole of order } k+1 \text{ at } z=\lambda\}.$$

In the case of a regular singular ODE, this is the set of pairs (λ,k) where λ is a root of the characteristic polynomial and k is its multiplicity. For simplicity, we will ignore issues of multiplicities in these lectures.

In the best of cases, solutions of $Pu = f$, with $f = 0$ or $f \in \dot{\mathcal{C}}^\infty(M)$, have asymptotic expansions at ∂M involving terms $\rho^\lambda u_\lambda$ where $\lambda \in \mathrm{spec}_\mathrm{b}(P)$ and $u_\lambda \in \ker N(P,\lambda)$. In the case of 'multiplicities ≥ 1', or when f itself has an expansion at ∂M, one should also allow for logarithmic terms $\rho^\lambda(\log\rho)^k u_{\lambda,k}$, just as in the case of regular singular ODEs.

2.2. **Conormality and asymptotic expansions.** A more permissive notion than smoothness on M is the following:

Definition 2.8 (Conormality)**.** Let $\rho \in \mathcal{C}^\infty(M)$ denote a boundary defining function. Let $\alpha \in \mathbb{R}$. We then define the space of *conormal functions on M* as

$$\mathcal{A}^\alpha(M) := \{u \in \mathcal{C}^\infty(M^\circ) \colon \forall\, m \in \mathbb{N}_0,\ P \in \mathrm{Diff}_\mathrm{b}^m(M),\ \chi \in \mathcal{C}_\mathrm{c}^\infty(M)\ \exists\, C \text{ s.t. } |\chi P u| \leq C\rho^\alpha\}.$$

(When M is compact, one can omit the cutoff function χ here.)

One can think of α as the order of vanishing at ∂M of elements of $\mathcal{A}^\alpha(M)$. For example, we have $\mathcal{C}^\infty(M) \subset \mathcal{A}^0(M)$, and $\rho^\alpha \in \mathcal{A}^\alpha(M)$, while for $k \in \mathbb{N}$ we have $\rho^\alpha(\log\rho)^k \in \mathcal{A}^{\alpha-\epsilon}(M)$ for all $\epsilon > 0$. As another example, if $f \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^3)$, then $\Delta^{-1}f = \frac{1}{4\pi|\cdot|} * f \in \mathcal{A}^1(\overline{\mathbb{R}^3})$.

A stronger notion than conormality is *polyhomogeneity*: the existence of (generalized) Taylor expansions. This is the best one can really hope for for the behavior of solutions of b-differential equations at ∂M.

Definition 2.9 (Polyhomogeneity)**.** An *index set* is a subset $\mathcal{E} \subset \mathbb{N}\times\mathbb{N}_0$ so that $(z,k) \in \mathcal{E}$ implies $(z+1,k) \in \mathcal{E}$ and also $(z,k-1) \in \mathcal{E}$ if $k \geq 1$; and so that for all $C \in \mathbb{R}$ there exist only finitely many $(z,k) \in \mathcal{E}$ with $\mathrm{Re}\, z < C$. The space $\mathcal{A}_\mathrm{phg}^\mathcal{E}(M)$ then consists of all

[9] This is often called the *Mellin-transformed normal operator family*, though I do not wish to say this out loud more than a handful of times.

[10] For general $P \in \mathrm{Diff}_\mathrm{b}^m(M)$, this need not be a discrete set. In all our applications however, it will be.

smooth functions u on M° for which, in a collar neighborhood $[0,\infty)_\rho \times \partial M$, there exist $u_{(z,k)} \in \mathcal{C}^\infty(\partial M)$, $(z,k) \in \mathcal{E}$, so that

$$u(\rho, y) \sim \sum_{(z,k)\in\mathcal{E}} \rho^z (\log\rho)^k u_{(z,k)}(y), \qquad \rho \searrow 0.$$

The notation '$\sim$' means the following: if $\chi \in \mathcal{C}^\infty_c([0,\infty)\times\partial M)$ is equal to 1 near $\{0\}\times\partial M$, then

$$u - \sum_{\substack{(z,k)\in\mathcal{E} \\ \operatorname{Re} z\leq C}} \chi(\rho, y)\rho^z(\log\rho)^k u_{(z,k)}(y) \in \mathcal{A}^C(M) \qquad \forall\, C \in \mathbb{R}. \tag{2.7}$$

We say that u is *polyhomogeneous with index set* $\mathcal{E}$, or $\mathcal{E}$*-smooth*.

The requirements on index sets guarantee that all sums in (2.7) are finite and the space $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M)$ is independent of the choice of collar neighborhood. As a special case, we have $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M) = \mathcal{C}^\infty(M)$ for $\mathcal{E} = \mathbb{N}_0 \times \{0\}$, and more generally $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M) = \rho^\alpha\mathcal{C}^\infty(M)$ for $\mathcal{E} = (\alpha + N_0) \times \{0\}$.

In Problem 2.13 you are tasked to show that solutions of regular singular ODEs $Pu = 0$ (see Example 2.4) are polyhomogeneous. Here, we prove a slightly weaker result, namely that solutions which are conormal are necessarily polyhomogeneous. (This re-proves Problem 2.13 once one shows that solutions of $Pu = 0$ are necessarily conormal; see Problem 2.14.) The key tool is the *Mellin transform*:

Definition 2.10 (Mellin transform)**.** The *Mellin transform* of a function u on $(0,\infty)$ is defined by

$$(\mathcal{M}u)(\lambda) = \int_0^\infty x^{-\lambda}u(x)\,\frac{\mathrm{d}x}{x}.$$

The *inverse Mellin transform* of a function $v = v(\lambda)$ is, for a choice of $\alpha \in \mathbb{R}$, defined by

$$(\mathcal{M}_\alpha^{-1}v)(x) := \frac{1}{2\pi}\int_{\alpha-i\infty}^{\alpha+i\infty} x^\lambda v(\lambda)\,\mathrm{d}\lambda.$$

When u is a function on $(0,\infty)\times Y$ where Y is a smooth manifold (possibly with boundary or corners), we define its Mellin transform $(\mathcal{M}u)(\lambda, y)$ parametrically in $y \in Y$.

Via the coordinate change $x = e^{-t}$, $\lambda = -i\sigma$, we have $(\mathcal{M}u)(\lambda) = \int_{-\infty}^\infty e^{-i\sigma t}u(e^{-t})\,\mathrm{d}t$, which is the Fourier transform of $t \mapsto u(e^{-t})$. Similarly, $\mathcal{M}_\alpha^{-1}v$ is the same, in logarithmic coordinates, as the inverse Fourier transform, with integration contour $\operatorname{Im}\sigma = \alpha$. For example then, if $u \in \mathcal{C}^\infty_c((0,\infty))$, then $(\mathcal{M}u)(\lambda)$ is well-defined for all $\lambda \in \mathbb{C}$, and it is indeed holomorphic. When $u \in L^2((0,\infty), \frac{\mathrm{d}x}{x})$, then $(\mathcal{M}u)(\lambda)$ is well-defined as an L^2-function of $\lambda \in \mathbb{R}$.

That the Mellin transform is a good tool for b-analysis is evidenced by the fact that

$$\mathcal{M}\Big(x\frac{\mathrm{d}}{\mathrm{d}x}u\Big)(\lambda) = \lambda(\mathcal{M}u)(\lambda). \tag{2.8}$$

This implies that $N(P,\lambda)$ is the conjugation of $N(P)$ by the Mellin transform.

Lemma 2.11 (Mellin transform characterization of conormality and polyhomogeneity)**.** *Let $C > 0$, and let $u\colon (0,\infty) \to \mathbb{C}$ be a function with $u(x) = 0$ for $x \geq C$ and $|u(x)| \leq Cx^\beta$ for some C, β.*[11]

[11] One can significantly weaken these assumptions to u an element of the dual space of $\dot{\mathcal{C}}^\infty([0,\infty))$.

(1) *We have* $u \in \bigcup_{\epsilon>0} \mathcal{A}^{\alpha-\epsilon}([0,\infty))$ *if and only if* $(\mathcal{M}u)(\lambda)$ *is holomorphic in* $\operatorname{Re}\lambda < \alpha$ *and satisfies* $|(\mathcal{M}u)(\lambda)| \leq C_N(1+|\operatorname{Im}\lambda|)^{-N}$ *for all* N*.*

(2) *Let* $\mathcal{E} \subset \mathbb{C}\times\mathbb{N}_0$ *be an index set. Then* $u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}([0,\infty))$ *if and only if* $(\mathcal{M}u)(\lambda)$ *extends from* $\operatorname{Re}\lambda \ll -1$ *to a meromorphic function in* $\lambda\in\mathbb{C}$ *so that*

$$\{(z,k)\in\mathbb{C}\times\mathbb{N}_0\colon (\mathcal{M}u)(\lambda) \text{ has a pole of order } k+1 \text{ at } \lambda = z\} \subset \mathcal{E},$$

and so that for all C *there exists* C' *so that* $|(\mathcal{M}u)(\lambda)| \leq C_N|\operatorname{Im}\lambda|^{-N}$ *for all* $\lambda\in\mathbb{C}$ *with* $|\operatorname{Re}\lambda| < C$ *and* $|\operatorname{Im}\lambda| > C'$*.*

Proof. The a priori assumptions on u ensure that $u = \mathcal{M}_\gamma^{-1}(\mathcal{M}u)$ for all $\gamma < \beta$, as follows from the Fourier inversion formula applied to $x^{-\gamma}u \in L^2([0,\infty), \frac{\mathrm{d}x}{x})$. We only prove one direction for part (2). Namely, if $(\mathcal{M}u)(\lambda)$ has the stated property, then in

$$u(x) = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} x^\lambda (\mathcal{M}u)(\lambda)\,\mathrm{d}\lambda$$

we can shift the contour of integration from $\gamma+i\mathbb{R}$ to $\gamma'+i\mathbb{R}$ for any $\gamma' > \gamma$ so that $(\mathcal{M}u)(\lambda)$ has no poles on $\gamma'+i\mathbb{R}$. The poles z of order $k+1$ of $(\mathcal{M}u)(\lambda)$ with $\gamma < \operatorname{Re}\lambda < \gamma'$ produce terms $\rho^z(\log\rho)^j$, $j \leq k$, by the residue theorem. The integral over the final contour lies in $\mathcal{A}^\gamma([0,\infty))$. (The conormal regularity follows from (2.8) and the rapid decay as $|\operatorname{Im}\lambda|\to\infty$.) $\square$

Proposition 2.12 (Conormal implies polyhomogeneous: regular singular ODE setting)**.** *Let* $P = \sum_{j=0}^m p_j(x)(x\frac{\mathrm{d}}{\mathrm{d}x})^j$*, with* $p_j \in \mathcal{C}^\infty([0,\infty))$ *and* $p_m(0)\neq 0$*, be a regular singular ordinary differential operator. Suppose that* $u\in\mathcal{A}^\alpha([0,\infty))$ *solves* $Pu=0$*. Then* u *is polyhomogeneous, with an index set depending only on the set of* $\lambda\in\operatorname{spec}_{\mathrm{b}}(P)$ *(with multiplicity, i.e. really on* $(\lambda,k)\in\operatorname{Spec}_{\mathrm{b}}(P)$*) with* $\operatorname{Re}\lambda\geq\alpha$*.*[12]

Proof. Let $\chi\in\mathcal{C}^\infty_{\mathrm{c}}([0,1))$ be equal to 1 near 0. Write the equation $Pu=0$ as

$$N(P)(\chi u) = f := [N(P),\chi]u - \chi(P-N(P))u. \tag{2.9}$$

The first term on the right lies in $\mathcal{C}^\infty_{\mathrm{c}}((0,1))$; the second term (by (2.4)) lies in $\mathcal{A}^{\alpha+1}([0,\infty))$. Thus $f\in\mathcal{A}^{\alpha+1}([0,\infty))$, with $\operatorname{supp} f\subset[0,1)$ compact. We now pass to the Mellin transform,

$$N(P,\lambda)(\mathcal{M}(\chi u))(\lambda) = (\mathcal{M}f)(\lambda), \qquad \operatorname{Re}\lambda<\alpha.$$

Therefore, we have

$$\mathcal{M}(\chi u)(\lambda) = N(P,\lambda)^{-1}(\mathcal{M}f)(\lambda). \tag{2.10}$$

The right hand side is meromorphic in the larger domain $\operatorname{Re}\lambda<\alpha+1$, with poles at $\lambda\in\operatorname{spec}_{\mathrm{b}}(P)$. Lemma 2.11 implies that χu is the sum of a polyhomogeneous function (with index set determined by $\operatorname{Spec}_{\mathrm{b}}(P)$) and a conormal function in $\bigcup_{\epsilon>0}\mathcal{A}^{\alpha+1-\epsilon}([0,\infty))$.

Plugging this improved information into (2.9) shows that f is the sum of a polyhomogeneous function and a function in $\bigcup_{\epsilon>0}\mathcal{A}^{\alpha+2-\epsilon}$, and thus so is χu in view of (2.10). We can iterate this argument any finite number of times, and thus deduce that u is polyhomogeneous (cf. the characterization (2.7)). $\square$

[12]More generally, this holds when $Pu=f\in\dot{\mathcal{C}}^\infty([0,\infty))$. More generally still, if $f\in\mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}([0,\infty))$ is itself polyhomogeneous, then so is u, and the index set of u takes into account both the boundary spectrum of P and the index set of $\mathcal{F}$. Careful inspection of the proof produces upper bounds on the index set of u.

2.3. **b-Sobolev spaces.** *Estimates* for b-differential operators take place naturally take place on function spaces which measure regularity with respect to b-vector fields.

Definition 2.13 (Weighted b-Sobolev spaces)**.** Let M be a *compact* manifold with boundary; let $\rho \in \mathcal{C}^\infty(M)$ denote a boundary defining function. Let μ_0 be a smooth positive density on M, and let $\mu = \rho^{w-1}\mu_0$ where $w \in \mathbb{R}$. Then $H_{\mathrm{b}}^0(M,\mu) := L^2(M^\circ,\mu)$. For $s \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$, we let

$$\begin{aligned} H_{\mathrm{b}}^s(M,\mu) &:= \bigl\{u \in H_{\mathrm{b}}^0(M,\mu) \colon Pu \in H_{\mathrm{b}}^0(M,\mu)\ \forall\, P \in \mathrm{Diff}_{\mathrm{b}}^s(M)\bigr\}, \\ H_{\mathrm{b}}^{s,\alpha}(M,\mu) &:= \bigl\{\rho^\alpha u \colon u \in H_{\mathrm{b}}^s(M,\mu)\bigr\}. \end{aligned}$$

When the density μ is clear from the context (or fixed but otherwise irrelevant), we omit it from the notation.

These spaces can be given the structure of Hilbert spaces: one can set $\|u\|^2_{H_{\mathrm{b}}^s(M,\mu)} = \sum_{P\in\mathscr{P}} \|Pu\|^2_{L^2(M^\circ,\mu)}$ where $\mathscr{P} \subset \mathrm{Diff}_{\mathrm{b}}^m(M)$ is a finite set of b-differential operators which spans $\mathrm{Diff}_{\mathrm{b}}^m(M)$ over $\mathcal{C}^\infty(M)$; and $\|u\|_{H_{\mathrm{b}}^{s,\alpha}(M,\mu)} := \|\rho^{-\alpha}u\|_{H_{\mathrm{b}}^s(M,\mu)}$. Directly from the definition, every $A \in \rho^{-\beta}\mathrm{Diff}_{\mathrm{b}}^m(M)$ defines a bounded linear map

$$A \colon H_{\mathrm{b}}^{s,\alpha}(M,\mu) \to H_{\mathrm{b}}^{s-m,\alpha-\beta}(M,\mu).$$

Let us make Definition 2.13 concrete in a coordinate patch $[0,\infty)_x \times \mathbb{R}^{n-1}_y$ near a point in ∂M. Let us take $\mu_0 = |\mathrm{d}x\,\mathrm{d}y|$ and $\mu = x^w\,|\frac{\mathrm{d}x}{x}\,\mathrm{d}y|$.[13] Pulling back along the map

$$\phi \colon (t,y) \mapsto (x,y) = (e^{-t},y) \tag{2.11}$$

gives $\phi^*\mu = e^{-wt}\,|\mathrm{d}t\,\mathrm{d}y|$. Suppose now u is a distribution on M° which has compact support in this chart; if $w = 0$, then $u \in H_{\mathrm{b}}^s(M,\mu)$ is equivalent to $\phi^*u \in H^s(\mathbb{R}^n)$. For general w,α, the membership $u \in H_{\mathrm{b}}^{s,\alpha}(M,\mu)$ is equivalent to $\phi^*(x^{\frac{w}{2}}x^{-\alpha}u) \in H^s(\mathbb{R}^n)$. Thus, weighted b-Sobolev spaces are the same as (exponentially weighted) Sobolev spaces on $\mathbb{R}^n$ in logarithmic coordinates.[14]

The following variant of Sobolev embedding links the L^2-based function spaces here with the L^∞-based spaces discussed in §2.2. It follows directly from the standard $\mathbb{R}^n$-version using the logarithmic change of coordinates (2.11).

Proposition 2.14 (b-Sobolev embedding)**.** *Let μ be as in Definition 2.13. If $s > \frac{\dim M}{2}$, then $H_{\mathrm{b}}^{s,\alpha}(M,\mu) \hookrightarrow \rho^{\alpha-\frac{w}{2}}L^\infty(M^\circ)$. Moreover,*

$$H_{\mathrm{b}}^{\infty,\alpha}(M,\mu) = \bigcap_{s\in\mathbb{R}} H_{\mathrm{b}}^{s,\alpha}(M,\mu) \hookrightarrow \mathcal{A}^{\alpha-\frac{w}{2}}(M).$$

Conversely, $\mathcal{A}^\beta(M) \subset \bigcap_{\epsilon>0} H_{\mathrm{b}}^{\infty,\beta+\frac{w}{2}-\epsilon}(M,\mu)$.

Similarly, the Rellich compactness theorem on $\mathbb{R}^n$ implies:

[13] The general case merely amounts to multiplying these by smooth positive functions of (x,y); the spaces $H_{\mathrm{b}}^{s,\alpha}(M,\mu)$ are unchanged by such modifications of the density.

[14] While this perspective is conceptually unappealing (since we are passing from a compact setting to a noncompact one), it gives us a convenient shortcut for b-analysis in these lectures.

Proposition 2.15 (Compact inclusions). *Let $s, s' \in \mathbb{N}_0$ and $\alpha, \alpha' \in \mathbb{R}$. Suppose that $s > s'$ and $\alpha > \alpha'$. Then the inclusion map*

$$H_{\mathrm{b}}^{s,\alpha}(M) \to H_{\mathrm{b}}^{s',\alpha'}(M)$$

is compact.

Example 2.16 (b-Sobolev spaces on the radial compactification). If $M = \overline{\mathbb{R}^n}$, then $\mu = |\mathrm{d}x| = r^{n-1}\,|\mathrm{d}r\,\mathrm{d}g_{\mathbb{S}^{n-1}}|$; near $\rho := r^{-1} = 0$, this is $\rho^{-n+1}|\frac{\mathrm{d}\rho}{\rho^2}\,\mathrm{d}g_{\mathbb{S}^{n-1}}|$. Thus, μ is of the form required in Definition 2.13 with $w = -n$. In the case $n = 3$, Proposition 2.14 gives (in the notation of (1.3)) for all $\beta \in \mathbb{R}$ and $\epsilon > 0$ the inclusions

$$H_{\mathrm{b}}^{\infty,-\frac{3}{2}+\beta} \subset \mathcal{A}^{\beta}(\overline{\mathbb{R}^3}) \subset H_{\mathrm{b}}^{\infty,-\frac{3}{2}+\beta-\epsilon}$$

Finally, we characterize b-Sobolev spaces on the 'model space' $[0,\infty) \times \partial M$ (where the normal operator of a b-differential operator lives as a dilation-invariant operator, cf. the expression (2.3)) using the Mellin transform:

Lemma 2.17 (Weighted b-Sobolev spaces and the Mellin transform). *Let ν_0 denote a positive density on ∂M, and define $H_{\mathrm{b}}^s([0,\infty) \times \partial M; x^w|\frac{\mathrm{d}x}{x}\,\mathrm{d}\nu_0|)$ to consist of all distributions u so that $(x\partial_x)^j P u \in L^2((0,\infty) \times \partial M; x^w|\frac{\mathrm{d}x}{x}\,\mathrm{d}\nu_0|)$ for all $j \le s$ and $P \in \mathrm{Diff}^{s-j}(\partial M)$. Then the Mellin transform is an isomorphism*

$$\begin{aligned}\mathcal{M}\colon x^\alpha H_{\mathrm{b}}^s\Big([0,\infty) \times \partial M; x^w\Big|\frac{\mathrm{d}x}{x}\,\mathrm{d}\nu_0\Big|\Big) \\ \to \Big\{ v \in L^2\Big(\Big\{\operatorname{Re}\lambda = \alpha - \frac{w}{2}\Big\}; H^s(\partial M)\Big) \colon \\ \langle\lambda\rangle^j v \in L^2\Big(\Big\{\operatorname{Re}\lambda = \alpha - \frac{w}{2}\Big\}; H^{s-j}(\partial M)\Big)\ \forall\, j \le s \Big\}.\end{aligned} \tag{2.12}$$

Proof. One can easily reduce this to the case $\alpha = w = 0$. For $s = 0$, this is then Plancherel's theorem in logarithmic coordinates. For $s \ge 1$, one uses the intertwining property (2.8). □

2.4. **Problems.**

Problem 2.1 (Smoothness on the radial compactification). Show that $\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ consists of all smooth functions on $\mathbb{R}^n$ which have an asymptotic expansion (2.1).

Problem 2.2 (Projective coordinates on the radial compactification). Let $n \ge 1$. For $j = 1, \ldots, n$, define on

$$\tilde U_{(j)}^{\pm} = \Big\{ x = (x^1, \ldots, x^n) \in \mathbb{R}^n \colon \pm x^j > \frac{1}{2}\max_{k \neq j} |x^k| \Big\} \subset \mathbb{R}^n$$

the functions $\rho_{(j)} = \frac{1}{|x^j|}$ and $\hat x_{(j)}^k = \frac{x^k}{|x^j|}$, $k \neq j$; set $\hat x_{(j)} = (\hat x_{(j)}^k)_{k \neq j} \in \mathbb{R}^{n-1}$. (Thus, we have $\rho_{(j)} \in (0,\infty)$ and $\hat x_{(j)}^k \in B_{\mathbb{R}^{n-1}}(0,2)$ on $\tilde U_{(j)}^{\pm}$.) Set

$$U_{(j)}^{\pm} = [0,\infty) \times B_{\mathbb{R}^{n-1}}(0,2),$$

and show that the map $(0,\infty) \times B_{\mathbb{R}^{n-1}}(0,2) \ni (\rho_{(j)}, \hat x_{(j)}) \mapsto x \in \tilde U_{(j)}^{\pm}$ extends to a smooth map $U_{(j)}^{\pm} \to \overline{\mathbb{R}^n}$ which is a diffeomorphism onto its image $C_{(j)}^{\pm} \subset \overline{\mathbb{R}^n}$. (In other words,

$(\rho_{(j)}, \hat{x}_{(j)})$ defines a smooth coordinate system on $C_{(j)}$.) Show that

$$\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \bigcup_{j=1}^{n} \bigcup_{\pm} C_{(j)}^{\pm}.$$

Problem 2.3 (Linear maps on $\mathbb{R}^n$)**.** Let $n \geq 1$, and let $A \in GL(n, \mathbb{R})$ be an invertible linear map on $\mathbb{R}^n$. Show that A extends, by continuity from the interior, to a diffeomorphism $\overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$. Conclude that the radial compactification of an n-dimensional real vector space is well-defined.

Problem 2.4 (Radial compactifications of vector bundles)**.** Let $E \to M$ denote a smooth real vector bundle of rank $n \in \mathbb{N}$ over the smooth manifold M. Show that its fiberwise radial compactification (or 'fiber-radial compactification') $\bar{E} = \bigsqcup_{p \in M} \overline{E_p} \to M$ can be given a structure of a smooth fiber bundle which is uniquely determined by the requirement that a trivialization $U \times \mathbb{R}^n$ of E over an open set $U \subset M$ extends by continuity from the interior of $\mathbb{R}^n \subset \overline{\mathbb{R}^n}$ to a smooth trivialization $U \times \overline{\mathbb{R}^n}$ of $\bar{E}$ over U.

Problem 2.5 (Diffeomorphisms on manifolds with boundary)**.** The goal of this problem is to show that b-vector fields are in essence the generators of families of diffeomorphisms of manifolds with boundary.

(1) Suppose $(-1, 1) \ni s \mapsto \phi_s$ is a smooth family of diffeomorphisms of a manifold M with boundary, with $\phi_0 = \mathrm{Id}$. For $p \in M$, set $V(p) := \frac{\mathrm{d}}{\mathrm{d}s}\phi_s(p)|_{s=0}$. Show that $V \in \mathcal{V}_{\mathrm{b}}(M)$.
(2) Conversely, if M is compact and $V \in \mathcal{V}_{\mathrm{b}}(M)$, show that the time s flow of V defines a smooth (1-parameter) family of diffeomorphisms of M.

Problem 2.6 (Lie algebra)**.** Show that $\mathcal{V}_{\mathrm{b}}(M)$ is a Lie algebra, where the Lie bracket is the vector field commutator. That is, show that $[V, W] \in \mathcal{V}_{\mathrm{b}}(M)$ whenever $V, W \in \mathcal{V}_{\mathrm{b}}(M)$.

Problem 2.7 (Conjugation by weights)**.** Prove Lemma 2.3.

Problem 2.8 (*-algebra of weighted b-differential operators)**.** Let M be a manifold with boundary, and let ρ denote a boundary defining function.

(1) Let $m_i \in \mathbb{N}_0$, $\alpha_i \in \mathbb{R}$, and $P_i \in \rho^{-\alpha_i}\mathrm{Diff}_{\mathrm{b}}^{m_i}(M)$. Show that
$$P_1 \circ P_2 \in \rho^{-\alpha_1-\alpha_2}\mathrm{Diff}_{\mathrm{b}}^{m_1+m_2}(M).$$
(2) Suppose M is equipped with a smooth positive density μ_0. If $P \in \rho^{-\alpha}\mathrm{Diff}_{\mathrm{b}}^{m}(M)$, show that $P^* \in \rho^{-\alpha}\mathrm{Diff}_{\mathrm{b}}^{m}(M)$. Show that this remains true for densities $\mu = \rho^w \mu_0$ for any $w \in \mathbb{R}$. Make this concrete in the special case $M = \overline{\mathbb{R}^n}$, $\mu = |\mathrm{d}x|$, and $P = \Delta$.

Problem 2.9 (Ellipticity)**.** Suppose $P \in \mathrm{Diff}_{\mathrm{b}}^m(M)$ is elliptic. Show that $N(P, \lambda)$ is elliptic for all $\lambda \in \mathbb{C}$, and its principal symbol is independent of λ.

Problem 2.10 (Boundary spectrum)**.** Let $n \geq 1$, and write $\Delta = -\sum_{j=1}^n \partial_{x^j}^2 \in \mathrm{Diff}^2(\mathbb{R}^n)$ for the Laplacian.

(1) Compute $\mathrm{spec}_{\mathrm{b}}(\langle x \rangle^2 \Delta)$.
(2) Let $V \in \langle x \rangle^{-2}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$ (i.e. $V = \langle x \rangle^{-2}V_0$ where $V_0 \in \mathcal{C}^\infty(\overline{\mathbb{R}^n})$). Suppose that $C := (|x|^2 V)|_{\partial\overline{\mathbb{R}^n}}$ is constant. (Thus, for $|x| > 1$ we have $V(x) = \frac{C}{|x|^2} + \mathcal{O}(|x|^{-3})$.) Compute $\mathrm{spec}_{\mathrm{b}}(\langle x \rangle^2(\Delta + V))$.

Problem 2.11 (Polyhomogeneous conormal functions)**.** Let M be a manifold with boundary.

(1) What is $\mathcal{A}^{\emptyset}_{\mathrm{phg}}(M)$?
(2) Let $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}_0$ be an index set. Determine the set of $\alpha \in \mathbb{R}$ for which $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M) \subset \mathcal{A}^{\alpha}(M)$.
(3) Show that the space $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M)$ is well-defined, i.e. it does not depend on the choice of collar neighborhood of ∂M.
(4) Given a function $u_{(z,k)} \in \mathcal{C}^\infty(\partial M)$ for each $(z,k) \in \mathcal{E}$, show that there exists $u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(M)$ so that $u \sim \sum_{(z,k)\in\mathcal{E}} \rho^z (\log \rho)^k u_{(z,k)}$.

Problem 2.12 (Mellin transform characterization)**.** Fill in the details in the proof of Lemma 2.11. Prove the same result only assuming that u is an element of the dual space of $\dot{\mathcal{C}}^\infty([0,\infty))$ with support in $x \le C$. (One can furthermore read off an upper bound for C from the growth properties of $(\mathcal{M}u)(s)$ as $\operatorname{Re} s \searrow -\infty$. Try to prove such a result, or look up the Paley–Wiener theorem.)

Problem 2.13 (Polyhomogeneity for regular singular ODEs)**.** Consider a solution u of the regular singular ODE

$$Pu = \sum_{j=0}^{m} p_j(x) \Big(x \frac{\mathrm{d}}{\mathrm{d}x}\Big)^j u = 0, \tag{2.13}$$

where $p_j \in \mathcal{C}^\infty([0,\infty))$ and $p_m(0) \neq 0$. Denote its characteristic polynomial by $N(P,\lambda) := \sum_{j=0}^m p_j(0)\lambda^j$.

(1) If the roots λ_j, $j = 1,\ldots,m$, of $N(P,\lambda)$ are pairwise distinct, construct m linearly independent solutions u_j of $Pu = 0$ with leading order term x^{λ_j}. Show that u_j is polyhomogeneous. Conclude that every solution u of (2.13) is polyhomogeneous.
(2) Generalize this result to the case that $N(P,\lambda)$ has repeated roots.

Problem 2.14 (Polyhomogeneity for regular singular ODEs, II)**.** Let P be as in the previous problem. Suppose $Pu = 0$. We shall give a new (and more robust) proof of the polyhomogeneity of u by reducing the situation to Proposition 2.12. To wit, show that there exists $\alpha \in \mathbb{R}$ so that $u \in \mathcal{A}^\alpha([0,\infty))$ as follows.

(1) Define $E(x) = \sum_{j=0}^{m-1} |(x\frac{\mathrm{d}}{\mathrm{d}x})^j u(x)|^2$. Show that there exists a constant $C > 0$ so that $-x\frac{\mathrm{d}}{\mathrm{d}x}E(x) \le CE(x)$ for $x \in (0,1]$. Conclude that $\int_0^1 x^{-2\alpha} \sum_{j=0}^{m-1} |(x\frac{\mathrm{d}}{\mathrm{d}x})^j u|^2 \frac{\mathrm{d}x}{x} < \infty$ for all $\alpha < -C$.
(2) Prove a similar estimate for the solution of $Pu = f$ when $\int_0^1 x^{-2\alpha}|u(x)|^2 \frac{\mathrm{d}x}{x} < \infty$.
(3) Writing $P(x\frac{\mathrm{d}}{\mathrm{d}x}u) = [P, x\frac{\mathrm{d}}{\mathrm{d}x}]u$, and similarly for higher derivatives, show that

$$\int_0^1 x^{-2\alpha} \Big| \Big(x\frac{\mathrm{d}}{\mathrm{d}x}\Big)^j u\Big|^2 \frac{\mathrm{d}x}{x} < \infty \quad \forall\, s \in \mathbb{N}.$$

Conclude that u is conormal.

Problem 2.15 (b-Sobolev spaces)**.** Show that the stated definition of the norm $\|\cdot\|_{H_{\mathrm{b}}^{s,\alpha}(M,\mu)}$ after Definition 2.13 is independent of the choice of $\mathscr{P}$ and ρ up to equivalence of norms.

3. Applications: I

3.1. **The Laplacian on $\mathbb{R}^3$ revisited.** We consider only the case of $n = 3$ dimensions here, and leave the general case to the problems. We already saw that

$$\Delta = \sum_{j=1}^{3} -\partial_{x^j}^2 \in \langle x\rangle^{-2}\mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^3}),$$

and we computed its boundary spectrum to be $\mathrm{spec}_\mathrm{b}(\langle x\rangle^2\Delta) = \mathbb{Z}$.

Theorem 3.1 (Laplacian on weighted b-Sobolev spaces)**.** *Let $s \geq 3$ and $\alpha \notin \frac{1}{2} + \mathbb{Z}$. Fix on $\overline{\mathbb{R}^3}$ the density $|\mathrm{d}x|$. Then $\Delta\colon H_\mathrm{b}^{s,\alpha}(\overline{\mathbb{R}^3}) \to H_\mathrm{b}^{s-2,\alpha+2}(\overline{\mathbb{R}^3})$ is Fredholm. For $\alpha > -\frac{3}{2}$, it is injective, and for $\alpha < -\frac{1}{2}$ it is surjective; in particular, it is invertible for $\alpha \in (-\frac{3}{2}, -\frac{1}{2})$.*

Remark 3.2 (Generalization). The Fredholm statement holds also for every elliptic b-operator which differs from Δ by an element of $\langle x\rangle^{-3}\mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^3})$. (Adding to Δ an element of $\langle x\rangle^{-3}\mathrm{Diff}_\mathrm{b}^2$ changes P by an element in $\langle x\rangle^{-1}\mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^3})$, and in particular leaves the normal operator of P unchanged.) This includes the Laplace operator on certain *asymptotically Euclidean spaces*, and also allows for the addition of terms $a\partial_{x^j}$ and V where $a \in \langle x\rangle^{-2}\mathcal{C}^\infty$ and $V \in \langle x\rangle^{-3}\mathcal{C}^\infty$.

As far the literature is concerned, a more general result was proved by Melrose in [Mel93, §5.17] (see also Theorem 3.3 below). The application to the Laplace operator appears to have been first given in Carron–Coulhon–Hassell in [CCH06, Lemma 3.2].

Proof of Theorem 3.1. We only consider $P = \langle x\rangle^2\Delta \in \mathrm{Diff}_\mathrm{b}^2(\overline{\mathbb{R}^3})$.

- Step 1. Elliptic estimate. Near a point in $\partial\overline{\mathbb{R}^3}$, we can pull back P via a logarithmic coordinate change to a uniformly elliptic operator on $\mathbb{R}\times\mathbb{R}^2$: schematically, $P = -(\rho\partial_\rho)^2 - \partial_y^2$ where $\rho = r^{-1}$ and $y \in \mathbb{R}^2$ denotes local coordinates on $\mathbb{S}^2$, so with $t = -\log\rho$ this becomes $P = -\partial_t^2 - \partial_y^2$. We can thus use (uniform) elliptic estimates on standard Sobolev spaces on $\mathbb{R}\times\mathbb{R}^2$ and pull them back to estimates on b-Sobolev spaces on $\overline{\mathbb{R}^3}$ to get the a priori estimate[15]

$$\|u\|_{H_\mathrm{b}^{s,\alpha}(\overline{\mathbb{R}^3})} \leq C\Big(\|Pu\|_{H_\mathrm{b}^{s-2,\alpha}(\overline{\mathbb{R}^3})} + \|u\|_{H_\mathrm{b}^{0,\alpha}(\overline{\mathbb{R}^3})}\Big). \tag{3.1}$$

Since the inclusion $H_\mathrm{b}^{s,\alpha} \hookrightarrow H_\mathrm{b}^{0,\alpha}$ is not compact, this estimate does not tell us much yet. (It neither implies the finite-dimensionality of $\ker P$ nor the closed range property.)

- Step 2. Inversion of the indicial family; semi-Fredholm estimate. With $\rho = r^{-1}$, let now $\chi \in \mathcal{C}_\mathrm{c}^\infty([0,\infty)_\rho \times \mathbb{S}^2)$ be identically 1 near $\rho = 0$. Then

$$\|u\|_{H_\mathrm{b}^{0,\alpha}} \leq \|\chi u\|_{H_\mathrm{b}^{0,\alpha}} + C_N\|(1-\chi)u\|_{H_\mathrm{b}^{0,-N}} \tag{3.2}$$

for any N, since $\mathrm{supp}(1-\chi)$ is disjoint from $\partial\overline{\mathbb{R}^3}$. The norm on χu here is, by Lemma 2.17 (with $s = 0$)

$$\|\chi u\|_{\rho^\alpha L^2\left([0,\infty)\times\mathbb{S}^2;\rho^{-3}|\frac{\mathrm{d}\rho}{\rho}\mathrm{d}g_{\mathbb{S}^2}|\right)} \leq C\|\mathcal{M}(\chi u)\|_{L^2\left(\{\mathrm{Re}\,\lambda=\alpha+\frac{3}{2}\};L^2(\mathbb{S}^2)\right)}.$$

[15]The weight α in this estimate is arbitrary. When $\alpha = -\frac{3}{2}$, then this indeed follows from estimates on unweighted Sobolev spaces on $\mathbb{R}\times\mathbb{R}^2$. One can reduce the case of general α to this special case by working with $\langle x\rangle^\beta P\langle x\rangle^{-\beta}$ instead of P for appropriate β.

But since $\alpha \notin \frac{1}{2}+\mathbb{Z}$, we have $\alpha+\frac{3}{2} \notin \operatorname{spec}_{\mathrm{b}}(P) = \{-l-1, l \colon l \in \mathbb{N}_0\}$. Therefore, $N(P,\lambda) = -\lambda^2+\lambda+\Delta_{\mathbb{S}^2} \colon H^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ is invertible for $\operatorname{Re}\lambda = \alpha+\frac{3}{2}$.[16] Moreover, there exists a constant C so that for $v \in L^2(\mathbb{S}^2)$ we have[17]

$$\|v\|_{L^2} \leq C\|N(P,\lambda)v\|_{L^2}, \qquad \operatorname{Re}\lambda = \alpha+\frac{3}{2}. \tag{3.3}$$

Indeed, for any bounded set of λ this follows (with the L^2-norm on the left replaced by the H^2-norm) from the invertibility of $N(P,\lambda)$. For $\lambda = (\alpha+\frac{3}{2})+is$, $s \in \mathbb{R}$, we have

$$-\lambda^2+\lambda = s^2+\mathcal{O}(|s|),$$

and therefore $N(P,\lambda) = \Delta_{\mathbb{S}^2}+1+c(\alpha,s)$ where $\operatorname{Re} c(\alpha,s) \geq 0$ when $|s| = |\operatorname{Im}\lambda|$ is sufficiently large, and the estimate (3.3) then follows (with $C=1$ for such λ) from $\langle N(P,\lambda)v, v\rangle_{L^2} \geq \|v\|_{L^2}^2$.

We can thus estimate

$$\|\mathcal{M}(\chi u)\|_{L^2(\{\operatorname{Re}\lambda=\alpha+\frac{3}{2}\};L^2(\mathbb{S}^2))} \leq C\|N(P,\cdot)\mathcal{M}(\chi u)\|_{L^2(\{\operatorname{Re}\lambda=\alpha+\frac{3}{2}\};L^2(\mathbb{S}^2))},$$

and therefore

$$\|\chi u\|_{H_{\mathrm{b}}^{0,\alpha}} \leq C\|N(P)(\chi u)\|_{H_{\mathrm{b}}^{0,\alpha}} \leq C\Big(\|\chi P u\|_{H_{\mathrm{b}}^{0,\alpha}} + \|\chi(P-N(P))u\|_{H_{\mathrm{b}}^{0,\alpha}} + \|[P,\chi]u\|_{H_{\mathrm{b}}^{0,\alpha}}\Big).$$

Since $\chi(P-N(P)) \in \rho\mathrm{Diff}_{\mathrm{b}}^2$, the second term is bounded by $C\|\rho u\|_{H_{\mathrm{b}}^{2,\alpha}} = \|u\|_{H_{\mathrm{b}}^{2,\alpha-1}}$. Moreover, $[P,\chi]$ is a first order operator whose coefficients are compactly supported in M°, and thus the third term is bounded by $C_N\|u\|_{H_{\mathrm{b}}^{1,-N}}$ for any N. Plugging this, together with (3.2), into (3.1) gives

$$\|u\|_{H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^3})} \leq C\Big(\|Pu\|_{H_{\mathrm{b}}^{s-2,\alpha}(\overline{\mathbb{R}^3})} + \|u\|_{H_{\mathrm{b}}^{2,\alpha-1}(\overline{\mathbb{R}^3})}\Big). \tag{3.4}$$

This is the *semi-Fredholm estimate* one is after in this business: it implies that $P \colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^3}) \to H_{\mathrm{b}}^{s-2,\alpha}(\overline{\mathbb{R}^3})$ has finite-dimensional kernel and closed range (see Problem 3.1).

- *Step 3. Fredholm property.* The L^2-orthogonal complement of the range $\Delta(H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^3})) \subset H_{\mathrm{b}}^{s-2,\alpha+2}(\overline{\mathbb{R}^3})$ is equal to the kernel of the formal adjoint Δ^* on $H_{\mathrm{b}}^{-s+2,-\alpha+2}(\overline{\mathbb{R}^3})$.[18] By elliptic regularity, elements of this kernel lie in $H_{\mathrm{b}}^{s',-\alpha+2}(\overline{\mathbb{R}^3})$ for all s'. By what we have already shown, we deduce that the range of Δ has finite codimension.

- *Step 4. Kernel and cokernel.* Suppose $\alpha > -\frac{3}{2}$ and $u \in H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^3}) \cap \ker\Delta$. Then $u \in H_{\mathrm{b}}^{s',\alpha}$ for all s' by elliptic regularity, and thus $u \in \mathcal{A}^{\alpha+\frac{3}{2}}(\overline{\mathbb{R}^3})$ by Sobolev embedding (Proposition 2.14). That is, u decays at infinity, and by the maximum principle it

[16] Here is one way to see this. Since $N(P,\lambda)$ is elliptic, it is Fredholm as an operator $H^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$. Elements of its kernel are automatically smooth; by decomposing an element in $\ker N(P,\lambda)$ into spherical harmonics and studying the action of $N(P,\lambda)$ on each piece, one concludes that $\ker N(P,\lambda) = \{0\}$, cf. the discussion after (1.4). Moreover, $N(P,\lambda)$ is a compact perturbation of the operator $\Delta_{\mathbb{S}^2}+1$ which is invertible (being symmetric and having trivial kernel); therefore, $N(P,\lambda)$ has index 0, which due to its injectivity implies its surjectivity.

[17] This estimate is technically correct, but morally wrong: $N(P,\lambda)$ is elliptic, so one expects Sobolev spaces to appear on both side. However, one should not use standard Sobolev spaces, but rather 'large parameter Sobolev spaces', with $\operatorname{Im}\lambda$ the large parameter; these are the spaces which secretly feature on the right hand side of (2.12).

[18] We have not defined this space appearing here; but using logarithmic coordinates it can be defined in terms of negative order Sobolev spaces on $\mathbb{R}\times\mathbb{R}^2$.

must be 0. Surjectivity of Δ for $\alpha < -\frac{1}{2}$ follows from the same argument since $-\alpha+2 > -\frac{3}{2}$ for $\alpha < -\frac{1}{2}$. □

Most of Theorem 3.1 is a special case of a general result about elliptic b-differential operators:

Theorem 3.3 (Fredholm property of elliptic b-differential operators)**.** *Let $P \in \mathrm{Diff}_\mathrm{b}^m(M)$ be an elliptic operator on the compact manifold M with boundary. Fix on M a smooth positive density as in Definition 2.13 with $w = 0$. Then:*

(1) $\mathrm{spec}_\mathrm{b}(P) \subset \mathbb{C}$ *is discrete; and for all C there exists C' so that all $\lambda \in \mathrm{spec}_\mathrm{b}(P)$ with $|\operatorname{Re}\lambda| < C$ have $|\operatorname{Im}\lambda| < C'$.*
(2) *Let $\alpha \in \mathbb{R}$ be such that $\operatorname{Re}\lambda \neq \alpha$ for all $\lambda \in \mathrm{spec}_\mathrm{b}(P)$. Let $s \geq m$. Then*
$$P\colon H_\mathrm{b}^{s,\alpha}(M) \to H_\mathrm{b}^{s-m,\alpha}(M)$$
is Fredholm.
(3) *Solutions of $Pu = f \in \dot{\mathcal{C}}^\infty(M)$ which lie in some weighted Sobolev space are polyhomogeneous.*

We do not give the proof here. It is similar to that of Theorem 3.1; the main technical task is to prove an estimate of the type (3.3) for the indicial family in this generality.

Solutions of elliptic b-differential equations are polyhomogeneous. We sketch this in the case of the Laplace equation on $\mathbb{R}^3$:

Proposition 3.4 (Polyhomogeneity of solutions of the Laplace equation)**.** *Suppose $u \in H_\mathrm{b}^{s,\alpha}(\overline{\mathbb{R}^3})$ satisfies $\Delta u = f \in \dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^3}) = \mathscr{S}(\mathbb{R}^3)$. Then u is polyhomogeneous with index set contained in $(\mathbb{Z} \cap \{\lambda > \alpha + \frac{3}{2}\}) \times \mathbb{N}_0$. This is true also if one replaces Δ by a more general operator as in Remark 3.2.*

Proof. Let $P = \langle x\rangle^2\Delta$. Elliptic regularity gives $u \in H_\mathrm{b}^{\infty,\alpha}(\overline{\mathbb{R}^3}) \subset \mathcal{A}^{\alpha+\frac{3}{2}}(\overline{\mathbb{R}^3})$. Write $Pu = \langle x\rangle^2 f$ using a cutoff $\chi \in \mathcal{C}^\infty(\overline{\mathbb{R}^3})$ which is 1 near $\partial\overline{\mathbb{R}^3}$ and 0 near 0 as
$$N(P)(\chi u) = \tilde{f} := \chi\langle x\rangle^2 f + [N(P), \chi]u - \chi(P - N(P))u \in \mathcal{A}^{\alpha+\frac{5}{2}}(\overline{\mathbb{R}^3}).$$
Therefore,
$$\chi u = \mathcal{M}_{\alpha-\epsilon}^{-1}\big(N(P,\lambda)^{-1}(\mathcal{M}\tilde{f})(\lambda)\big).$$
The key point is that $N(P,\lambda)^{-1}\colon \mathcal{C}^\infty(\mathbb{S}^2) \to \mathcal{C}^\infty(\mathbb{S}^2)$ is meromorphic in λ, with simple poles at $\lambda = -l, l+1$, and satisfies appropriate uniform estimates when $\operatorname{Re}\lambda$ is bounded and $|\operatorname{Im}\lambda| \to \infty$. Thus, one can shift the contour in the inverse Mellin transform to $\operatorname{Re}\lambda = \alpha + 1 - \epsilon$, which is how one picks up terms in the polyhomogeneous expansion of u corresponding to the poles of $N(P,\lambda)^{-1}$ with $\alpha \leq \operatorname{Re}\lambda < \alpha + 1$. □

3.2. **Waves in the interior of a Schwarzschild black hole.** The metric of a Schwarzschild black hole with mass $\mathfrak{m} > 0$ is
$$g = -\Big(1 - \frac{2\mathfrak{m}}{r}\Big)\,\mathrm{d}t^2 + \Big(1 - \frac{2\mathfrak{m}}{r}\Big)^{-1}\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2},$$
defined on $\mathcal{M}_\mathrm{int} \cup \mathcal{M}_\mathrm{ext}$ where
$$\mathcal{M}_\mathrm{int} = \mathbb{R}_t \times (0, 2\mathfrak{m})_r \times \mathbb{S}^2, \qquad \mathcal{M}_\mathrm{ext} = \mathbb{R}_t \times (2\mathfrak{m}, \infty)_r \times \mathbb{S}^2.$$

One can perform a change of variables near $r = 2\mathfrak{m}$ to extend the metric smoothly across the 'event horizon' $r = 2\mathfrak{m}$. We shall not discuss the global geometric properties of the Schwarzschild metric, such as its Ricci-flatness or its maximal analytic extension, and refer the interested reader to the lecture notes [DR08].

On $\mathcal{M}_{\rm int}$ where $1-\frac{2\mathfrak{m}}{r} < 0$, the metric g is $\sim -\mathrm{d}r^2 + \mathrm{d}t^2 + r^2 g_{\mathbb{S}^2}$. (Compare this with the Minkowski metric $-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}$!) so r is a time coordinate and t is a space coordinate. This reflects the fact that 'time' r is monotone for freely falling observers in $\mathcal{M}_{\rm int}$, and indeed $r \searrow 0$ along all timelike geodesics in $\mathcal{M}_{\rm int}$.

We will study the wave equation

$$\Box_g u = \Big[\Big(1-\frac{2\mathfrak{m}}{r}\Big)^{-1}\partial_t^2 - r^{-2}\partial_r r^2\Big(1-\frac{2\mathfrak{m}}{r}\Big)\partial_r + r^{-2}\Delta_{\mathbb{S}^2}\Big]u(t,r,\omega) = 0$$

on $\mathcal{M}_{\rm int}$ following Fournodavlos–Sbierski [FS20].[19] Multiplying through by $-r^3$ gives

$$Pu = 0, \qquad P = (-2\mathfrak{m} + r)(r\partial_r)^2 + r^2\partial_r - r\Delta_{\mathbb{S}^2} + \frac{r^4}{2\mathfrak{m} - r}\partial_t^2.$$

This is now a b-differential operator on

$$M := [0, 2\mathfrak{m})_r \times \mathbb{R}_t \times \mathbb{S}^2,$$

with normal operator at $\partial M = r^{-1}(0)$ and indicial family given by

$$N(P) = -2\mathfrak{m}(r\partial_r)^2, \qquad N(P,\lambda) = -2\mathfrak{m}\lambda^2.$$

Note however that it is *very degenerate* even as a b-operator: it is really built from $r^2\partial_t$ and $r^{1/2}V$ ($V \in \mathcal{V}(\mathbb{S}^2)$). (This is also reflected by the fact that $N(P,\lambda)$, which for general $P \in \mathrm{Diff}_{\mathrm{b}}^2(M)$ is an element of $\mathrm{Diff}^2(\partial M) = \mathrm{Diff}^2(\mathbb{R}_t \times \mathbb{S}^2)$, is actually an operator of order 0.) It is thus rather fortuitous that we can analyze it using b-techniques only. As we will see, the reason is that P commutes with the (much stronger) b-vector fields ∂_t and $\mathcal{V}(\mathbb{S}^2)$. This (or at least the weaker property of 'almost commuting' with b-vector fields) is fairly common; we give another, and in a way simpler, example in §3.3 below.

Theorem 3.5 (Asymptotics of waves near the Schwarzschild singularity)**.** *Fix* $r_0 \in (0, 2\mathfrak{m})$*. Suppose* u *solves the initial value problem*

$$\Box_g u = 0, \qquad (u|_{r=r_0}, \partial_r u|_{r=r_0}) = (u_0, u_1), \tag{3.5}$$

where $u_0, u_1 \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}_t \times \mathbb{S}^2)$*. Then* u *is polyhomogeneous at* $r = 0$*, with*

$$u \sim \sum_{j=0}^\infty \Big(A_j r^j(\log r) + B_j r^j\Big), \qquad A_j, B_j \in \mathcal{C}_{\rm c}^\infty(\mathbb{R}_t \times \mathbb{S}^2). \tag{3.6}$$

Moreover, there is a bijection between initial data (u_0, u_1) *and the leading order coefficients* (A_0, B_0)*.*

Proof. The existence of a smooth solution u on $(0, r_0) \times \mathbb{R} \times \mathbb{S}^2$ follows from the standard theory of wave equations. Moreover, using the finite speed of propagation for solutions of wave equations, the solution $u(r,t,\omega)$ vanishes for $|t| > T$, $r \in (0, r_0]$, where $T < \infty$ can in principle be computed explicitly from the support of the initial data u_0, u_1.

[19] Unlike [FS20], we work here strictly in the interior, and do not study the behavior near the event horizon $r = 2\mathfrak{m}$.

- Energy estimate. Write $\not\nabla$ for the gradient on $(\mathbb{S}^2, g_{\mathbb{S}^2})$. Define the energy

$$E(r) := \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{S}^2} (2\mathfrak{m}-r)|r\partial_r u(r,t,\omega)|^2 + |r^{\frac{1}{2}}\not\nabla u(r,t,\omega)|^2 + \frac{1}{2\mathfrak{m}-r}|r^2\partial_t u(r,t,\omega)|^2 \,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2}.$$

We then compute

$$\begin{aligned}
-r\frac{\mathrm{d}}{\mathrm{d}r}E(r) &= -\iint_{\mathbb{R}\times\mathbb{S}^2} (2\mathfrak{m}-r)(r\partial_r u)((r\partial_r)^2 u) + \Big(\frac{1}{2}|r^{\frac{1}{2}}\not\nabla u|^2 + r^{\frac{1}{2}}\not\nabla u\cdot(r^{\frac{1}{2}}\not\nabla r\partial_r u)\Big) \\
&\qquad + \Big(\frac{r}{(2\mathfrak{m}-r)^2}|r^2\partial_t u|^2 + \frac{2}{2\mathfrak{m}-r}|r^2\partial_t u|^2 + \frac{1}{2\mathfrak{m}-r}(r^2\partial_t u)(r^2\partial_t r\partial_r u)\Big)\,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2} \\
&= -\iint_{\mathbb{R}\times\mathbb{S}^2} (r\partial_r u)\Big((2\mathfrak{m}-r)(r\partial_r)^2 + r\Delta_{\mathbb{S}^2} - \frac{1}{2\mathfrak{m}-r}(r^2\partial_t)^2\Big)u \\
&\qquad\qquad + \frac{1}{2}|r^{\frac{1}{2}}\not\nabla u|^2 + \frac{4\mathfrak{m}-r}{(2\mathfrak{m}-r)^2}|r^2\partial_t u|^2\,\mathrm{d}r\,\mathrm{d}g_{\mathbb{S}^2} \\
&= -\iint_{\mathbb{R}\times\mathbb{S}^2} (r\partial_r u)(-r^3\Box_g u + r^2\partial_r u) + \frac{1}{2}|r^{\frac{1}{2}}\not\nabla u|^2 + \frac{4\mathfrak{m}-r}{(2\mathfrak{m}-r)^2}|r^2\partial_t u|^2\,\mathrm{d}r\,\mathrm{d}g_{\mathbb{S}^2} \\
&\leq 0.
\end{aligned}$$

Therefore, $E(r) \leq E(r_0)$.[20] We also need to control u itself; but since

$$u(r,t,\omega) = u_0(t,\omega) + \int_{r_0}^{r} s\partial_s u(s,t,\omega)\,\frac{\mathrm{d}s}{s},$$

we have

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{S}^2} |u(r,t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2} \\
&\qquad \leq 2\iint_{\mathbb{R}\times\mathbb{S}^2} |u_0(t,\omega)|^2\,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2} + 2\iint_{\mathbb{R}\times\mathbb{S}^2}\Big(\int_{r_0}^{r}\frac{\mathrm{d}s}{s}\Big)\Big(\int_{r_0}^{r} CE(s)\,\frac{\mathrm{d}s}{s}\Big)\,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2} \\
&\qquad \leq C_0 + C_1 E(r_0)(\log r)^2.
\end{aligned}$$

Altogether, we conclude that for any $\alpha < 0$,

$$\iiint_{(0,r_0]\times\mathbb{R}\times\mathbb{S}^2} r^{-2\alpha}\Big(|u|^2 + |r\partial_r u|^2 + |r^2\partial_t u|^2 + |r^{\frac{1}{2}}\not\nabla u|^2\Big)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2} \leq C, \tag{3.7}$$

where C only depends on $\|u_0\|_{H^1}$ and $\|u_1\|_{L^2}$.

- Higher regularity. Since $\Box_g(\partial_t u) = \partial_t\Box_g u = 0$ and $\Box_g\Omega u = \Omega\Box_g u$ for every rotation vector field $\Omega \in \mathcal{V}(\mathbb{S}^2)$, we have (3.7) also for $\partial_t^j\Omega^\gamma u$ for all $j,k\in\mathbb{N}_0$ and $|\gamma|$-fold compositions Ω^γ of rotation vector fields. Moreover, we can express $(r\partial_r)^2 u$ in terms of $Pu = 0$ and t- and spherical derivatives and first order derivatives in r:

$$(-2\mathfrak{m}+r)(r\partial_r)^2 u = Pu - r^2\partial_r u + r\Delta_{\mathbb{S}^2}u - \frac{r^4}{2\mathfrak{m}-r}\partial_t^2.$$

[20] We would be content here even with an estimate $-r\frac{\mathrm{d}}{\mathrm{d}r}E(r) \leq CE(r)$, giving $E(r) \leq E(r_0)(\frac{r}{r_0})^{-C}$, for any constant C.

Therefore, we also obtain the bound (3.7) for derivatives of u along (any number of) $r\partial_r$. Thus, we have full b-regularity: for $s \in \mathbb{N}$,

$$\|u\|^2_{H_{\mathrm{b}}^{s,\alpha}} := \sum_{i+j+|\gamma|\leq s} \iiint_{(0,r_0]\times\mathbb{R}\times\mathbb{S}^2} r^{-2\alpha}|(r\partial_r)^i\partial_t^j\Omega^\gamma u|^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}t\,\mathrm{d}g_{\mathbb{S}^2} < \infty.$$

By Sobolev embedding (Proposition 2.14), this gives

$$u \in \mathcal{A}^\alpha(M), \qquad \alpha < 0.$$

• Asymptotic expansion. We now proceed as in the proof of Proposition 2.12. If $\chi \in \mathcal{C}^\infty_{\mathrm{c}}([0,r_0))$ is equal to 1 near 0, we write

$$N(P)(\chi u) = f := [N(P),\chi]u - \chi(P - N(P))u. \tag{3.8}$$

Thus, $f \in \mathcal{A}^{\alpha+1}(M)$. We now pass to the Mellin transform, with $\mathcal{M}(\chi u)(\lambda,t,\omega)$ initially defined for $\lambda < 0$; but $(\mathcal{M}f)(\lambda,t,\omega)$ is defined on the larger region $\lambda < \alpha+1$, while $N(P,\lambda)^{-1} = -\frac{1}{2\mathfrak{m}}\lambda^{-2}$ has a double pole at 0. Shifting the contour in the formula $\chi u = \mathcal{M}_\alpha^{-1}(N(P,\lambda)^{-1}(\mathcal{M}f)(\lambda))$ from $\operatorname{Re}\lambda = \alpha$ to $\operatorname{Re}\lambda = \alpha+1$ gives

$$\chi u = A(t,\omega)\log r + B(t,\omega) + \tilde u(r,t,\omega), \qquad \tilde u \in \bigcap_{\epsilon>0}\mathcal{A}^{1-\epsilon}(M).$$

One can then plug this information back into (3.8) and iterate the argument to obtain the full expansion of u.

The final statement is left to the reader; see Problem 3.8. □

3.3. **The wave equation on de Sitter space.** One popular description of the de Sitter spacetime is $\mathbb{R}_t \times \mathbb{R}^n_x$ with metric $g = -\mathrm{d}t^2 + e^{2t}\mathrm{d}x^2$ (or e^{2t} replaced by e^{2Ht} for the *Hubble constant* $H > 0$). We replace here the spatial manifold by the (compact) n-torus $\mathbb{T}^n$. The initial value problem for the wave equation for $u = u(t,x)$ reads

$$\begin{aligned} \Box_g u &= \bigl(e^{-nt}\partial_t e^{nt}\partial_t - e^{-2t}\partial_x^2\bigr)u = 0, \\ (u,\partial_t u)|_{t=1} &= (u_0,u_1), \end{aligned} \tag{3.9}$$

where $u_0,u_1 \in \mathcal{C}^\infty(\mathbb{T}^n)$. Being interested in the asymptotic behavior of $u(t,x)$ as $t\to\infty$, we compactify the spacetime manifold $\mathbb{R}_t\times\mathbb{T}^n_x$ at $t=\infty$ by defining

$$M := [0,\infty)_\tau \times \mathbb{T}^n_x$$

and identifying M° with $(0,\infty)\times\mathbb{T}^n$ via $\tau = e^{-t}$. Thus $\partial_t = -\tau\partial_\tau$, and therefore

$$\Box_g = (\tau\partial_\tau)^2 - n\tau\partial_\tau - (\tau\partial_x)^2.$$

This is thus a b-differential operator, but again a degenerate one since x-derivatives appear only via $\tau\partial_x$.[21] Nonetheless, we can again study solutions of (3.9) using b-techniques:

[21] Thus, the underlying Lie algebra of vector fields is the Lie algebra of *uniformly degenerate vector fields*, or *0-vector fields* in the terminology of Mazzeo–Melrose [MM87]: these are the smooth vector fields on M which vanish at ∂M.

Theorem 3.6 (Asymptotics of waves on a de Sitter type spacetime)**.** *Let $n \geq 1$. Suppose u is a solution of the initial value problem* (3.9)*. Then u is polyhomogeneous on M. More precisely, for suitable $A_j, B_j \in \mathcal{C}^\infty(\mathbb{T}^n)$ we have*

$$u \sim \begin{cases} \sum_{j=0}^\infty \bigl(A_j \tau^j + B_j \tau^{n+j}\bigr), & n \text{ odd}, \\ \sum_{j=0}^\infty \bigl(A_j \tau^j + B_j \tau^{n+j} \log \tau\bigr), & n \text{ even}. \end{cases}$$

Moreover, there is a bijection between initial data (u_0, u_1) and the leading order coefficients (A_0, B_0).

This can easily be generalized to more general metrics g. (One can also include lower order terms in the wave operator.) See [Vas10], where also the precise analytic structure of the map $(u_0, u_1) \mapsto (A_0, B_0)$ is described (it is a Fourier integral operator). In these lectures, we follow the presentation of [HX22, §2].

Proof of Theorem 3.6*.* The proof is very similar to that of Theorem 3.6, and hence we shall be brief. Defining the energy

$$E(\tau) := \frac{1}{2} \int_{\mathbb{T}^n} |\tau\partial_\tau u(\tau, x)|^2 + |\tau\partial_x u(\tau, x)|^2 \, \mathrm{d}x,$$

one finds by direct differentiation and usage of the wave equation for u that

$$-\tau \frac{\mathrm{d}}{\mathrm{d}\tau} E(\tau) \leq 0.$$

Furthermore, one can compute $u(\tau, x)$ in terms of $u(1, x) = u_0(x)$ by integrating $\tau\partial_\tau u$. In this manner, one finds

$$\iint_{(0,1]\times\mathbb{T}^n} \tau^{-2\alpha} \bigl(|u|^2 + |\tau\partial_\tau u|^2 + |\tau\partial_x u|^2\bigr) \frac{\mathrm{d}\tau}{\tau} \, \mathrm{d}x \leq C$$

for any $\alpha < 0$, where C only depends on $\|u_0\|_{H^1}$ and $\|u_1\|_{L^2}$ (and α).

Next, since $[\Box, \partial_x] = 0$, one obtains the same estimate for $\partial_x^\beta u$, $\beta \in \mathbb{N}_0^n$. More generally, since $(\tau\partial_\tau)^2 u = \Box_g u + n\tau\partial_\tau u + (\tau\partial_x)^2 u$, one obtains the same estimate for $(\tau\partial_\tau)^i \partial_x^\beta u$ for all $i \in \mathbb{N}_0$, $\beta \in \mathbb{N}_0^n$. That is,

$$\|u\|_{H_{\mathrm{b}}^{s,\alpha}(M)} := \sum_{i+|\beta|\leq s} \iint_{(0,1]\times\mathbb{T}^n} \tau^{-2\alpha} |(\tau\partial_\tau)^i \partial_x^\beta u|^2 \frac{\mathrm{d}\tau}{\tau} \, \mathrm{d}x < \infty$$

for all $s \in \mathbb{N}_0$. Sobolev embedding gives

$$u \in \mathcal{A}^\alpha(M), \qquad \alpha < 0.$$

The asymptotic expansion of u can be extracted by inverting $N(\Box_g, \lambda) = \lambda(\lambda - n)$ (with poles at $\lambda = 0, n$). The appearance of $\log \tau$ terms for $n = 2k$ even is due to an 'integer coincidence': the leading order term $A_0\tau^0 = A_0(x)$ produces error terms (via $(\tau\partial_x)^2$) with two, four, six, etc. more powers of τ in the second, third, fourth, etc. step in the extraction of the expansion. But since $2k$ is an indicial root, the solution of $((\tau\partial_\tau)^2 - 2k\tau\partial_\tau)v = \tau^{2k}\tilde{A}_k(x)$ in the k-th step has a logarithmic singularity $\sim \tau^{2k} \log \tau$. The reader is asked to fill in the details in Problem 3.9. □

3.4. **Problems.**

Problem 3.1 (Semi-Fredholm estimates)**.** Let X, Y, Z be Banach spaces, $P \in L(X, Y)$, and suppose $R\colon X \to Z$ is a compact operator. Suppose that there exists a constant C so that

$$\|Px\|_Y \leq C\big(\|x\|_X + \|Rx\|_Z\big).$$

(1) Show that $\dim \ker P < \infty$.
(2) Denote by $V \subset X$ a topological complement of $\ker P$. Show that there exists a constant $C' < \infty$ so that $\|Px\|_Y \leq C'\|x\|_X$ for all $x \in V$. Deduce that $\operatorname{ran} P \subset Y$ is closed.

Problem 3.2 (Failure of Fredholmness)**.** Let $s \geq 3$. Show that the operator $\Delta\colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^3}) \to H_{\mathrm{b}}^{s-2,\alpha+2}(\overline{\mathbb{R}^3})$ is *not* Fredholm when $\alpha \in \frac{1}{2} + \mathbb{Z}$. *Hint.* What properties does the operator family $\langle x\rangle^{2+\alpha}\Delta\langle x\rangle^{-\alpha} \in L(H_{\mathrm{b}}^s, H_{\mathrm{b}}^{s-2})$ have?

Problem 3.3 (Lower regularity)**.** Show that the Fredholm property in Theorem 3.1 and Problem 3.2 holds also for $s = 2$.

Problem 3.4 (General dimensions)**.** Let $n \geq 1$. Find the set of weights $\alpha \in \mathbb{R}$ for which $\Delta\colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n}) \to H_{\mathrm{b}}^{s-2,\alpha+2}(\overline{\mathbb{R}^n})$ (with $s \geq 3$) is Fredholm. For which α is it injective, for which surjective? Find the set of weights (depending on n) for which $\Delta\colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n}) \to H_{\mathrm{b}}^{s-2,\alpha+2}(\overline{\mathbb{R}^n})$ is an isomorphism.

Problem 3.5 (Inverse square potentials)**.** Let $V \in \langle x\rangle^{-2}\mathcal{C}^\infty(\overline{\mathbb{R}^n})$. Prove that the operator $\Delta + V\colon H_{\mathrm{b}}^{s,\alpha}(\overline{\mathbb{R}^n}) \to H_{\mathrm{b}}^{s-2,\alpha+2}(\overline{\mathbb{R}^n})$ is Fredholm for all $\alpha \in \mathbb{R}$ which avoid the discrete set $\{\operatorname{Re}\lambda\colon \lambda \in \operatorname{spec}_{\mathrm{b}}(\Delta + V)\}$.

Problem 3.6 (Fredholm theory for elliptic b-differential operators)**.** Prove Theorem 3.3.

Problem 3.7 (Polyhomogeneity)**.** Fill in the details in the proof of Proposition 3.4. Prove the polyhomogeneity of u when $\Delta u = f$ where $f \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}(\overline{\mathbb{R}^3})$ is itself polyhomogeneous.

Problem 3.8 (Scattering data for the wave equation in the interior of a Schwarzschild black hole)**.** Show that there is a linear bijection between initial data $(u_0, u_1) \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}_t \times \mathbb{S}^2) \oplus \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}_t \times \mathbb{S}^2)$ for the wave equation (3.5) and leading order coefficients $(A_0, B_0) \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}_t \times \mathbb{S}^2) \oplus \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}_t \times \mathbb{S}^2)$. *Hint.* Show that if u is of the form (3.6), then A_j, B_j for $j \geq 1$ are uniquely determined by A_0, B_0. Show moreover that if the expansion (3.6) holds with $A_0 = B_0$ (and thus $A_j = B_j = 0$ for all j, so $u \in \dot{\mathcal{C}}^\infty(M)$), then u vanishes identically; to do this, prove an energy estimate $r\frac{\mathrm{d}}{\mathrm{d}r}E(r) \leq CE(r)$ (which thus estimates the growth of energy in the direction of increasing r, i.e. in the backward time direction).

Problem 3.9 (Wave equation on a de Sitter like space)**.** Fill in the details of the proof of Theorem 3.6. Prove also the statement about the bijection between initial data (u_0, u_1) and scattering data (A_0, B_0).

4. Manifolds with corners and blow-ups

In these lectures, we will only encounter manifolds with codimension 2 corners. In general, points on an n-dimensional manifold with corners come in two flavors: points in the manifold interior M°, with open neighborhoods that are diffeomorphic to $\mathbb{R}^n$ as usual;

and points on the manifold boundary ∂M. We require that for each $p \in \partial M$ there exists a number $k \in \mathbb{N}$ so that a neighborhood of p is diffeomorphic to

$$\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}.$$

The closure of the set of points in ∂M where k is a fixed number is denoted $\partial_k M$. Thus, $\partial M = \partial_1 M$, while $\partial_2 M$ is the closure of the set of points which lie in a codimension 2 corner, and so on. A *boundary hypersurface* is the closure of a connected component of the set of points with $k = 1$. We require that all boundary hypersurfaces of M are embedded submanifolds. See Figure 4.1.

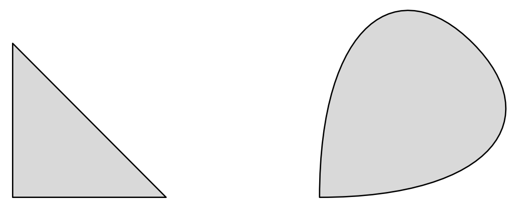

FIGURE 4.1. *On the left:* a manifold with corners. *On the right:* a space which has the correct local structure, but a non-embedded boundary hypersurface. We do *not* regard this as a manifold with corners.

Manifolds with boundary are special cases, with $\partial_j M = \emptyset$ for $j \geq 2$. The space $\mathbb{R}^n_k$ itself is of course a manifold with corners; writing the standard coordinates as $x^1, \ldots, x^k \geq 0$, $y^1, \ldots, y^{n-k} \in \mathbb{R}$, the boundary hypersurfaces are $\{x^j = 0\}$ for $j = 1, \ldots, k$. Another simple example is the cube $[0,1]^n$.

Definition 4.1 (Boundary defining functions)**.** If $H \subset M$ is a boundary hypersurface, then a smooth nonnegative function $\rho \in \mathcal{C}^\infty(M)$ is a *defining function of H* if $H = \rho^{-1}(0)$ and $\mathrm{d}\rho(p) \neq 0$ for all $p \in H$.

Example 4.2 (Boundary defining function, I). A defining function of $x^1 = 0$ in $\mathbb{R}^n_k$, $k \geq 1$, is x^1.

In §1.2, we encountered the idea of blowing up a submanifold (in that case, a 0-dimensional one consisting of just a single point) of a given manifold in order to desingularize a PDE (and thus the behavior of its solutions). We proceed to describe this in general.

Definition 4.3 (Blow-up of the origin in $\mathbb{R}^n$)**.** Let $n \geq 1$. Then $[\mathbb{R}^n; \{0\}]$ is defined to be the manifold with boundary

$$[\mathbb{R}^n; \{0\}] := [0,\infty) \times \mathbb{S}^{n-1},$$

together with the *blow-down map* $\beta \colon [0,\infty) \times \mathbb{S}^{n-1} \ni (r,\omega) \mapsto r\omega$. We call $\{0\} \times \mathbb{S}^{n-1}$ the *front face* of the blow-up, written $\mathrm{ff}[\mathbb{R}^n; \{0\}]$.

Thus β restricts to the complement of the front face to a diffeomorphism

$$[\mathbb{R}^n; \{0\}] \setminus \mathrm{ff}[\mathbb{R}^n; \{0\}] \xrightarrow{\cong} \mathbb{R}^n \setminus \{0\}.$$

In this sense, the manifolds $\mathbb{R}^n$ and $[\mathbb{R}^n; \{0\}]$ are the same *except* $\{0\}$ is replaced by a copy of $\mathbb{S}^{n-1}$.

Example 4.4 (Boundary defining function, II). A defining function of the front face of $[\mathbb{R}^n; \{0\}]$ is the polar coordinate r.

Example 4.5 (Trading analytic for geometric complexity)*.* In the spirit of Theme II, consider on $\mathbb{R}^2_{x,y}$ the PDE

$$(x\partial_x + y\partial_y)u(x,y) = 0, \qquad u|_{\mathbb{S}^1} = x, \tag{4.1}$$

where $\mathbb{S}^1 = \{x^2+y^2=1\}$ is the unit circle. The solution is given by

$$u(x,y) = \frac{x}{(x^2+y^2)^{1/2}}$$

and thus has a rather complicated singularity at $(x,y)=(0,0)$. For instance, u is constant along $(r\cos\theta, r\sin\theta)$, $r>0$, for fixed θ, but all these rays intersect at the origin. If, however, we instead work on $[\mathbb{R}^2;\{(0,0)\}] = [0,\infty)\times\mathbb{S}^1$, then the PDE becomes

$$r\partial_r u(r,\theta) = 0, \qquad u(1,\theta) = \cos\theta,$$

and its solution is $u(r,\theta)=\cos\theta$. This is smooth on $[\mathbb{R}^2;\{(0,0)\}]$.[22]

It is often computationally convenient to work with projective coordinates. For example, in the subset of $\mathbb{R}^n$ where $x^1 > c\max_{j\neq 1}|x^j|$, $c>0$, the functions

$$\rho := x^1, \qquad \hat x^j = \frac{x^j}{x^1} \quad (j=2,\ldots,n) \tag{4.2}$$

extend to a smooth local coordinate system $[0,1)_\rho \times B_{\mathbb{R}^{n-1}}(0,c^{-1})$ on $[\mathbb{R}^n;\{0\}]$. See Figure 4.2.

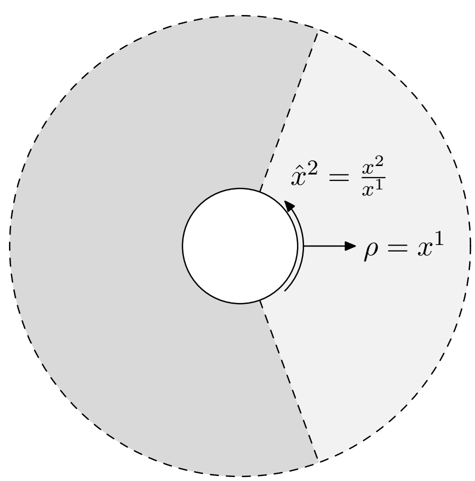

FIGURE 4.2. Illustration of the blow-up $[\mathbb{R}^2;\{0\}]$, together with local coordinates (4.2).

Lemma 4.6 (Lifts of invertible maps)**.** *Let $A\in GL(n,\mathbb{R})$. Then $A\colon \mathbb{R}^n\setminus\{0\}\to\mathbb{R}^n\setminus\{0\}$ extends by continuity to a diffeomorphism of $[\mathbb{R}^n;\{0\}]$. More generally, if $\phi\colon\mathbb{R}^n\to\mathbb{R}^n$ is a diffeomorphism with $\phi(0)=0$, then ϕ extends from $\mathbb{R}^n\setminus\{0\}$ to a diffeomorphism of $[\mathbb{R}^n;\{0\}]$.*

[22] In this particular example, where we blow up a point in the interior of the manifold $\mathbb{R}^2$ on which the PDE is posed, there is a small but occasionally important technical caveat: the original PDE (4.1) makes sense on all of $\mathbb{R}^2$ in the sense of distributions. Working on $[\mathbb{R}^2;\{(0,0)\}]$ on the other hand, the standard distributional formulation of the PDE $r\partial_r u=0$ involves integration against test functions which are smooth and compactly supported in the manifold interior $\mathbb{R}^2\setminus\{(0,0)\}$ of $[\mathbb{R}^2;\{(0,0)\}]$. Thus, upon passage to the blown-up manifold, one loses information about possible (differentiated) δ-distributions supported at the point $(0,0)$ one blows up.

Proof. See Problem 4.1. □

The first part of this lemma implies that the blow-up of the origin of a finite-dimensional vector space is well-defined. The second part allows us to define blow-ups of points p in a smooth n-dimensional manifold M: we define

$$[M;\{p\}] := (M \setminus \{p\}) \sqcup \mathbb{S}^{n-1}$$

as a smooth manifold with boundary $\mathbb{S}^{n-1}$ in the following way: if $\phi\colon \mathbb{R}^n \to U$ is a diffeomorphism to a neighborhood $U \subset M$ of p, then ϕ extends by continuity from $\mathbb{R}^n \setminus \{0\} \to U \setminus \{p\}$ to a diffeomorphism $[\mathbb{R}^n;\{0\}] \to (U \setminus \{p\}) \cup \mathbb{S}^{n-1}$.[23] (Lemma 4.6 ensures that the smooth structure of $[M;\{p\}]$ does not depend on the choice of ϕ.) Thus $\mathbb{S}^{n-1}$ is the front face, and the blow-down map

$$\beta\colon [M;\{p\}] \to M$$

is the identity on $M \setminus \{p\}$ and the map $\mathbb{S}^{n-1} \to \{p\}$ on the front face.

If $S \subset M$ is any subset, then the *lift of S* is defined as

$$\beta^* S = \begin{cases} \beta^{-1}(S), & S \subset \{p\}, \\ \overline{\beta^{-1}(S \setminus \{p\})}, & S \not\subset \{p\}. \end{cases}$$

The blow-up procedure can be generalized in three successive ways.

(1) Blow up $\{0\} \subset \mathbb{R}^n_k$: this is the lift of $\mathbb{R}^n_k$ to $[\mathbb{R}^n;\{0\}]$. For example, $[\mathbb{R}^2_2;\{0\}] = [[0,\infty)^2;\{0\}] = [0,\infty)_r \times [0,\frac{\pi}{2}]_\theta$, with blow-down map $(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$. One can then also define the blow-up of points in the boundary of a manifold with corners.
(2) Blow up *p-submanifolds* of a manifold with corners: these are submanifolds $X \subset M$ so that for each point $p \in X$ there exist local coordinates $x^1,\ldots,x^k \geq 0$, $y^1,\ldots,y^{n-k} \in \mathbb{R}$ on M near p so that $X = \{x^1 = \ldots = x^p = 0,\ y^1 = \ldots = y^q = 0\}$ for some $0 \leq p \leq k$, $0 \leq q \leq n-k$. In such coordinates, the blow-up $[M;X]$ is then given locally by $([[0,\infty)^p;\{0\}] \times [0,\infty)^{k-p}) \times ([\mathbb{R}^q;\{0\}] \times \mathbb{R}^{n-k-q})$.
(3) Iterated blow-ups: if $S,T \subset M$ are two p-submanifolds so that the lift β^*T of T to $[M;S]$ is a p-submanifold, then one can define $[M;S;T] := [[M;S];\beta^*T]$. This can be iterated to arbitrary depths. We will not need such iterated blow-ups here, but they feature frequently in the resolution of the compactified manifold underlying PDEs with complicated (iterated) singular behavior.

As a special case, we can blow up points in the boundary of a manifold with corners.

Example 4.7 (Blowing up the origin in the half space). Let $n \geq 1$ and consider the space

$$M := [\mathbb{R}^n_x \times [0,\infty)_s; \{(0,0)\}].$$

(See Figure 4.3 for the case $n = 1$.) We proceed to describe its boundary hypersurfaces.

(1) The front face ff of M is the closed hemisphere $\mathbb{S}^n_1 := \mathbb{S}^n \cap \mathbb{R}^{n+1}_1$. A more useful way to think about ff is as follows: as local coordinates on M near the interior ff$^\circ$ of the front face, we may use the projective coordinates (cf. Problem 2.2)

$$s \geq 0, \qquad \hat{x} := \frac{x}{s} \in \mathbb{R}^n.$$

[23] Simply put, the smooth structure of $[M;\{p\}]$ is obtained by regarding polar coordinates $r > 0$, $\omega \in \mathbb{S}^{n-1}$ around p as valid down to $r = 0$.

On the other hand, near the corner of M and where $x^1 > c\max_{j\neq 1}|x^j|$, we can use

$$\hat\rho := x^1, \qquad \hat\omega^j := \frac{x^j}{x^1} \quad (j=2,\ldots,n), \qquad \rho_\circ := \frac{s}{x^1}.$$

On ff (i.e. at $s=0$, resp. $\hat\rho=0$) and on the overlap of the two charts, we have the relationship

$$\hat\omega = \frac{\hat x^j}{\hat x^1}, \qquad \rho_\circ = \frac{1}{\hat x^1},$$

and $\hat x^1 > c\max_{j\neq 1}|\hat x^j|$. But this precisely means that $\hat\omega$, $\rho_\circ$ are themselves projective local coordinates near the boundary at infinity of the radial compactification of $\mathbb{R}^n_{\hat x}$! Therefore, we have shown that

$$\mathrm{ff} = \overline{\mathbb{R}^n_{\hat x}}.$$

(2) The lift H of the 'original boundary' $\mathbb{R}^n\times\{0\}$ is then $[\mathbb{R}^n;\{0\}]$ in the sense that the 'identity' map $\mathbb{R}^n\setminus\{0\}\to(\mathbb{R}^n\setminus\{0\})\times\{0\}\subset H$ extends by continuity to a diffeomorphism $[\mathbb{R}^n;\{0\}]\to H$. Indeed, this needs to be verified in a neighborhood $\mathcal{U}$ of the front face ff, with $\mathcal{U}$ diffeomorphic to $[0,\infty)\times\mathbb{S}^n_1$; and $H\cap\mathcal{U}$ is diffeomorphic to $[0,\infty)\times\mathbb{S}^{n-1}=[\mathbb{R}^n;\{0\}]$ where $\mathbb{S}^{n-1}\subset\mathbb{S}^n_1$ is the equator.

A defining function of ff is $\sqrt{|x|^2+s^2}$, and a defining function of H is $\frac{s}{(|x|^2+s^2)^{1/2}} = (1+|\hat x|^2)^{-1/2}$. See Problem 4.4 for a geometric generalization.

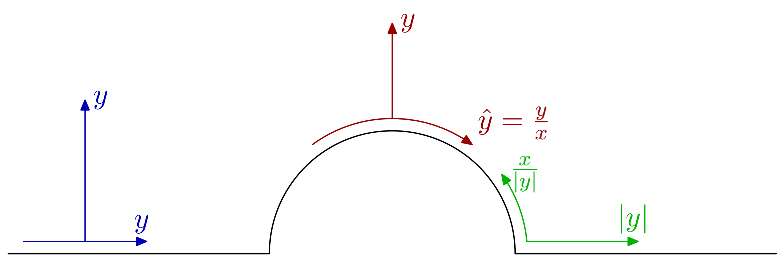

FIGURE 4.3. Blow-up of $\mathbb{R}_x\times[0,\infty)_y$ at $\{(0,0)\}$, with local coordinate charts.

On manifolds M with corners, one can again study b-vector fields and b-differential operators; here,

$$\mathcal{V}_\mathrm{b}(M) = \bigl\{V\in\mathcal{V}(M)\colon V \text{ is tangent to each boundary hypersurface of } M\bigr\}.$$

In local coordinates $x^1,\ldots,x^k\geq 0$ and $y^1,\ldots,y^{n-k}\in\mathbb{R}$ near a boundary point as above, b-vector fields are linear combinations, with smooth coefficients, of

$$x^i\partial_{x^i} \quad (i=1,\ldots,k), \qquad \partial_{y^j} \quad (j=1,\ldots,n-k).$$

4.1. **Problems.**

Problem 4.1 (Lifts of invertible maps)**.** Prove Lemma 4.6.

Problem 4.2 (Lifts of vector fields, I)**.** Let M be a smooth manifold, and let $p\in M$. Show that $\mathcal{V}_\mathrm{b}([M;\{p\}])$ is spanned over $\mathcal{C}^\infty([M;\{p\}])$ by the lifts of all smooth vector fields on M which vanish at p. (Here, the lift of $V\in\mathcal{V}(M)$ is simply defined to be V on the interior $M\setminus\{p\}$ of $[M;\{p\}]$. Convince yourself that this extends to a smooth vector field on $[M;\{p\}]$ if and only if $V(p)=0$.)

Problem 4.3 (Lifts of vector fields, II)**.** Let M be a smooth manifold with corners. Let $F \subset M$ be a *boundary face*, i.e. an intersection of boundary hypersurfaces.

(1) Show that the space of lifts of $\mathcal{V}_{\mathrm{b}}(M)$ to $[M; F]$ spans $\mathcal{V}_{\mathrm{b}}([M; F])$ over $\mathcal{C}^\infty([M; F])$.
(2) Conclude that $u\colon M^\circ \to \mathbb{C}$ is bounded conormal on M (i.e. $Pu \in L^\infty(M^\circ)$ for all $P \in \mathrm{Diff}_{\mathrm{b}}^m(M)$, $m \in \mathbb{N}_0$) if and only if u is bounded conormal on $[M; F]$.

Problem 4.4 (Blow-up of boundary points: geometric setting)**.** Let X be a smooth manifold without boundary, and let $p \in X$. Set $M = [[0, 1) \times X; \{(0, p)\}]$.

(1) Show that the lift of $\{0\} \times X$ to M is equal to $[X; \{p\}]$.
(2) Show that there exists a 'natural' diffeomorphism of the front face ff of M and the radial compactification[24] $\overline{T_p}X$ of T_pX. More precisely, show that the following map $\psi\colon T_pX \to \mathrm{ff}$ is well-defined and extends by continuity to a diffeomorphism $\overline{T_p}X \to \mathrm{ff}$: given $V \in T_pX$, let $\gamma\colon [0, 1] \to X$ be a smooth curve with $\gamma(0) = p$, $\gamma'(0) = V$, and define the smooth curve $(0, 1) \ni t \mapsto \tilde\gamma(t) = (t, \gamma(t)) \in M$; set then $\psi(V) := \lim_{t\searrow 0} \tilde\gamma(t) \in \mathrm{ff}$.

Problem 4.5 (Diffeomorphisms on manifolds with corners)**.** Prove the analogue of Problem 2.5 on manifolds with corners.

Problem 4.6 (One point compactification)**.** Recall the stereographic projection

$$\phi\colon \mathbb{S}^n \setminus \{N\} = \{(\omega', \omega_{n+1}) \in \mathbb{R}^n \times \mathbb{R}\colon |\omega'|^2 + \omega_{n+1}^2 = 1,\ (\omega', \omega_{n+1}) \neq N := (0, 1)\} \to \mathbb{R}^n,$$

$$\phi(\omega', \omega_{n+1}) = \frac{\omega'}{1 - \omega_{n+1}},$$

with inverse

$$\phi^{-1}(x) = \Big(\frac{2x}{|x|^2 + 1}, \frac{|x|^2 - 1}{|x|^2 + 1}\Big).$$

(1) Show that $X = \phi^*(\frac{x}{|x|^2})$ is a smooth $\mathbb{R}^n$-valued function on $\mathbb{S}^n$ near $N = (0, 1)$ which defines a coordinate system on $\mathbb{S}^n$ near N.
(2) Show that ϕ extends from $\mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ to a diffeomorphism $[\mathbb{S}^n; \{N\}] \to \overline{\mathbb{R}^n}$.
(3) Compute the form of $P := \Delta_{\mathbb{R}^n_x}$ in terms of X using polar coordinates $X = R\Omega$, $R = |X|$, $\Omega = \frac{X}{|X|} \in \mathbb{S}^{n-1}$. Show that $R^{-4}P$ is a smooth coefficient differential operator on $\mathbb{S}^n$ near N if and only if $n = 2$.[25] Show that $R^{-2}P$ is a smooth coefficient b-differential operator on $[\mathbb{S}^n; \{N\}]$ near the front face.

5. Applications: II

5.1. **The Laplacian with a singular potential.** As a very simple example of blow-ups, we consider the operator

$$\Delta - \frac{\mathsf{Z}}{|x|}$$

from §1.2 as an operator on $\mathbb{R}^3$. Blowing up the origin, this becomes

$$-\partial_r^2 - \frac{2}{r}\partial_r + r^{-2}\Delta_{\mathbb{S}^2} - \frac{\mathsf{Z}}{r}$$

[24] Since T_pX is a vector space, this is well-defined; the obvious notation would be $\overline{T_pX}$, but we write $\overline{T_p}X$ as it is more commonly used.

[25] This is related to the conformal invariance properties of the Laplace equation in 2 dimensions.

on $[\mathbb{R}^3;\{0\}]$, and r^2 times it is the elliptic b-differential operator

$$P = -(r\partial_r)^2 - r\partial_r + \Delta_{\mathbb{S}^2} - \mathsf{Z}r \in \mathrm{Diff}_{\mathrm{b}}^2([\mathbb{R}^3;\{0\}]).$$

The indicial family

$$N(P,\lambda) = -\lambda^2 - \lambda + \Delta_{\mathbb{S}^2} \tag{5.1}$$

is invertible unless $\lambda = -l-1, l$, $l \in \mathbb{N}_0$, in which case the kernel consists of spherical harmonics of degree l; one can thus show that conormal solutions of $Pu = 0$ or $Pu = f \in \dot{\mathcal{C}}^\infty([\mathbb{R}^3;\{0\}])$ are polyhomogeneous at $\mathrm{ff}[\mathbb{R}^3;\{0\}]$.

One can make this a bit more exciting by fixing $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^3)$, $\chi \geq 0$, to be equal to 1 near 0 and considering

$$Q = r^2\Big(\Delta - \frac{\mathsf{Z}}{|x|}\chi\Big) \in \mathrm{Diff}_{\mathrm{b}}^2\big([\overline{\mathbb{R}^3};\{0\}]\big).$$

Since $[\overline{\mathbb{R}^3};\{0\}]$ has two boundary hypersurfaces, $r = 0$ and $\rho = r^{-1} = 0$, the analysis of Q should take place on function spaces with two weights,

$$H_{\mathrm{b}}^{s,\alpha_0,\alpha_\infty}([\overline{\mathbb{R}^3};\{0\}]) := \rho_0^{\alpha_0}\rho_\infty^{\alpha_\infty} H_{\mathrm{b}}^s([\overline{\mathbb{R}^3};\{0\}]), \qquad \rho_0 = \frac{r}{r+1},\ \rho_\infty = \frac{1}{1+r},$$

where we use the Euclidean volume density to define the underlying L^2-space.

Theorem 5.1 (Mapping properties). *Let $s \geq 3$ and $\alpha_0, \alpha_\infty \notin \frac12 + \mathbb{Z}$. Then*

$$\Delta - \frac{\mathsf{Z}}{|x|}\chi\colon H_{\mathrm{b}}^{s,\alpha_0,\alpha_\infty}([\overline{\mathbb{R}^3};\{0\}]) \to H_{\mathrm{b}}^{s-2,\alpha_0-2,\alpha_\infty+2}([\overline{\mathbb{R}^3};\{0\}])$$

is Fredholm. If $\mathsf{Z} < 0$, $\alpha_0 \in (\frac12, \frac32)$, and $\alpha_\infty \in (-\frac32, -\frac12)$, it is invertible.

Proof. The proof combines elliptic estimates on the one hand and the inversion of the indicial families $-\lambda^2 - \lambda + \Delta_{\mathbb{S}^2}$ at $\mathrm{ff}[\overline{\mathbb{R}^3};\{0\}]$ (with $\operatorname{Re}\lambda = \frac32 + \alpha_0$) and $-\lambda^2 + \lambda + \Delta_{\mathbb{S}^2}$ at $\partial\overline{\mathbb{R}^3}$ (with $\operatorname{Re}\lambda = -\frac32 + \alpha_\infty$ as in the proof of Theorem 3.1) on the other hand.

The injectivity for $\mathsf{Z} < 0$ and $\alpha_0 > \frac12$, $\alpha_\infty > -\frac32$ follows from the fact that u in the kernel of $\Delta - \frac{\mathsf{Z}}{|x|}\chi$ lies in $\rho_0^{\alpha_0-\frac32}\rho_\infty^{\alpha_\infty+\frac32}\mathcal{A}^{0,0}([\overline{\mathbb{R}^3};\{0\}])$ (where $\mathcal{A}^{0,0}$ denotes bounded conormal functions on $[\overline{\mathbb{R}^3};\{0\}]$). This suffices to justify integration by parts in the L^2-pairing

$$0 = \Big\langle \Big(\Delta - \frac{\mathsf{Z}}{|x|}\chi\Big)u, u\Big\rangle = \|\nabla u\|^2 + \bigg\|\Big(\frac{-\mathsf{Z}}{|x|}\chi\Big)^{1/2}u\bigg\|^2,$$

which gives $u = 0$. The surjectivity for $\mathsf{Z} < 0$ and $\alpha_0 < \frac32$, $\alpha_\infty < -\frac12$, follows by duality.

The reader is asked to fill in the details in Problem 5.1. $\square$

5.2. **The Laplace equation on polygonal domains.** Let $\mathcal{M} \subset \mathbb{R}^2$ be a (connected) bounded polygonal domain with vertices $p_1, \ldots, p_N \in \mathcal{M}$ and edges $e_i = \overline{p_i p_{i+1}}$ where $p_{N+1} := p_1$. That is, all points in $\mathcal{M}$ come in three types:

(1) points in $\mathcal{M}^\circ$ which have neighborhoods in $\mathcal{M}$ diffeomorphic to $\mathbb{R}^2$;
(2) points $p \in \bigcup e_i^\circ$ (where e_i° is the interior of e_i) which have neighborhoods $\mathcal{U}$ so that $\mathcal{M} \cap \mathcal{U} = H \cap \mathcal{U}$ where $H \subset \mathbb{R}^2$ is a half space; and
(3) the points p_j, $j = 1, \ldots, N$, which have neighborhoods $\mathcal{U}$ so that $\mathcal{M} \cap \mathcal{U} = \mathcal{U} \cap \{p_j + r(\cos\theta, \sin\theta) \colon \theta_j \leq \theta \leq \theta_j + \alpha_j\}$ for some $\theta_j \in \mathbb{R}$ and $\alpha_j \in (0, 2\pi)$.

The numbers α_j are the interior angles of $\mathcal{M}$ at p_i.

Lemma 5.2 (Polygonal domains as manifolds with corners)**.** (1) *$\mathcal{M}$ is a manifold with corners if and only if all cone angles α_j, $j = 1, \ldots, N$, satisfy $\alpha_j \in (0, \pi]$.*

(2) *The lift M of $\mathcal{M}$ to $[\mathbb{R}^2; \{p_1, \ldots, p_N\}]$ is a manifold with corners (no matter what the cone angles are).*

See Figure 5.1.

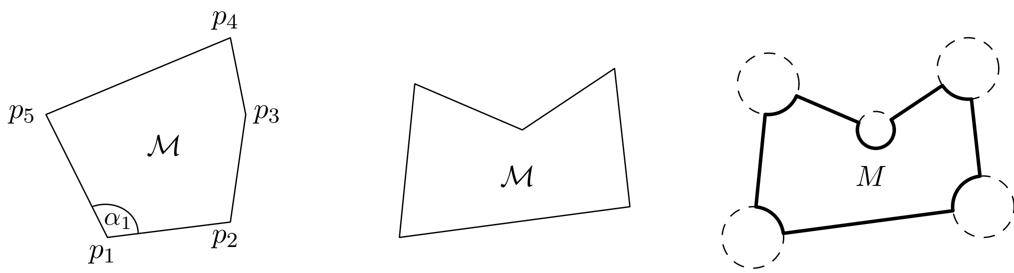

FIGURE 5.1. *On the left:* a polygonal domain with cone angles $< \pi$. *In the middle:* a polygonal domain $\mathcal{M}$ with one cone angle $> \pi$ (not a manifold with corners). *On the right:* the lift M of $\mathcal{M}$ to blow-up of $\mathbb{R}^2$ at the vertices.

Proof of Lemma 5.2. For part (1), the main observation is that the map $(x, y) \mapsto (x - (\cot\alpha)y, y)$ is a diffeomorphism

$$\{(r\cos\theta, r\sin\theta)\colon r \geq 0,\ 0 \leq \theta \leq \alpha\} \to [0, \infty)^2.$$

This is well-defined for $\alpha \in (0, \pi)$. (Vertices with interior angles of π can be discarded of course.)

For part (2), one observes that near the lift of $\{p_j\}$, M is diffeomorphic to a neighborhood of $\{0\} \times [0, \alpha_j]$ inside $[0, \infty) \times [0, \alpha_j]$ via the lift of $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. □

Theorem 5.3 (Full asymptotics for the Laplace equation on polygonal domains)**.** *Let $\mathcal{M}$ be a bounded polygonal domain with vertices $p_1, \ldots, p_N$ and interior angles $\alpha_1, \ldots, \alpha_N \in (0, 2\pi)$. Let M be the lift of $\mathcal{M}$ to $[\mathbb{R}^2; \{p_1, \ldots, p_N\}]$. Suppose $f \in \dot{\mathcal{C}}^\infty(\mathcal{M})$, i.e. f is the restriction to $\mathcal{M}$ of a smooth function which vanishes to infinite order at $\partial\mathcal{M}$.*[26] *Let*[27] *$u \in H_0^1(\mathcal{M}^\circ)$ be the unique weak solution of $\Delta u = f$, i.e. $\int_{\mathcal{M}^\circ} \nabla u \cdot \nabla\phi \,\mathrm{d}x = \int_{\mathcal{M}^\circ} f\phi \,\mathrm{d}x$ for all $\phi \in \mathcal{C}_\mathrm{c}^\infty(\mathcal{M}^\circ)$. Then the lift of u to M is smooth at the lift of $\partial\mathcal{M}$ and polyhomogeneous at the lift ff_j of $\{p_j\}$. More precisely, in local coordinates $r \geq 0$ and $\theta \in [0, \alpha_j]$ near p_j*[28] *there exist $u_{j,k} \in \mathcal{C}^\infty([0, \alpha_j])$, $k \in \mathbb{N}$, with $u_{j,k}(0) = u_{j,k}(\alpha_j) = 0$ so that*

$$u\big(p_j + (r\cos(\theta_j + \theta), r\sin(\theta_j + \theta))\big) \sim \sum_{k\in\mathbb{N}} c_{k,j} r^{\frac{k\pi}{\alpha_j}} \sin\Big(\frac{k\pi}{\alpha_j}\theta\Big), \qquad r \searrow 0, \tag{5.2}$$

for some constants $c_{j,k}$.

[26] All elements of $\mathcal{C}_\mathrm{c}^\infty(\mathcal{M}^\circ)$ satisfy this property.

[27] Recall that $H_0^1(\mathcal{M}^\circ)$ is the closure of $\mathcal{C}_\mathrm{c}^\infty(\mathcal{M}^\circ)$ in the norm $\|u\|_{H_0^1(\mathcal{M}^\circ)} := \|\nabla u\|_{L^2}$.

[28] By this we mean that $\{p_j + (r\cos(\theta_j + \theta), r\sin(\theta_j + \theta))\colon 0 \leq r < r_j,\ \theta \in [0, \alpha_j]\}$ is an open neighborhood of p_j in M for some $\theta_j \in \mathbb{R}$ and some small $r_j > 0$.

The reader is tasked with generalizing this to the case that the edges are not straight lines anymore in Problem 5.6. In the right choice of local coordinates near the vertices, the result is 'the same' except for the description of the coefficients $u_{j,k}$ of the polyhomogeneous expansion and the possibility of additional factors of $(\log r)^l$, $0 \leq l \leq l_{\max}(j,k)$.

Proof of Theorem 5.3. We first motivate the form of the result by working in local polar coordinates $r \geq 0$ and $\theta \in [0, \alpha_j]$ near the boundary hypersurface $\mathrm{ff}_j \subset M$. In these coordinates,

$$-r^2\Delta = -r^2\Big(-\partial_r^2 - \frac{1}{r}\partial_r - r^{-2}\partial_\theta^2\Big) = (r\partial_r)^2 + \partial_\theta^2.$$

The indicial family is $N(-r^2\Delta, \lambda) = \lambda^2 + \partial_\theta^2 \in \mathrm{Diff}^2([0, \alpha_j])$. Since we are considering the Dirichlet problem, we only study the action of $N(-r^2\Delta, \lambda)$ on functions $v(\theta)$ vanishing at 0 and α_j; but then $N(-r^2\Delta, \lambda)v = 0$ and $v(0) = v(\alpha_j) = 0$ requires v to be a multiple of $\sin(\lambda\theta)$, with $\lambda\alpha_j \in \mathbb{Z}\pi$, so $\lambda = \frac{k\pi}{\alpha_j}$ for some $k \in \mathbb{Z}$. Therefore, $N(-r^2\Delta, \lambda)$ has a nontrivial nullspace only for $\lambda = \frac{k\pi}{\alpha_j}$ for $k \in \mathbb{Z} \setminus \{0\}$, spanned by $\sin(\frac{k\pi}{\alpha_j}\theta)$. The corresponding (local) solution

$$r^{\frac{k\pi}{\alpha_j}} \sin\Big(\frac{k\pi}{\alpha_j}\theta\Big) \in \ker\Delta$$

lies in $L^2((0,1)_r \times [0, \alpha_j)_\theta, r|\mathrm{d}r\,\mathrm{d}\theta|)$ provided $\frac{2k\pi}{\alpha_j} > -2$, and its derivative along ∂_r lies in L^2 iff $2(\frac{k\pi}{\alpha_j} - 1) > -2$, i.e. $k > 0$.

In order to show that u indeed has the structure (5.2), it suffices to show that $V_1 \cdots V_I u \in L^2(\mathcal{M}^\circ)$ for all $I \in \mathbb{N}$ and smooth vector fields $V_i \in \mathcal{V}(M)$ which are tangent to all ff_j: this says that u is conormal at each ff_j and smooth at the lift of $\partial\mathcal{M}$, and then the standard Mellin transform argument produces the asymptotic expansion of u. (Since $-r^2\Delta$ is already dilation-invariant in r, i.e. equal to $N(-r^2\Delta)$, the terms in the expansion lie in $\ker N(-r^2\Delta, \lambda)$.)

One can further simplify the task: we claim that it suffices to show

$$V_1 \cdots V_I u \in H_0^1(\mathcal{M}^\circ) \quad \forall\, I \in \mathbb{N},\ V_1, \ldots, V_I \in \mathcal{V}_{\mathrm{b}}(M). \tag{5.3}$$

This follows from a standard trick in the regularity theory for the Dirichlet problem: tangential regularity implies normal regularity since $\Delta u = 0$ roughly speaking expresses two normal derivatives of u in terms of two tangential derivatives. Concretely, in the coordinates $r \geq 0$, $\theta \in [0, \alpha_j]$ as above, and near $\theta = 0$,

$$\dot{\mathcal{C}}^\infty(\mathcal{M}) \ni -r^2 f = -r^2\Delta u = \partial_\theta^2 u + (r\partial_r)^2 u.$$

Thus, if $(r\partial_r)^2 u \in H_0^1$ (locally near the vertex), then $\partial_\theta^2 u \in H_0^1$. Higher order normal derivatives are controlled by applying (powers of) ∂_θ to $\Delta u = f$.

Finally, in order to prove (5.3), recall that

$$\Delta\colon H_0^1(\mathcal{M}^\circ) \to H^{-1}(\mathcal{M}^\circ) := (H_0^1(\mathcal{M}^\circ))^*, \qquad (\Delta u)(v) = \langle \nabla u, \nabla v\rangle_{L^2},$$

is an isomorphism. Given $V \in \mathcal{V}_{\mathrm{b}}(M)$, the idea is to approximate Vu by difference quotients $D_h u$, $0 < h < 1$, which gives $\Delta D_h u = D_h f + [\Delta, D_h]u$. If $[\Delta, D_h]\colon H_0^1 \to H^{-1}$ is uniformly bounded, and $D_h f \to Vf$ in $\dot{\mathcal{C}}^\infty(M) = \dot{\mathcal{C}}^\infty(\mathcal{M})$, we find that $D_h u \in H_0^1$ is uniformly bounded and converges in distributions to Vu; therefore $Vu \in H_0^1$. Higher derivatives are handled inductively. It suffices to define D_h when V is supported near ff_j. Indeed we only need to consider $V = r\partial_r$, in which case one can take

$$(D_h u)(r,\theta) = h^{-1}\big(u(e^h r,\theta) - u(r,\theta)\big),$$

and $V = \theta\partial_\theta$ in $\theta < \frac{3}{4}\alpha_j$ (and similarly $V = (\alpha_j - \theta)\partial_\theta$ in $\theta > \frac{1}{4}\alpha_j$), in which case one can take

$$(D_h u)(r,\theta) = h^{-1}\big(u(r, e^h\theta) - u(r,\theta)\big).$$

We leave the detailed implementation of these arguments to the reader. □

5.3. **The wave equation on Minkowski space.** Perhaps somewhat surprisingly, geometric singular analysis perspectives on the wave equation on Minkowski space which are robust (i.e. works also for suitable perturbations) are rather nontrivial to find; see [Wan13, BVW15, HV20, HV23, Hin23] for several options. For a general perspective in the exterior region $r > t$, we refer the reader to [Hin23] (see in particular [Hin23, §1.2] for the main vector field multiplier in a basic energy estimate); see [HV23] for a microlocal treatment. Here, we shall merely work on *exact* Minkowski space in $(n+1)$ dimensions,

$$(\mathbb{R}_t \times \mathbb{R}^n_x, g), \qquad g = -\mathrm{d}t^2 + \sum_{j=1}^n (\mathrm{d}x^j)^2,$$

and study the solution u of the initial value problem

$$\Box_g u = \Big(\partial_t^2 - \sum_{j=1}^n \partial_{x^j}^2\Big)u = 0, \qquad (u, \partial_t u)|_{t=0} = (u_0, u_1), \tag{5.4}$$

to verify that a certain compactification of $\mathbb{R}^{n+1}$ is the 'correct' one on which u is polyhomogeneous when the Cauchy data u_0, u_1 are.

Consider first the radial compactification $\overline{\mathbb{R}^{n+1}}$ of $\mathbb{R}^{n+1}$, with boundary defining function $\varrho = (1 + t^2 + |x|^2)^{-1/2}$. This is defined directly by analogy with the case of the Laplace operator; and indeed one has $\Box_g \in \varrho^2 \mathrm{Diff}^2_{\mathrm{b}}(\overline{\mathbb{R}^{n+1}})$. However, solutions of $\Box_g u = 0$ are known to be asymptotic as $r = |x| \to \infty$ and $t - r = \mathcal{O}(1)$ to $r^{-1}F(t-r,\omega)$ where $\omega = \frac{x}{|x|} \in \mathbb{S}^{n-1}$, and F is the *radiation field*. However, if $\omega \in \mathbb{S}^{n-1}$ is fixed, the future null-geodesics $t - r = c$, $r \to \infty$, being parallel, limit to the same point in $\partial\overline{\mathbb{R}^{n+1}}$ depending only on ω but not on $c \in \mathbb{R}$. The set of all these points is the future 'light cone at infinity',

$$Y^+ := \{\varrho = 0,\ t/r = +1\} \subset \overline{\mathbb{R}^{n+1}}.$$

Local coordinates near it are $\rho = r^{-1}$, $v = \frac{t-r}{r}$, $\omega \in \mathbb{S}^{n-1}$. If we blow up Y (which is locally given by $\rho = v = 0$), then coordinates in the interior of the front face are $\frac{v}{\rho} = t - r$, i.e. one resolves the radiation field. Defining Y^- similarly, we thus set

$$M := \big[\overline{\mathbb{R}^{n+1}}; Y^- \cup Y^+\big]. \tag{5.5}$$

We write $\mathscr{I}^\pm$ (future and past null infinity) for the lift of $Y^\pm$. The lift of $\partial\overline{\mathbb{R}^{n+1}}$ has three connected components which we label I^-, I^0, I^+. For example, a chart on $\overline{\mathbb{R}^4}$ in a neighborhood of Y^+ is $[0,1) \times (-1,1) \times \mathbb{S}^{n-1}$ with coordinates $1/r$, $(r-t)/r$, $\omega \in \mathbb{S}^{n-1}$; therefore, a chart on M near $I^0 \cap \mathscr{I}^+$ is

$$[0,1)_{\rho_I} \times [0,1)_{\rho_0} \times \mathbb{S}^{n-1}, \qquad \rho_I := \frac{r-t}{r}, \quad \rho_0 := \frac{1/r}{(r-t)/r} = \frac{1}{r-t}. \tag{5.6}$$

See Figure 5.2.

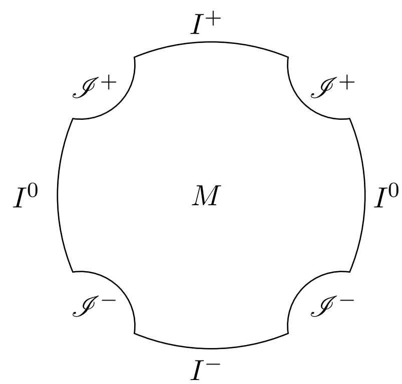

FIGURE 5.2. The compactification M of $\mathbb{R}^{n+1}$ (see (5.5)) on which solutions of the wave equation on Minkowski space are polyhomogeneous.

The compactification M has nice properties related to the Minkowski metric:

Proposition 5.4 (Poincaré group, scalings, and b-vector fields)**.** *Denote by $\Omega_{\mu\nu}$ ($0 \leq \mu, \nu \leq 3$) the generators of the Lorentz group,*

$$\Omega_{0j} = t\partial_{x^j} + x_j\partial_t \quad (1 \leq j \leq n), \qquad \Omega_{ij} = x^i\partial_{x^j} - x^j\partial_{x^i} \quad (1 \leq i \neq j \leq n).$$

Then the vector fields $\Omega_{\mu\nu}$ and the translation vector fields ∂_t, ∂_{x^j} ($1 \leq j \leq n$)—which taken together are the generators of the Poincaré group—together with the scaling vector field $S = t\partial_t + \sum_{j=1}^n x^j\partial_{x^j}$ span $\mathcal{V}_{\mathrm{b}}(M)$ over $\mathcal{C}^\infty(M)$.

Proof. We check this only near a point in $I^0 \cap \mathscr{I}^+$ using the coordinates (5.6).

The vector fields Ω_{ij} span $\mathcal{V}(\mathbb{S}^{n-1})$, and we have $S = t\partial_t + r\partial_r = -\rho_0\partial_{\rho_0}$. If we work near a point where $x^1 > c\max\{|x^2|, \ldots, |x^n|\}$, $c > 0$, we may use $\omega^k = \frac{x^k}{x^1}$, $2 \leq k \leq n$, as coordinates on $\mathbb{S}^{n-1}$. In the coordinates ρ_I, ρ_0, ω^k ($2 \leq k \leq n$), we then compute $r = (\rho_I\rho_0)^{-1}$, $t = \frac{1-\rho_I}{\rho_I\rho_0}$, and $\partial_{x^1} r = \frac{x^1}{r} = \langle\omega\rangle^{-1} := (1 + (\omega^2)^2 + (\omega^3)^2)^{-1/2}$, so $\frac{1}{x^1} = \rho_0\rho_I\langle\omega\rangle$, $\partial_{x^k} r = \frac{x^k}{r} = \frac{\omega^k}{\langle\omega\rangle}$ and $x^k = \frac{\omega^k}{\rho_0\rho_I\langle\omega\rangle}$ for $2 \leq k \leq n$, and

$$\begin{aligned}
\partial_t &= \rho_0(\rho_0\partial_{\rho_0} - \rho_I\partial_{\rho_I}), \\
\partial_{x^1} &= \frac{tx^1}{r^3}\partial_{\rho_I} - \frac{x^1}{r(r-t)^2}\partial_{\rho_0} - \frac{\omega^2}{x^1}\partial_{\omega^2} - \frac{\omega^3}{x^1}\partial_{\omega^3} \\
&= \rho_0\Big(\langle\omega\rangle^{-1}\big((1-\rho_I)\rho_I\partial_{\rho_I} - \rho_0\partial_{\rho_0}\big) - \rho_I\langle\omega\rangle(\omega^2\partial_{\omega^2} + \omega^3\partial_{\omega^3}\Big), \\
\partial_{x^k} &= \frac{tx^k}{r^3}\partial_{\rho_I} - \frac{x^k}{r(r-t)^2}\partial_{\rho_0} + \rho_0\rho_I\langle\omega\rangle\partial_{\omega^k} \\
&= \rho_0\Big(\langle\omega\rangle^{-1}\omega^k\big((1-\rho_I)\rho_I\partial_{\rho_I} - \rho_0\partial_{\rho_0}\big) + \rho_I\langle\omega\rangle\partial_{\omega^k}\Big),
\end{aligned}$$

where $2 \leq k \leq n$. This shows that ∂_t, ∂_{x^i} ($1 \leq i \leq n$) lie in $\mathcal{V}_{\mathrm{b}}(M)$; and moreover

$$\begin{aligned}
t\partial_{x^1} + x^1\partial_t &= \langle\omega\rangle^{-1}\big(\rho_0\partial_{\rho_0} - (2-\rho_I)\rho_I\partial_{\rho_I}\big) - (1-\rho_I)\langle\omega\rangle(\omega^2\partial_{\omega^2} + \omega^3\partial_{\omega^3}), \\
t\partial_{x^k} + x^k\partial_t &= -\langle\omega\rangle^{-1}\omega^k\big((1-\rho_I)\rho_I\partial_{\rho_I} - \rho_0\partial_{\rho_0}\big) + (1-\rho_I)\langle\omega\rangle\partial_{\omega^k},
\end{aligned}$$

so $\Omega_{0j} \in \mathcal{V}_{\mathrm{b}}(M)$ for $1 \leq j \leq n$, and $\rho_I \partial_{\rho_I}$ can be expressed as a linear combination with coefficients in $\mathcal{C}^\infty(M)$ of Ω_{01} and the vector fields $S = -\rho_0 \partial_{\rho_0}$ and $\partial_{\omega^k} \in \mathcal{V}(\mathbb{S}^{n-1})$, $2 \leq k \leq n$.

The verification near other points of M is left to the reader. $\square$

Since $[\Box_g, \Omega_{\mu\nu}] = 0$ and $[\Box_g, S] = 2\Box_g$, Proposition 5.4 implies that a bound for the solution u of the initial value problem for $\Box_g$ in any spacetime L^2-space[29] implies full b-regularity and thus conormality of u (and therefore pointwise estimates for u), albeit without further work with not particularly strong weights, if the initial data are conormal on $\overline{\mathbb{R}^3}$.

Corollary 5.5 (A variant of Klainerman–Sobolev embedding)**.** *(See* [Kla85] *or* [Sog95, Chapter II.1].[30]*) Let $\mathscr{V} = \{\Omega_{\mu\nu}, \partial_t, \partial_{x^j}\}$. Then for $t > 0$,*

$$(1 + |t| + |x|)^{\frac{n-1}{2}} (1 + |t - r|)^{\frac{1}{2}} |u(t, x)| \leq C \sum_{|\alpha| \leq \lceil \frac{n+2}{2} \rceil} \|\mathscr{V}^\alpha u\|_{(1+|t|+|x|)^{1/2} L^2} \tag{5.7}$$

for all u for which the right hand side is finite.

Proof. In brief, this follows from Sobolev embedding for b-Sobolev spaces in view of Proposition 5.4; the weights in the estimate arise from the weights of the density $|\mathrm{d}t\,\mathrm{d}x|$ at the various boundary hypersurfaces from M (cf. w in Definition 2.13 and Proposition 2.14).

We makes this concrete near $I^0 \cap \mathscr{I}^+$ in the coordinates (5.6), so

$$|\mathrm{d}t\,\mathrm{d}x| = r^{n-1} |\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}g_{\mathbb{S}^{n-1}}| = \rho_0^{-n-1} \rho_I^{-n} \Big| \frac{\mathrm{d}\rho_I}{\rho_I} \frac{\mathrm{d}\rho_0}{\rho_0} \mathrm{d}g_{\mathbb{S}^{n-1}} \Big|.$$

Therefore, for $s > \frac{n+1}{2}$ (i.e. $s = \lceil \frac{n+2}{2} \rceil$ works), elements of $(1 + \langle t \rangle + \langle x \rangle)^{1/2} H^s_{\mathrm{b}}(M, |\mathrm{d}t\,\mathrm{d}x|)$ locally lie in $(1 + \langle t \rangle + \langle x \rangle)^{1/2} \rho_0^{\frac{n+1}{2}} \rho_I^{\frac{n}{2}} L^\infty$ by a simple generalization of Proposition 2.14. Since $(1 + \langle t \rangle + \langle x \rangle)^{1/2} \sim \rho_0^{-1/2} \rho_I^{-1/2}$, this gives (5.7). $\square$

Proposition 5.6 (Radiation field)**.** *Suppose $u_0, u_1 \in \mathscr{S}(\mathbb{R}^n) = \mathcal{A}^\emptyset_{\mathrm{phg}}(\overline{\mathbb{R}^n})$. Then the solution u of* (5.4)*, expressed in the coordinates* (5.6)*, is*

$$u = u(\rho_I, \rho_0, \omega) \sim \sum_{j=0}^\infty \rho_I^{\frac{n-1}{2}+j} u_j(\rho_0, \omega), \qquad u_j \in \dot{\mathcal{C}}^\infty([0,1)_{\rho_0} \times \mathbb{S}^{n-1}).$$

The radiation field *is* $(r^{\frac{n-1}{2}} u)|_{\mathscr{I}^+} = (r - t)^{\frac{n-1}{2}} u_1(\frac{1}{r-t}, \omega) \in \dot{\mathcal{C}}^\infty(\mathscr{I}^+ \cap \{\rho_0 < 1\})$.

Proof. Since $r = \rho_0^{-1} \rho_I^{-1}$ and

$$\partial_t = \rho_0(\rho_0 \partial_{\rho_0} - \rho_I \partial_{\rho_I}), \qquad \partial_r = \rho_0\big(-\rho_0 \partial_{\rho_0} + (1 - \rho_I)\rho_I \partial_{\rho_I}\big),$$

---

[29] Such a bound follows e.g. from something as simple as energy conservation.

[30] Compared to the references, we use a weighted spacetime L^2-norm on the right hand side here (allowing for $(1+|t|+|x|)^{1/2}$ growth relative to $L^2(\mathbb{R}^n, |\mathrm{d}t\,\mathrm{d}x|)$), instead of the $L^2(\mathbb{R}^n_x)$ norm on a fixed t-level set, and we use 1 derivative more on the right hand side when n is odd. One can apply this estimate to wu where w is a product of real powers of defining functions of the boundary hypersurfaces of M to get a weighted version.

one finds

$$\begin{aligned}\Box_g = \partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r + r^{-2}\Delta_{\mathbb{S}^{n-1}} &= P_0 + \tilde{P}, \\ P_0 &= -2\rho_0^2\rho_I(\rho_0\partial_{\rho_0} - \rho_I\partial_{\rho_I})\Bigl(\rho_I\partial_{\rho_I} - \frac{n-1}{2}\Bigr), \\ \tilde{P} &= -(\rho_0\rho_I^2\partial_{\rho_I})^2 + (n-1)\rho_0^2\rho_I^2\rho_I\partial_{\rho_I} + \rho_0^2\rho_I^2\Delta_{\mathbb{S}^{n-1}}.\end{aligned}$$

Thus $P_0 \in \rho_0^2\rho_I\mathrm{Diff}_{\mathrm{b}}^2([0,1)_{\rho_I}\times[0,1)_{\rho_0}\times\mathbb{S}^{n-1})$, while $\tilde{P} \in \rho_0^2\rho_I^2\mathrm{Diff}_{\mathrm{b}}^2$ is of lower order (in the sense of decay) as a b-differential operator. Similarly to the settings considered in §§3.2 and 3.3, the wave operator here is degenerate even as a (weighted) b-differential operator, but can nonetheless be studied using b-techniques. (For a nondegenerate perspective, see [HV23].) We only sketch the argument here.

- *Step 1. Energy estimate.* In this step, the term $\rho_0^2\rho_I\cdot\rho_I\Delta_{\mathbb{S}^{n-1}}$ in $\tilde{P}$ still plays a crucial role. Using a suitable vector field multiplier (essentially: $\rho_0^{\beta_0}\rho_I^{\beta_I}((1-c)\rho_0\partial_{\rho_0} - \rho_I\partial_{\rho_I})$, $0<c\ll 1$), one can prove an energy estimate

$$\|\rho_0\partial_{\rho_0}u\|_H^2 + \|\rho_I\partial_{\rho_I}u\|_H^2 + \|\rho_I^{1/2}\nabla^{\mathbb{S}^{n-1}}u\|_H^2 \le C, \tag{5.8}$$

where C depends only on the initial data, and

$$\|u\|_H := \|\rho_0^{-\alpha_0}\rho_I^{-\alpha_I}u\|_{L^2}$$

for any fixed[31] $\alpha_0\in\mathbb{R}$ and $\alpha_I < \min(\alpha_0+\frac12, -\frac12)$.

- *Step 2. Conormal regularity.* Commuting Poincaré vector fields and the scaling vector field through $\Box_g u = 0$, one obtains (5.8) also for all derivatives $\mathscr{V}^\alpha u$. Thus, one obtains full b-regularity, and by Sobolev embedding

$$u \in \rho_0^{\alpha_0+\frac{n+1}{2}}\rho_I^{\alpha_I+\frac{n}{2}}\mathcal{A}^{0,0},$$

where $\mathcal{A}^{0,0}$ denotes the space of bounded conormal functions on (5.6). Since α_0 is arbitrary, we obtain

$$u \in \rho_0^N\rho_I^{\frac{n-1}{2}-\epsilon}\mathcal{A}^{0,0}\quad \forall\,N\in\mathbb{R},\ \epsilon>0.$$

We remark that in this step one does not actually need the Poincaré invariance of $\Box_g$; it suffices to commute with b-vector fields which leave the structure of the leading order terms $P_0 + \rho_0^2\rho_I^2\Delta_{\mathbb{S}^{n-1}}$ intact (insofar as they play a role in the energy estimate (5.8)).

- *Step 3. Expansion at $\mathscr{I}^+$.* We now rewrite $0 = \Box_g u = P_0 u + \tilde{P} = 0$ as

$$(\rho_0\partial_{\rho_0} - \rho_I\partial_{\rho_I})\Bigl(\rho_I\partial_{\rho_I} - \frac{n-1}{2}\Bigr)u = \tilde{Q}u, \qquad \tilde{Q} = \frac12\rho_0^{-2}\rho_I^{-1}\tilde{P} \in \rho_I\mathrm{Diff}_{\mathrm{b}}^2. \tag{5.9}$$

Thus, if $u\in\rho_0^N\rho_I^{\frac{n-1}{2}-\epsilon}\mathcal{A}^{0,0}$, then $\tilde{Q}u \in \rho_0^N\rho_I^{\frac{n-1}{2}+1-\epsilon}\mathcal{A}^{0,0}$ has an extra order of vanishing at $\mathscr{I}^+ = \{\rho_I = 0\}$. One can integrate the vector field $\rho_0\partial_{\rho_0} - \rho_I\partial_{\rho_I}$ starting at the Cauchy surface $\rho_0 = 1$ to obtain

$$\Bigl(\rho_I\partial_{\rho_I} - \frac{n-1}{2}\Bigr)u \in \rho_0^N\rho_I^{\frac{n-1}{2}+1-\epsilon}\mathcal{A}^{0,0}.$$

Integrating the Fuchsian vector field $\rho_I\partial_{\rho_I} - \frac{n-1}{2}$ from $\rho_I = 1$ towards $\rho_I = 0$ gives

$$u = \rho_I^{\frac{n-1}{2}}u_0(\rho_0,\omega) + \tilde{u}, \qquad \tilde{u}\in\rho_0^N\rho_I^{\frac{n-1}{2}+1-\epsilon}\mathcal{A}^{0,0}$$

[31] If the initial data are not Schwartz, then there is an upper bound for α_0.

and thus produces the leading order term of u at $\mathscr{I}^+$. Plugging this improved information back into (5.9) and iterating the argument proves the result. □

5.4. Problems.

Problem 5.1 (Laplacian with potential)**.** Fill in the details of the proof of Theorem 5.1.

Problem 5.2 (Polygonal domains)**.** Fill in the details in the proof of Lemma 5.2.

Problem 5.3 (Regularity of solutions of the Laplace equation on polygonal domains)**.** Use the notation of Theorem 5.3; let $\alpha_{\max} = \max\{\alpha_j : j = 1, \ldots, N\}$. Let $k \in \mathbb{N}_0$, $\beta \in [0,1)$ with $\frac{\pi}{\alpha_{\max}} = k + \beta$ denote the integer and fractional parts of $\frac{\pi}{\alpha_j}$.

(1) Show that the solution $u \in H_0^1(\mathcal{M}^\circ)$ of $\Delta u = f \in \dot{\mathcal{C}}^\infty(\mathcal{M})$ lies in $\mathcal{C}^{k,\beta}(\mathcal{M})$. In particular, if all interior angles are $< \pi$, then $u \in \mathcal{C}^2(\mathcal{M})$ is a classical solution all the way down to $\partial\mathcal{M}$.
(2) Show that there exists $f \in \dot{\mathcal{C}}^\infty(\mathcal{M})$ so that $u \notin \mathcal{C}^{1,\beta+\epsilon}(\mathcal{M})$ for any $\epsilon > 0$. *Hint.* Fix $\chi \in \mathcal{C}^\infty([0, r_0))$ to be equal to 1 near 0, where r_0 is smaller than the distance from p_j to the next non-adjacent edge or vertex. Then $f_0 := \Delta(\chi(r) r^{\pi/\alpha_j} \sin(\frac{\pi}{\alpha_j}\theta)) = [\Delta, \chi](r^{\pi/\alpha_j} \sin(\frac{\pi}{\alpha_j}\theta))$ is supported near the smooth part of $\partial\mathcal{M}$; construct then (the Taylor series at $\partial\mathcal{M}$ of) a smooth function u_0 on $\mathcal{M}$ with support in $\operatorname{supp} \mathrm{d}\chi$ so that $f_0 - \Delta u_0 \in \dot{\mathcal{C}}^\infty(\mathcal{M})$, and conclude.

Problem 5.4 (Eigenfunctions on polygonal domains)**.** In the notation of Theorem 5.3, show that all Dirichlet eigenfunctions of Δ have a polyhomogeneous expansion at ff_j of the form

$$u \sim \sum r^{\frac{k\pi}{\alpha_j}} u_{j,k}(\theta), \qquad r \searrow 0,$$

where $u_{j,k}(\theta) = \sum_{l=0}^{\lfloor \frac{k}{2} \rfloor} c_{j,k,l} \sin(\frac{(k-2l)\pi}{\alpha_j}\theta)$ for some constants $c_{j,k,l}$.

Problem 5.5 (Curvilinear polygonal domains)**.** Generalize Lemma 5.2 to *curvilinear polygonal domains* $\mathcal{M} \subset \mathbb{R}^2$. That is, $\mathcal{M}$ is a bounded connected subset of $\mathbb{R}^2$, and every point in $\mathcal{M}$ is either

- an interior point (with a neighborhood in $\mathcal{M}$ that is diffeomorphic to $\mathbb{R}^2$),
- a boundary point (with a neighborhood in $\mathcal{M}$ that is diffeomorphic to the half space $[0,\infty) \times \mathbb{R}$), or
- a vertex (for which there exists a diffeomorphism $\phi \colon \mathcal{U} \cap \mathcal{M} \to \{r(\cos\theta, \sin\theta) : r \geq 0,\ 0 \leq \theta \leq \alpha\}$ which is defined on an open neighborhood $\mathcal{U} \subset \mathbb{R}^2$ of the vertex *in* $\mathbb{R}^2$, with α denoting the[32] interior angle at the vertex).

Problem 5.6 (Dirichlet problem on curvilinear polygonal domains)**.** Generalize Theorem 5.3 to the case of the Laplace equation $\Delta u = f \in \dot{\mathcal{C}}^\infty(\mathcal{M})$, $u|_{\partial\mathcal{M}} = 0$, when $\mathcal{M}$ is a curvilinear polygonal domain. *Hint.* The polyhomogeneous expansion near the front face ff_j corresponding to the vertex p_j should be expressed in the local coordinates which locally straighten out $\mathcal{M}$.

[32] The reader should convince themselves that α is well-defined.

Problem 5.7 (Dirichlet problem on the stadium)**.** Define the *Bunimovich stadium* $\mathcal{M}$ to be a rectangle capped by two semidisks. Normalizing one side length to be 2 and the other $a > 0$, we may take

$$\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \colon x \le -a,\ (x+a)^2 + y^2 \le 1\} \\ \cup \big([-a,0] \times [-1,1]\big) \cup \{(x,y) \in \mathbb{R}^2 \colon x \ge 0,\ x^2 + y^2 \le 1\}.$$

Define M to be the lift of $\mathcal{M}$ to the blow-up $[\mathbb{R}^2; \{-a,0\} \times \{-1,1\}]$ of $\mathbb{R}^2$ at the four corners (which are the only four points in $\partial\mathcal{M}$ where $\mathcal{M}$ fails to be $\mathcal{C}^\infty$).

(1) Show that a local coordinate system near the lift of $\{(0,-1)\}$ is given by
$$[0, \min(a,2)) \times [0,\pi] \ni (r,\alpha) \mapsto (0,-1) + (r\cos\theta(r,\alpha), r\sin\theta(r,\alpha)),$$
where
$$\theta(\alpha) := \alpha\Big(1 - \frac{\arcsin(r/2)}{\pi}\Big).$$
(2) Compute the expression for $-r^2\Delta$ in the coordinates (r,α).
(3) Let $f \in \dot{\mathcal{C}}^\infty(\mathcal{M})$. Show that the unique solution $u \in H_0^1(\mathcal{M}^\circ)$ of $\Delta u = f \in \dot{\mathcal{C}}^\infty(\mathcal{M})$ is polyhomogeneous on M, and compute the first few terms in the expansion of u. *Hint.* One has $u(r,\alpha) \sim r u_0(\alpha) + r^2(\log r) u_1(\alpha) + r^2 u_2(\alpha) + \ldots$, where u_1 can be computed explicitly in terms of u_0.
(4) Show that u lies in $\mathcal{C}^{1,\beta}(\mathcal{M})$ for all $\beta < 1$, but for some f fails to lie in $\mathcal{C}^2(\mathcal{M})$.

Problem 5.8 (Explicit verification of the polyhomogeneity of solutions of the wave equation in $n = 3$ spatial dimensions)**.** Let $\mathcal{F} \subset \mathbb{C} \times \mathbb{N}_0$ be an index set with $\operatorname{Re} z > 1$,[33] and suppose $\langle x\rangle u_1 \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(\overline{\mathbb{R}^3})$. Let u be the unique solution of $\Box_g u = 0$ with $(u, \partial_t u)|_{t=0} = (0, u_1)$. Let $\mathcal{E}_{I^0} = \mathcal{E}_{I^-} = \mathcal{E}_{I^+} = \mathcal{F}$ and $\mathcal{E}_{\mathscr{I}^-} = \mathcal{E}_{\mathscr{I}^+} = \mathcal{F} \cup ((\mathbb{N}_0 + 1) \times \{0\})$. Then u is polyhomogeneous on M with index set $\mathcal{E}_H$ at the hypersurface $H \subset M$, $H \in \{I^-, \mathscr{I}^-, I^0, \mathscr{I}^+, I^+\}$ of M.[34] *Hint:* use the formula
$$u(t,x) = -\frac{1}{2\pi t}\int_{\partial B_t(x)} u_1(y)\,\mathrm{d}\sigma(y),$$
where $\partial B_t(x) = \{y \in \mathbb{R}^3 \colon |y - x| = t\}$ is the boundary of the t-ball around x.

Problem 5.9 (Generators of the Poincaré group and scalings)**.** Complete the proof of Proposition 5.4. Deduce that every element of the Poincaré group extends by continuity from $\mathbb{R}^4$ to diffeomorphisms of M.

6. Singular limits via geometric singular analysis

We consider here only one type of setting to illustrate the basic strategy for studying singular limits using manifolds with corners and blow-up techniques. The simplest case is as follows. Let X denote manifold without boundary, and suppose we are given a smooth family
$$(0,1] \ni \epsilon \mapsto P_\epsilon \in \mathrm{Diff}^m(X)$$

[33] Without this assumption, there may be log terms at $\mathscr{I}^\pm$.

[34] We have not formally defined polyhomogeneity on manifolds with corners. The rough definition is clear: at each boundary hypersurface H, one has an asymptotic expansion as in (2.7), with ρ a defining function of H and with $\mathcal{E}$ replaced by the index set $\mathcal{E}_H$ associated with H, where the coefficients $u_{(z,k)}(y)$ themselves are polyhomogeneous on H, with index set $\mathcal{E}_{H'}$ at the hypersurface $H' \cap H$ of H. See [Maz91, Mel96] for precise definitions.

of differential operators; if $x \in \mathbb{R}^n$ are local coordinates near a point in X, this simply means that $P_\epsilon = \sum_{|\alpha|\leq m} a(\epsilon, x)\partial_x^\alpha$ where $a \in \mathcal{C}^\infty((0,1] \times \mathbb{R}^n)$. We combine all P_ϵ into a single differential operator on the *total space* $(0,1] \times X$,

$$\widetilde{P} \in \mathrm{Diff}^m((0,1] \times X),$$

which acts on $\widetilde{u} \in \mathcal{C}^\infty((0,1] \times X)$ via $(\widetilde{P}\widetilde{u})(\epsilon, \cdot) = P(\widetilde{u}(\epsilon, \cdot))$. Thus, $\widetilde{P}$ is a 'vertical' differential operator in that it does not involve differentiation in ϵ.

We assume that P_ϵ becomes singular as $\epsilon \searrow 0$ only at a distinguished point $p \in X$ in the following precise sense. Define the *resolved total space*

$$\widetilde{X} := \big[\,[0,1] \times X; \{(0,p)\}\big], \tag{6.1}$$

and write

- $X_\circ \subset \widetilde{X}$ for the lift of $\{0\} \times X$.
- $\hat{X} \subset \widetilde{X}$ for the front face of $\widetilde{X}$.

Problem 4.4 gives natural diffeomorphisms $X_\circ = [X; \{p\}]$ and $\hat{X} = \overline{T_p}X$. Notice what happens here: every level set of ϵ inside the interior $\widetilde{X}^\circ = (0,1) \times X$ is a copy of X; but as $\epsilon \searrow 0$, the level sets degenerate, and at $\epsilon = 0$ the fiber of $\widetilde{X}$ over $\epsilon = 0$ (i.e. the preimage $\epsilon^{-1}(0)$) is no longer a smooth manifold, but rather the union of the two manifolds with boundary $\hat{X}$ and $X_\circ$. See Figure 6.1.

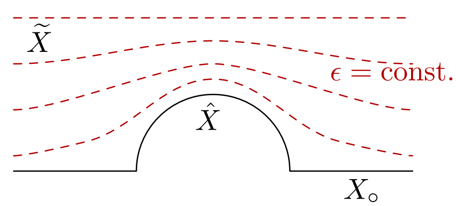

FIGURE 6.1. Illustration of the resolved total space (6.1), its boundary hypersurfaces, and some level sets of ϵ.

Write $\rho_\circ$, $\hat{\rho} \in \mathcal{C}^\infty(\widetilde{X})$ for defining functions of $X_\circ$, $\hat{X}$ respectively. (Thus $\hat{\rho}\rho_\circ$ is a smooth positive multiple of ϵ.) Then we require that $\widetilde{P} \in \mathrm{Diff}_\mathrm{b}^m(\widetilde{X})$, or more generally

$$\widetilde{P} \in \hat{\rho}^{-\alpha}\mathrm{Diff}_\mathrm{b}^m(\widetilde{X}). \tag{6.2}$$

Since $\widetilde{P}$ does not involve differentiation in ϵ (which can be formally expressed e.g. as $[\widetilde{P}, \epsilon] = 0$), its behavior at $\epsilon = 0$ can be captured by two operators

$$P_\circ \in \hat{\rho}^{-\alpha}\mathrm{Diff}_\mathrm{b}^m(X_\circ), \qquad \hat{P} \in \rho_\circ^{\alpha}\mathrm{Diff}_\mathrm{b}^m(\hat{X}). \tag{6.3}$$

Here, we define $P_\circ u$ for $u \in \mathcal{C}_\mathrm{c}^\infty(X_\circ) = \mathcal{C}_\mathrm{c}^\infty(X \setminus \{p\})$ to be equal to $(\widetilde{P}\widetilde{u})|_{X_\circ}$ where $\widetilde{u} \in \mathcal{C}^\infty(\widetilde{X})$ satisfies $\widetilde{u}|_{X_\circ} = u$; and similarly $\hat{P}u = (\epsilon^\alpha \widetilde{P}\widetilde{u})|_{\hat{X}}$ for $u \in \mathcal{C}_\mathrm{c}^\infty(\hat{X}^\circ)$ where $\widetilde{u} \in \mathcal{C}^\infty(\widetilde{X})$ satisfies $\widetilde{u}|_{\hat{X}} = u$.

Example 6.1 (Laplacians with degenerating potentials). Let $X = \mathbb{R}^n$, $p = 0$. Then $\widetilde{X} = [[0,1] \times \mathbb{R}^n; \{(0,0)\}]$; we already studied this manifold (with $[0,1]$ replaced by $[0,\infty)$) in Example 4.7, where we saw in particular that $X_\circ = [\mathbb{R}^n_x; \{0\}]$ and $\hat{X} = \overline{\mathbb{R}^n_{\hat{x}}}$ where $\hat{x} = \frac{x}{\epsilon}$. Let $W, V \in \mathcal{C}_\mathrm{c}^\infty(\mathbb{R}^n)$ and set $V_\epsilon(x) = V(\frac{x}{\epsilon})$. Then

$$P_\epsilon = \Delta + W + \epsilon^{-2}V_\epsilon$$

fits into the above setting with $\alpha = 2$. Indeed, when $|x| < R\epsilon$ where $R > 0$ is twice the radius of an open ball containing $\operatorname{supp} V$, then local coordinates on $\widetilde{X}$ are $\epsilon \in [0,1]$, $\hat{x} = \frac{x}{\epsilon}$. We then compute[35]

$$\epsilon^2 \widetilde{P} = \epsilon^2(\Delta + W + V_\epsilon) = -\sum_{j=1}^n (\epsilon\partial_{x^j})^2 + V_\epsilon(x) + \epsilon^2 W(x) = -\sum_{j=1}^n \partial_{\hat{x}^j}^2 + V(\hat{x}) + \epsilon^2 W(\epsilon\hat{x}).$$

A neighborhood of the corner of $\widetilde{X}$ is $[0,1)_r \times [0, 2R^{-1})_{\rho_\circ} \times \mathbb{S}^{n-1}$ where $r = |x|$ and $\rho_\circ = \frac{\epsilon}{|x|}$, so

$$\widetilde{P} = \Delta + W = r^{-2}\bigl(-(r\partial_r - \rho_\circ\partial_{\rho_\circ})^2 - (r\partial_r - \rho_\circ\partial_{\rho_\circ}) + \Delta_{\mathbb{S}^2} + r^2 W\bigr),$$

as required. The operators $P_\circ$ and $\hat{P}$ in (6.3) are then

$$P_\circ = \Delta_x + W(x) \in \hat{\rho}^{-2}\mathrm{Diff}_\mathrm{b}^2(X_\circ), \qquad \hat{P} = \Delta_{\hat{x}} + V(\hat{x}) \in \rho_\circ^2\mathrm{Diff}_\mathrm{b}^2(\hat{X}).$$

Note that $\widetilde{P} - P_\circ \in \hat{\rho}^{-2}\rho_\circ\mathrm{Diff}_\mathrm{b}^2(\widetilde{X})$ vanishes at $X_\circ$ as a b-differential operator, and similarly $\epsilon^2\widetilde{P} - \hat{P} \in \hat{\rho}\rho_\circ^2\mathrm{Diff}_\mathrm{b}^2(\widetilde{X})$ vanishes at $\hat{X}$ as a b-differential operator. In this sense, $P_\circ$ and $\hat{P}$ are accurate models for $\widetilde{P}$ in the two asymptotic regimes $X_\circ$, $\hat{X}$.

6.1. **q-analysis.** It is useful to have more precise description of $\widetilde{P}$ than (6.2) which explicitly records the fact that $\widetilde{P}$ does not involve differentiation in ϵ:

Definition 6.2 (q-vector fields and q-differential operators)**.** On the manifold $\widetilde{X}$ defined in (6.1), we write $\mathcal{V}_\mathrm{q}(\widetilde{X}) \subset \mathcal{V}_\mathrm{b}(\widetilde{X})$ for the space of all b-vector fields V on $\widetilde{X}$ with $V\epsilon = 0$. We write $\mathrm{Diff}_\mathrm{q}^m(\widetilde{X})$ for locally finite sums of up to m-fold compositions of elements of $\mathcal{V}_\mathrm{q}(\widetilde{X})$.

Thus, $\widetilde{P} \in \hat{\rho}^{-\alpha}\mathrm{Diff}_\mathrm{q}^m(\widetilde{X})$. This was implicitly introduced in [HX22] and explicitly in [Hin21], with earlier variants appearing already in [HMM95]. See Problem 6.1 for some properties of these spaces of vector fields and differential operators. We shall use in the sequel that $\mathcal{V}_\mathrm{q}(\widetilde{X})$ is spanned over $\mathcal{C}^\infty(\widetilde{X})$ by the vector fields $\hat{\rho}W$ where $W \in \mathcal{V}(X)$ is regarded as an ϵ-independent vector field on $(0,1] \times X$. Using $\mathcal{V}_\mathrm{q}(\widetilde{X})$, we can define a corresponding scale of Sobolev spaces:

Definition 6.3 (q-Sobolev spaces)**.** Suppose X is a compact manifold without boundary, and define $\widetilde{X}$ by (6.1); fix boundary defining functions $\rho_\circ, \hat{\rho} \in \mathcal{C}^\infty(\widetilde{X})$ of $X_\circ, \hat{X}$. Fix a finite collection $\mathscr{V} = \{V_1, \ldots, V_N\} \subset \mathcal{V}(X)$ of vector fields which spans $\mathcal{V}(X)$ over $\mathcal{C}^\infty(X)$. Let $s \in \mathbb{N}_0$. For $\epsilon \in (0,1]$, we define $H^s_{\mathrm{q},\epsilon}(X) = H^s(X)$ but with ϵ-dependent norm

$$\|u\|^2_{H^s_{\mathrm{q},\epsilon}(X)} := \sum_{|\alpha| \le s} \|\hat{\rho}^{|\alpha|}\mathscr{V}^\alpha u\|^2_{L^2(X)}.$$

For weights $\alpha_\circ, \hat{\alpha} \in \mathbb{R}$, we define $H^{s,\alpha_\circ,\hat{\alpha}}_{\mathrm{q},\epsilon}(X) = H^s(X)$ but with ϵ-dependent norm

$$\|u\|_{H^{s,\alpha_\circ,\hat{\alpha}}_{\mathrm{q},\epsilon}(X)} := \|\rho_\circ^{-\alpha_\circ}\hat{\rho}^{-\hat{\alpha}}u\|_{H^s_{\mathrm{q},\epsilon}(X)}.$$

These are the natural function spaces on which to study q-differential operators uniformly as $\epsilon \searrow 0$. Indeed, every $\widetilde{P} \in \mathrm{Diff}_\mathrm{q}^m(\widetilde{X})$ defines a uniformly bounded family of maps

$$H^s_{\mathrm{q},\epsilon}(X) \to H^{s-m}_{\mathrm{q},\epsilon}(X), \quad \epsilon \in (0,1].$$

[35] In this region, ϵ is a smooth positive multiple of a defining function of $\hat{X}$.

(See Problem 6.3 for a generalization.) Furthermore, since the two model operators $P_\circ$ and $\hat{P}$ corresponding to $\widetilde{P}$ are b-differential operators, one expects there to be a relationship between q-Sobolev spaces on X and b-Sobolev spaces on $X_\circ$ and $\hat{X}$. We explain this in the simplest case in local coordinates and leave the general case to the reader (Problem 6.5).

Thus, suppose $u = u(\epsilon, x)$ is defined for $\epsilon > 0$ and $x \in \mathbb{R}^n$, $|x| < 1$. Then we can take $\hat{\rho} = (\epsilon^2 + |x|^2)^{1/2}$ and upon changing variables via $x = \epsilon\hat{x}$ compute

$$\begin{aligned} \|u\|^2_{H^1_{\mathrm{q},\epsilon}} &= \|u(\epsilon,\cdot)\|^2_{L^2} + \|\hat{\rho}\partial_x u(\epsilon,\cdot)\|^2_{L^2} \\ &= \int_{\mathbb{R}^n} |u(\epsilon,x)|^2 + (\epsilon^2 + |x|^2)|\partial_x u(\epsilon,x)|^2 \,\mathrm{d}x \\ &= \epsilon^n \int_{\mathbb{R}^n} |u(\epsilon,\epsilon\hat{x})|^2 + (1+|\hat{x}|^2)|(\partial_{\hat{x}} u)(\epsilon,\epsilon\hat{x})|^2 \,\mathrm{d}\hat{x}. \end{aligned}$$

Note that $(1+|\hat{x}|^2)^{1/2}\partial_{\hat{x}}$ is a b-vector field on the radial compactification $\overline{\mathbb{R}^n_{\hat{x}}}$. Thus, if we define $\hat{u}(\epsilon,\hat{x}) = u(\epsilon,\epsilon\hat{x})$ to be the family of functions $u(\epsilon,\cdot)$ expressed in $\hat{x}$-coordinates, then

$$\|u\|^2_{H^1_{\mathrm{q},\epsilon}} = \epsilon^n \|\hat{u}(\epsilon,\cdot)\|^2_{H^1_{\mathrm{b}}(\overline{\mathbb{R}^n_{\hat{x}}})}. \tag{6.4}$$

When u is equal to 0 for $|x| \leq R\epsilon$ for some $R > 0$, then $\epsilon \leq R^{-1}|x|$ on $\operatorname{supp} u$, so $|x|^2 \leq \epsilon^2 + |x|^2 \leq (1+R^{-2})|x|^2$, and therefore the $H^1_{\mathrm{q},\epsilon}$ and $H^1_{\mathrm{b}}([\mathbb{R}^n;\{0\}])$ norms of u are comparable:

$$C(R)^{-1}\|u\|^2_{H^1_{\mathrm{q},\epsilon}} \leq \int_{\mathbb{R}^n} |u(\epsilon,x)|^2 + |x|^2|\partial_x u(\epsilon,x)|^2 \,\mathrm{d}x = \|u(\epsilon,\cdot)\|^2_{H^1_{\mathrm{b}}([\mathbb{R}^n;\{0\}])} \leq C(R)\|u\|^2_{H^1_{\mathrm{q},\epsilon}}. \tag{6.5}$$

The strategy for the uniform analysis of $\widetilde{P} \in \mathrm{Diff}^m_{\mathrm{q}}(\widetilde{X})$ is then to prove uniform estimates on q-Sobolev spaces in a fashion similar to how the Laplacian on $\mathbb{R}^n$ was analyzed in Theorem 3.1: in the simplest case, one combines elliptic estimates for $\widetilde{P}$ on q-Sobolev spaces with the *invertibility* of the two model operators $P_\circ$ and $\hat{P}$ on the matching b-Sobolev spaces to obtain a uniform estimate of the sort

$$\|u\|_{H^{s,\alpha_\circ,\hat{\alpha}}_{\mathrm{q},\epsilon}(X)} \leq C\Big(\|P_\epsilon u\|_{H^{s-m,\alpha_\circ,\hat{\alpha}}_{\mathrm{q},\epsilon}(X)} + \|u\|_{H^{0,\alpha_\circ-1,\hat{\alpha}-1}_{\mathrm{q},\epsilon}}\Big)$$

(cf. the estimate (3.4)). For small $\epsilon > 0$, this implies the injectivity of $P_\epsilon \in \mathrm{Diff}^m(X)$ (the restriction of $\widetilde{P}$ to the ϵ-level set); see Problem 6.4.[36] We will implement this strategy in the setting of Example 6.1 in §7.1, and in a general relativity context in §7.2.

The perspective explained above can also be used for *constructing* families of solutions of (nonlinear) PDEs which become singular in a prescribed fashion; see [Hin22] for an example.

6.2. Problems.

Problem 6.1 (q-vector fields and q-differential operators)**.** Fixing a smooth manifold X and a point $p \in X$, define $\widetilde{X} = [[0,1] \times X; \{(0,p)\}]$. Write $X_\circ$ and $\hat{X}$ for the lift of $\{0\} \times X$ and the front face, respectively, and $\rho_\circ$ and $\hat{\rho} \in \mathcal{C}^\infty(\widetilde{X})$ for defining functions. Prove the following statements about $\mathcal{V}_{\mathrm{q}}(\widetilde{X})$ and $\mathrm{Diff}^m_{\mathrm{q}}(\widetilde{X})$.

[36] If one proves analogous estimates for the adjoint $\widetilde{P}^*$, one can obtain the surjectivity of P_ϵ for small $\epsilon > 0$.

(1) $\mathcal{V}_{\mathrm{q}}(\widetilde{X})$ is equal to the $\mathcal{C}^\infty(\widetilde{X})$-span of all vector fields of the form $\hat\rho W$ where $W \in \mathcal{V}(X)$ is regarded as an ϵ-independent vector field on $(0,1] \times X$.
(2) $\mathcal{V}_{\mathrm{q}}(\widetilde{X})$ is a Lie algebra: $V, W \in \mathcal{V}_{\mathrm{q}}(\widetilde{X})$ implies $[V,W] \in \mathcal{V}_{\mathrm{q}}(\widetilde{X})$.
(3) There are well-defined and surjective restriction maps $N_{\hat X}\colon \mathcal{V}_{\mathrm{q}}(\widetilde{X}) \to \mathcal{V}_{\mathrm{b}}(\hat X)$ and $N_{X_\circ}\colon \mathcal{V}_{\mathrm{q}}(\widetilde{X}) \to \mathcal{V}_{\mathrm{b}}(X_\circ)$.
(4) The maps $N_{\hat X}$, $N_{X_\circ}$ extend to surjective algebra homomorphisms $\mathrm{Diff}_{\mathrm{q}}(\widetilde{X}) \to \mathrm{Diff}_{\mathrm{b}}(\hat X)$ and $\mathrm{Diff}_{\mathrm{q}}(\widetilde{X}) \to \mathrm{Diff}_{\mathrm{b}}(X_\circ)$. The operators $P_\circ$ and $\hat P$ in (6.3) are equal to $N_{X_\circ}(\widetilde{P})$ and $N_{\hat X}(\widetilde{P})$, respectively.
(5) Let $\widetilde{P} \in \mathrm{Diff}_{\mathrm{q}}^m(\widetilde{X})$. Then $N_{X_\circ}(\widetilde{P}) = 0$ if and only if $\widetilde{P} \in \rho_\circ \mathrm{Diff}_{\mathrm{q}}^m(\widetilde{X})$; and $N_{\hat X}(\widetilde{P}) = 0$ if and only if $\widetilde{P} \in \hat\rho \mathrm{Diff}_{\mathrm{q}}^m(\widetilde{X})$. (Thus, $N_{X_\circ}(\widetilde{P})$ and $N_{\hat X}(\widetilde{P})$ capture $\widetilde{P}$ to leading order at $X_\circ$ and $\hat X$.)

Problem 6.2 (q-Sobolev spaces)**.** Show that the norms on weighted q-Sobolev spaces do not depend on any choices up to *uniform* equivalence of norms. That is, if $\|\cdot\|_{\epsilon,1}$, $\|\cdot\|_{\epsilon,2}$ are the ϵ-dependent norms $\|\cdot\|_{H^{s,\alpha_\circ,\hat\alpha}_{\mathrm{q},\epsilon}(X)}$ defined with respect to two different choices of $\rho_\circ, \hat\rho, \mathscr{V}$, show that there exists $C > 0$ so that for all $\epsilon \in (0,1]$ we have $\|u\|_{\epsilon,1} \leq C\|u\|_{\epsilon,2}$ and $\|u\|_{\epsilon,2} \leq C\|u\|_{\epsilon,1}$ for all u.

Problem 6.3 (Mapping properties)**.** Let X be compact, and let $\widetilde{P} \in \rho_\circ^{-\beta_\circ}\hat\rho^{-\hat\beta}\mathrm{Diff}_{\mathrm{q}}^m(\widetilde{X})$ be a weighted q-differential operator. Show that for all $s \geq m$ and $\alpha_\circ, \hat\alpha \in \mathbb{R}$, $\widetilde{P}$ defines a uniformly bounded family of maps

$$H^{s,\alpha_\circ,\hat\alpha}_{\mathrm{q},\epsilon}(X) \to H^{s-m,\alpha_\circ-\beta_\circ,\hat\alpha-\hat\beta}_{\mathrm{q},\epsilon}(X).$$

Problem 6.4 (Smallness)**.** Let X be compact, $s \in \mathbb{N}_0$, and $\alpha_\circ, \hat\alpha < 0$. Suppose $\epsilon_j > 0$ and $u_j \in H^s(X)$, $j \in \mathbb{N}$ are sequences with $\epsilon_j \searrow 0$ and such that there exists a constant $C > 0$ so that

$$\|u_j\|_{H^s_{\mathrm{q},\epsilon}(X)} \leq C\|u_j\|_{H^{s,\alpha_\circ,\hat\alpha}_{\mathrm{q},\epsilon}(X)}.$$

Show that there exists $j_0 \in \mathbb{N}$ so that $u_j = 0$ for $j \geq j_0$.

Problem 6.5 (q-Sobolev spaces and their relationships to b-Sobolev spaces)**.** (See [Hin21, Proposition 2.13].) Let X be a compact n-dimensional manifold without boundary, $p \in X$. Fix local coordinates $x \in \mathbb{R}^n$ on X around p, with $x = 0$ at p. Let $s \in \mathbb{N}_0$ and $\alpha_\circ, \hat\alpha \in \mathbb{R}$. Let $\chi_\circ, \hat\chi \in \mathcal{C}^\infty(\widetilde{X})$, with $\chi_\circ = 0$ when $|x| < c\epsilon$ for some $c > 0$, and $\operatorname{supp}\hat\chi \subset \{|x| < C\}$, $C > 0$.

(1) Given a function $u = u(\epsilon, x)$, write $\hat u = \hat u(\epsilon, \hat x) := u(\epsilon, \epsilon\hat x)$ for its pullback along $(\epsilon, \hat x) \mapsto (\epsilon, \epsilon\hat x)$. (That is, we are expressing u in terms of the variables $\hat x = \frac{x}{\epsilon}$.) Show that

$$\|\hat\chi u(\epsilon,\cdot)\|_{H^{s,\alpha_\circ,\hat\alpha}_{\mathrm{q},\epsilon}(X)} \sim \epsilon^{\frac{n}{2}-\hat\alpha}\|\hat\chi\hat u(\epsilon,\cdot)\|_{H^{s,\alpha_\circ-\hat\alpha}_{\mathrm{b}}(\hat X)},$$

in the sense that there exists a constant C, independent of ϵ, u, so that the left hand side is bounded by C times the right hand side, and vice versa.
(2) Given $u = u(\epsilon, x)$, show that

$$\|\chi_\circ u(\epsilon,\cdot)\|_{H^{s,\alpha_\circ,\hat\alpha}_{\mathrm{q},\epsilon}(X)} \sim \epsilon^{-\alpha_\circ}\|\chi_\circ u\|_{H^{s,\hat\alpha-\alpha_\circ}_{\mathrm{b}}(X_\circ)}.$$

7. Applications: III

7.1. **Laplacians with degenerating potentials.** We return to the setting from §1.2 and Example 6.1.

Theorem 7.1 (Spectrum under singular perturbations)**.** *Let (X, g) be a compact 3-dimensional*[37] *Riemannian manifold without boundary, $W \in \mathcal{C}^\infty(X)$. Let $p \in X$, and denote normal coordinates around p by $x \in \mathbb{R}^3$, $|x| < \delta$. Let $V \in \mathcal{C}^\infty_c(\mathbb{R}^3)$. For $\epsilon > 0$, consider the operator*

$$P_\epsilon := \Delta_g + W + \epsilon^{-2} V\Big(\frac{x}{\epsilon}\Big) \in \mathrm{Diff}^2(X).$$

Assume that $(\Delta_{\mathbb{R}^3} + V)u = 0$ and $|u(x)| \to 0$ as $|x| \to \infty$ implies $u = 0$. Suppose that $\lambda_0 \notin \sigma(\Delta_g + W)$, i.e. $\Delta_g + W - \lambda_0 \colon H^2(X) \to L^2(X)$ is invertible. Then there exist $\epsilon_0 > 0$ and a neighborhood $\Lambda \subset \mathbb{C}$ of λ_0 so that

$$P_\epsilon - \lambda \colon H^2(X) \to L^2(X)$$

is invertible for all $\epsilon \in (0, \epsilon_0)$ and $\lambda \in \Lambda$.

This result says that the limit of $\sigma(P_\epsilon)$ as $\epsilon \searrow 0$ is contained in $\sigma(\Delta_g + W)$. The converse is true as well, but we shall not prove this here. Typically, for any fixed $\epsilon > 0$, P_ϵ will have eigenvalues far from $\sigma(\Delta_g + W)$; but these eigenvalues are of size $\gtrsim \epsilon^{-2}$, i.e. they 'disappear' to infinity in the limit $\epsilon \searrow 0$.

Proof of Theorem 7.1. To simplify the notation, we shall only prove the invertibility of $P_\epsilon - \lambda_0$. (Allowing λ to vary near λ_0 merely adds another parameter on which all operators depend smoothly.) By adding a constant to W, we may moreover assume that $\lambda_0 = 0$.

As we essentially already saw in Example 6.1, the total family $\widetilde{P} = (P_\epsilon)_{\epsilon \in (0,1]}$ satisfies

$$\widetilde{P} \in \hat\rho^{-2} \mathrm{Diff}^2_{\mathrm{q}}(\widetilde{X});$$

its restriction to $X_\circ$ is $P_\circ = \Delta_g + W$, and the restriction of $\epsilon^2 \widetilde{P}$ to $\hat{X}$ is $\hat{P} = \Delta_{\mathbb{R}^3} + V$.

- *Step 1. Uniform elliptic estimate.* Let $s \geq 3$ and $\hat\alpha \in \mathbb{R}$. We claim that there exists $C > 0$ so that

$$\|u\|_{H^{s,0,\hat\alpha}_{\mathrm{q},\epsilon}(X)} \leq C\Big(\|P_\epsilon u\|_{H^{s-2,0,\hat\alpha-2}_{\mathrm{q},\epsilon}(X)} + \|u\|_{H^{2,0,\hat\alpha}_{\mathrm{q},\epsilon}(X)} \Big). \tag{7.1}$$

The norm on $P_\epsilon u$ is equal to $\|\hat\rho^2 P_\epsilon u\|_{H^{s-2,0,\hat\alpha}_{\mathrm{q},\epsilon}(X)}$. This estimate is insensitive to lower order terms of P_ϵ; moreover we can reduce to the case $\hat\alpha = 0$ by considering $\hat\rho^{-\hat\alpha} P_\epsilon \hat\rho^{\hat\alpha}$ instead of P_ϵ. Away from $\hat{X}$, this estimate is then a standard elliptic estimate. Near $\hat{X}$, note that $-\hat\rho^2 \Delta$ is, to leading order, a positive definite quadratic form in $\hat\rho \partial_x$. An adaptation of standard elliptic regularity estimates then shows that u has two more $\hat\rho\partial_x$-derivatives than $\hat\rho^2 P_\epsilon u$, which is the content of (7.1).

- *Step 2. Inversion of the $X_\circ$-normal operator.* Let $\chi_\circ \in \mathcal{C}^\infty(\widetilde{X})$ be equal to 1 near $X_\circ$, and equal to 0 for $|x| < \epsilon$. Then for any $\eta > 0$

$$\begin{aligned} \|u\|_{H^{2,0,\hat\alpha}_{\mathrm{q},\epsilon}(X)} &\leq \|\chi_\circ u\|_{H^{2,0,\hat\alpha}_{\mathrm{q},\epsilon}(X)} + \|(1-\chi_\circ)u\|_{H^{2,0,\hat\alpha}_{\mathrm{q},\epsilon}(X)} \\ &\leq C_\eta\Big(\|\chi_\circ u\|_{H^{2,\hat\alpha}_{\mathrm{b}}(X_\circ)} + \|u\|_{H^{2,-\eta,\hat\alpha}_{\mathrm{q},\epsilon}(X)} \Big), \end{aligned} \tag{7.2}$$

[37] This holds in any dimension $n \geq 3$, as the reader is tasked with proving in Problem 7.1.

since $\operatorname{supp}(1-\chi_\circ)\cap X_\circ=\emptyset$, hence the weight in the norm on $(1-\chi_\circ)u$ can be changed arbitrarily (and then we drop $1-\chi_\circ$, which is a uniformly bounded multiplication operator on every weighted q-Sobolev space). We now specify that

$$\hat\alpha\in\Bigl(\frac12,\frac32\Bigr).$$

In this case, the arguments of Theorem 5.1 apply and show that

$$P_\circ=\Delta_g+W\colon H_{\mathrm{b}}^{2,\hat\alpha}(X_\circ)\to H_{\mathrm{b}}^{0,\hat\alpha-2}(X_\circ) \tag{7.3}$$

is Fredholm. We claim that it is invertible; this is not automatic since we are considering this operator on a space of functions which allows for singular behavior at $r^{-1}(0)=\partial X_\circ$ (where we write $r=|x|$ as usual). If $u\in H_{\mathrm{b}}^{2,\hat\alpha}(X_\circ)$, then by elliptic regularity and Sobolev embedding $u\in\mathcal{A}^{-\frac32+\hat\alpha}(X_\circ)$. Since $N(r^2P_\circ,\lambda)=-\lambda^2-\lambda+\Delta_{\mathbb{S}^2}$ (cf. (5.1)) has indicial roots $l,-l-1$ for $l\in\mathbb{Z}$, and since $-\frac32+\hat\alpha>-1$, we conclude that u is polyhomogeneous and in fact $u(r,\omega)=u_0+\tilde u(r,\omega)$ where $\tilde u\in\mathcal{A}^{1-\epsilon}(X_\circ)$ for all $\epsilon>0$. Therefore, 0 is a removable singularity of u, and the continuous extension of u to X is automatically smooth and thus vanishes since it lies in $\ker(\Delta_g+W)$.

To prove the surjectivity of (7.3), we note that $W\colon H_{\mathrm{b}}^{2,\hat\alpha}\to H_{\mathrm{b}}^{2,\hat\alpha}\hookrightarrow H_{\mathrm{b}}^{0,\hat\alpha-2}$ is compact by Proposition 2.15, so we can change W to 1 without affecting the index; so $P_\circ$ has the same index as $\Delta_g+1\colon H_{\mathrm{b}}^{2,\hat\alpha}(X_\circ)\to H_{\mathrm{b}}^{0,\hat\alpha-2}(X_\circ)$. This operator is invertible (it is injective due to $\hat\alpha>\frac12$ by an integration by parts argument, and the adjoint is injective since $-(\hat\alpha-2)>\frac12$ also). Thus, (7.3) has index 0 and therefore is invertible.

We can thus estimate

$$\begin{aligned}\|\chi_\circ u\|_{H_{\mathrm{b}}^{2,\hat\alpha}(X_\circ)}&\le C\|P_\circ(\chi_\circ u)\|_{H_{\mathrm{b}}^{0,\hat\alpha-2}(X_\circ)}\\&\le\|\chi_\circ P_\epsilon u\|_{H_{\mathrm{q},\epsilon}^{0,0,\hat\alpha-2}(X)}+\|[P_\epsilon,\chi_\circ]u\|_{H_{\mathrm{q},\epsilon}^{0,0,\hat\alpha-2}(X)}+\|\chi_\circ(P_\epsilon-P_\circ)u\|_{H_{\mathrm{q},\epsilon}^{0,0,\hat\alpha}(X)}\Bigr)\\&\le C\Bigl(\|\chi_\circ P_\epsilon u\|_{H_{\mathrm{q},\epsilon}^{0,0,\hat\alpha-2}(X)}+\|u\|_{H_{\mathrm{q},\epsilon}^{1,-\eta,\hat\alpha}(X)}+\|u\|_{H_{\mathrm{q},\epsilon}^{2,-1,\hat\alpha}(X)}\Bigr),\end{aligned}$$

where for the second term we use that the coefficients of $[\widetilde P,\chi_\circ]\in\hat\rho^{-2}\mathrm{Diff}_{\mathrm{q}}^1(\widetilde X)$ vanish near $X_\circ$ (so the weight there is arbitrary), and $\chi_\circ(\widetilde P-P_\circ)\in\hat\rho^{-2}\rho_\circ\mathrm{Diff}_{\mathrm{q}}^2(\widetilde X)$ since $P_\circ$ is the restriction of $\widetilde P$ to $X_\circ$.

Altogether, we have now improved the estimate (7.1) to the uniform estimate

$$\|u\|_{H_{\mathrm{q},\epsilon}^{s,0,\hat\alpha}(X)}\le C\Bigl(\|P_\epsilon u\|_{H_{\mathrm{q},\epsilon}^{s-2,0,\hat\alpha-2}(X)}+\|u\|_{H_{\mathrm{q},\epsilon}^{2,-\eta,\hat\alpha}(X)}\Bigr) \tag{7.4}$$

for any $0<\eta\le1$. Thus, the weight $(-\eta)$ in the norm on the error term at $X_\circ$ is weaker than that (0) on the left hand side.

- *Step 3. Inversion of the $\hat X$-normal operator.* We now take $\hat\chi\in\mathcal{C}^\infty(\widetilde X)$ to be equal to 1 near $\hat X$, with support in $|x|<\delta$. Write $\hat u(\epsilon,\hat x)=u(\epsilon,\epsilon\hat x)$. Then analogously to (7.2) we have

$$\|u\|_{H_{\mathrm{q},\epsilon}^{2,-\eta,\hat\alpha}(X)}\le C\Bigl(\epsilon^{\frac32-\hat\alpha}\|\hat\chi\hat u\|_{H_{\mathrm{b}}^{2,-\eta-\hat\alpha}(\hat X)}+\|u\|_{H_{\mathrm{q},\epsilon}^{2,-\eta,\hat\alpha-\eta'}(X)}\Bigr)$$

for any fixed $\eta'>0$. If we take $\eta>0$ so small that $\beta:=-\hat\alpha-\eta\in(-\frac32,-\frac12)$, we have

$$\|\hat\chi\hat u\|_{H_{\mathrm{b}}^{2,-\eta-\hat\alpha}(\hat X)}\le C\|\hat P(\hat\chi\hat u)\|_{H_{\mathrm{b}}^{0,-\eta-\hat\alpha+2}(\hat X)}$$

since

$$\hat{P} = \Delta + V \colon H_{\mathrm{b}}^{2,\beta}(\overline{\mathbb{R}^3}) \to H_{\mathrm{b}}^{0,\beta+2}(\overline{\mathbb{R}^3})$$

is invertible: this operator is injective since elements of its nullspace lie in $\mathcal{A}^{\frac{3}{2}+\beta}(\overline{\mathbb{R}^3})$ by elliptic regularity, thus decay at infinity and therefore vanish by assumption; and it has Fredholm index 0 since it is a compact perturbation of Δ.

Using that $\hat{\chi}(P_\epsilon - \epsilon^{-2}\hat{P}) \in \hat{\rho}^{-2}\tilde{\rho}\mathrm{Diff}_{\mathrm{q}}^2(\widetilde{X})$, we can thus estimate $\hat{P}(\hat{\chi}\hat{u})$ in terms of $\hat{\chi}P_\epsilon u$, analogously to the earlier arguments involving $P_\circ$. We finally obtain the improvement

$$\|u\|_{H_{\mathrm{q},\epsilon}^{s,0,\hat{\alpha}}(X)} \leq C\Big(\|P_\epsilon u\|_{H_{\mathrm{q},\epsilon}^{s-2,0,\hat{\alpha}-2}(X)} + \|u\|_{H_{\mathrm{q},\epsilon}^{2,-\eta,\hat{\alpha}-1}(X)}\Big) \tag{7.5}$$

of the estimate (7.4). For $\epsilon_0 > 0$, the error term here is *small* compared to the left hand side when $\epsilon < \epsilon_0$ and can therefore be dropped; this gives

$$\|u\|_{H_{\mathrm{q},\epsilon}^{s,0,\hat{\alpha}}(X)} \leq 2C\|P_\epsilon u\|_{H_{\mathrm{q},\epsilon}^{s-2,0,\hat{\alpha}-2}(X)}, \qquad \epsilon < \epsilon_0,$$

and therefore the injectivity of P_ϵ on $H^s(X)$. The surjectivity follows from the fact that P_ϵ has index 0. $\square$

7.2. **Quasinormal modes of Schwarzschild–de Sitter black holes in the vanishing mass limit.** We follow here [HX22, Hin21]. Fix the 'cosmological constant' $\Lambda > 0$. The metric of a Schwarzschild–de Sitter (SdS) black hole with cosmological constant Λ and mass $\mathfrak{m} > 0$ is then

$$g_{\mathfrak{m}} = -\mu_{\mathfrak{m}}(r)\,\mathrm{d}t^2 + \frac{1}{\mu_{\mathfrak{m}}(r)}\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}, \qquad \mu_{\mathfrak{m}}(r) := 1 - \frac{2\mathfrak{m}}{r} - \frac{\Lambda r^2}{3}.$$

One requires that $\mu_{\mathfrak{m}}$ has two distinct positive roots $r_+(\mathfrak{m}) < r_{\mathcal{C}}(\mathfrak{m})$, which happens if and only if $0 < 9\Lambda\mathfrak{m}^2 < 1$; the *black hole exterior* is then the manifold

$$\mathcal{M}_{\mathfrak{m}} := \mathbb{R}_t \times \big(r_+(\mathfrak{m}), r_{\mathcal{C}}(\mathfrak{m})\big) \times \mathbb{S}^2.$$

The set $\mathrm{QNM}(\mathfrak{m})$ of *quasinormal modes* (*QNMs* for short) of such a black hole is the set of all $\sigma \in \mathbb{C}$ so that there exists a *mode solution* of $\Box_{g_{\mathfrak{m}}}$ with frequency σ: this means

$$\Box_{g_{\mathfrak{m}}}\big(e^{-i\sigma t}u(r,\omega)\big) = 0, \tag{7.6}$$

where u is smooth and satisfies for $\bullet = +, \mathcal{C}$ the 'outgoing (boundary) conditions'

$$u(r,\omega) = |r - r_\bullet(\mathfrak{m})|^{-i\sigma/2\kappa_\bullet(\mathfrak{m})}u_\bullet(r,\omega), \qquad \kappa_\bullet(\mathfrak{m}) := \frac{1}{2}|\mu_{\mathfrak{m}}'(r_\bullet)|,$$

where $u_\bullet$ is smooth *down to* $r = r_\bullet$. One can show that $\mathrm{QNM}(\mathfrak{m}) \subset \mathbb{C}$ is a discrete set.

Remark 7.2 (Relevance of QNMs). General solutions of the wave equation $\Box_{g_{\mathfrak{m}}}u = 0$ with smooth initial data (with appropriate behavior near $r = r_+(\mathfrak{m}), r_{\mathcal{C}}(\mathfrak{m})$, e.g. vanishing) have asymptotic expansions as $t \to \infty$ of the form

$$u(t,r,\omega) \sim \sum_{\sigma \in \mathrm{QNM}(\mathfrak{m})} e^{-i\sigma t}u_\sigma(r,\omega)$$

(ignoring the possibility of terms including polynomial factors of t) where u_σ is a mode solution with frequency σ [BH10, Dya12]. This is *not* a polyhomogeneous expansion: the set of $\sigma \in \mathrm{QNM}(\mathfrak{m})$ with $\mathrm{Im}\,\sigma > -C$ is infinite for all sufficiently large C.

Theorem 7.3 (Convergence of QNMs in the vanishing mass limit)**.** *The set* $\mathrm{QNM}(\mathfrak{m})$ *converges locally uniformly*[38] *to the set*[39]

$$\mathrm{QNM}_{\mathrm{dS}} := -i\frac{\Lambda}{3}\mathbb{N}_0. \tag{7.7}$$

The set $\mathrm{QNM}_{\mathrm{dS}}$ is the set of quasinormal modes of de Sitter space with cosmological constant $\Lambda > 0$; the metric is

$$g_{\mathrm{dS}} = -\mu_{\mathrm{dS}}(r)\,\mathrm{d}t^2 + \frac{1}{\mu_{\mathrm{dS}}(r)}\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}, \qquad \mu_{\mathrm{dS}}(r) = 1 - \frac{\Lambda r^2}{3}. \tag{7.8}$$

The metric studied in §3.3 is locally isometric to the de Sitter metric with cosmological constant $\Lambda = 3$ when $n = 3$, and the integer exponents in the expansion in Theorem 3.6 correspond (as in Remark 7.2) to the set (7.7) in this case. See [HX21] and [HV18, Appendix C] for details on this connection. Unlike $g_{\mathfrak{m}}$, the metric g_{dS} is singular only at $r = r_{\mathrm{dS}} = \sqrt{3/\Lambda}$, while g_{dS} is smooth at $r = 0$ when expressed in Cartesian coordinates $r\omega$.

Proof of Theorem 7.3. The equation for u in (7.6) reads

$$\Big(-r^{-2}\partial_r r^2\mu_{\mathfrak{m}}(r)\partial_r + r^{-2}\Delta_{\mathbb{S}^2} - \frac{1}{\mu_{\mathfrak{m}}(r)}\sigma^2\Big)u(r,\omega) = 0.$$

We shall only consider the case that $u(r,\omega) = v(r)Y_{lm}(\omega)$ where Y_{lm} is a spherical harmonic of some fixed degree $l \in \mathbb{N}_0$: one can then replace $\Delta_{\mathbb{S}^2}$ by $l(l+1)$. In fact, since the analysis for all values of l is the same, we shall only consider $l = 0$, i.e. spherically symmetric $u = u(r)$; this leads to the ODE

$$\begin{aligned} P_{\mathfrak{m}}u &:= r^{-2}\partial_r r^2\mu_{\mathfrak{m}}(r)\partial_r u + \frac{1}{\mu_{\mathfrak{m}}(r)}\sigma^2 u \\ &= r^{-2}\partial_r r^2\Big(1 - \frac{2\mathfrak{m}}{r} - \frac{\Lambda r^2}{3}\Big)\partial_r u(r) + \Big(1 - \frac{2\mathfrak{m}}{r} - \frac{\Lambda r^2}{3}\Big)^{-1}\sigma^2 u(r) \\ &= 0 \end{aligned} \tag{7.9}$$

on $(r_+(\mathfrak{m}), r_{\mathcal{C}}(\mathfrak{m}))$. We shall (of course!) analyze the operator $P_{\mathfrak{m}}$ as $\mathfrak{m} \searrow 0$ by studying the total family $\widetilde{P} = (P_{\mathfrak{m}})_{\mathfrak{m}\in(0,\mathfrak{m}_0)}$ where $9\Lambda\mathfrak{m}_0^2 < 1$.

- Step 1. *Resolution of the domain.* One can check that $r_+(\mathfrak{m})$, $r_{\mathcal{C}}(\mathfrak{m})$ are smooth functions $\mathfrak{m} \in [0,\mathfrak{m}_0)$, and in fact $\frac{r_+(\mathfrak{m})}{\mathfrak{m}} \to 2$ and $r_{\mathcal{C}}(\mathfrak{m}) \to r_{\mathrm{dS}}$ as $\mathfrak{m} \searrow 0$. To resolve the regime where $r \simeq \mathfrak{m}$, we thus work on

$$\widetilde{X} := \big[[0,\mathfrak{m}_0)_{\mathfrak{m}} \times [0,\infty)_r; \{(0,0)\}\big], \tag{7.10}$$

with front face denoted $\hat{X}$ and the lift of $\mathfrak{m} = 0$ denoted $X_\circ$. See Figure 7.1.

Thus, the closures of $\{(\mathfrak{m}, r_+(\mathfrak{m}))\colon 0 < \mathfrak{m} < \mathfrak{m}_0\}$ and $\{(\mathfrak{m}, r_{\mathcal{C}}(\mathfrak{m}))\colon 0 < \mathfrak{m} < \mathfrak{m}_0\}$ (the dashed lines in Figure 7.1) are smooth submanifolds of $\widetilde{X}$ which are transversal to $\hat{X} = [0,\infty]_{\hat{r}}$ (where $\hat{r} = \frac{r}{\mathfrak{m}}$) and $X_\circ = [0,\infty)$, respectively.

[38] The convergence is in fact uniform in half spaces, as shown in [Hin21].

[39] By this we mean that for every bounded open subset $\mathcal{U} \subset \mathbb{C}$ with $\partial\mathcal{U} \cap \mathrm{QNM}_{\mathrm{dS}} = \emptyset$, the set $\mathrm{QNM}(\mathfrak{m}) \cap \mathcal{U}$ converges to $\mathrm{QNM}_{\mathrm{dS}} \cap \mathcal{U}$ in the Hausdorff distance sense. More simply put: for every $\sigma_0 \in \mathrm{QNM}_{\mathrm{dS}}$ there exists a sequence $\sigma_j \in \mathrm{QNM}(\mathfrak{m}_j)$, $\mathfrak{m}_j \searrow 0$, with $\sigma_j \to \sigma_0$; and conversely if $\sigma_j \in \mathrm{QNM}(\mathfrak{m}_j)$ is a convergent sequence and $\mathfrak{m}_j \searrow 0$, then $\lim \sigma_j \in \mathrm{QNM}_{\mathrm{dS}}$.

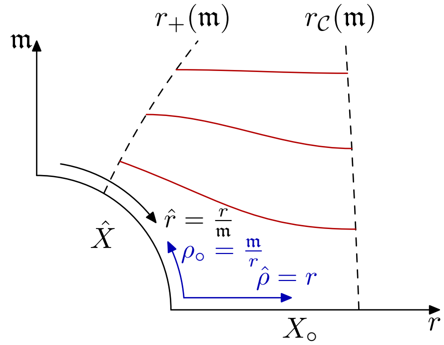

FIGURE 7.1. The resolved space (7.10), some local coordinates, and its boundary hypersurfaces. The red curves are the subspaces $\{\mathfrak{m}\} \times (r_+(\mathfrak{m}), r_{\mathcal{C}}(\mathfrak{m}))$ for three different values of $\mathfrak{m}$.

• *Step 2. Structure of $\widetilde{P}$.* We claim that $\widetilde{P} \in r^{-2}\mathrm{Diff}^2_{\mathrm{q}}$ (where it is defined). Since on the interval $(r_+(\mathfrak{m}), r_{\mathcal{C}}(\mathfrak{m}))$, we have $\mathfrak{m} \lesssim r \lesssim 1$, we can use r as a defining function of $\hat{X}$, and then q-vector fields are locally spanned over $\mathcal{C}^\infty(\widetilde{X})$ by $r\partial_r$. Note then that

$$\mu_{\mathfrak{m}}(r) = 1 - \frac{2\mathfrak{m}}{r} - \frac{\Lambda r^2}{3}$$

is a smooth function of $(\mathfrak{m}, r)$ when $r \geq r_0 > 0$, while near $\hat{X}$ we can use the coordinates $\hat{\rho} = r$ and $\rho_\circ = \frac{\mathfrak{m}}{r}$ in which $\mu_{\mathfrak{m}} = 1 - 2\rho_\circ - \frac{\Lambda\hat{\rho}^2}{3}$ is smooth as well. In view of (7.9), we thus indeed see that $(r^2 P_{\mathfrak{m}})_{\mathfrak{m}\in(0,\mathfrak{m}_0)}$ is an element of $\mathrm{Diff}^2_{\mathrm{q}}$.

The models of $\widetilde{P}$ are obtained by taking $\mathfrak{m} \searrow 0$ for fixed $r > 0$ (giving $P_\circ$) or by rescaling by $\mathfrak{m}^2$ and letting $\mathfrak{m} \searrow 0$ for fixed $\hat{r} = \frac{r}{\mathfrak{m}}$ (giving $\hat{P}$), so

$$\begin{aligned} P_\circ &= r^{-2}\partial_r r^2\Big(1 - \frac{\Lambda r^2}{3}\Big)\partial_r + \Big(1 - \frac{\Lambda r^2}{3}\Big)^{-1}\sigma^2, \\ \hat{P} &= \hat{r}^{-2}\partial_{\hat{r}}\hat{r}^2\Big(1 - \frac{2}{\hat{r}}\Big)\partial_{\hat{r}}. \end{aligned}$$

But $P_\circ$ is precisely the expression for the de Sitter wave operator $\Box_{g_{\mathrm{dS}}}$ applied to a function of the form $e^{-i\sigma t}u(r)$, as can be easily checked using (7.8); and $\hat{P}$ is the expression for the wave operator for a Schwarzschild black hole of mass 1 applied to a function of r only (no time dependence).[40]

• *Step 3. Strategy for the rest of the proof.* Having found the correct setup for the uniform analysis of $P_{\mathfrak{m}}$ as $\mathfrak{m} \searrow 0$, the remainder of the proof is rather similar to that of Theorem 7.1.[41] Besides estimates for solutions $\widetilde{u}$ of the equation $\widetilde{P}\widetilde{u} = \widetilde{f}$ on weighted q-Sobolev spaces which only use the principal symbol of $\widetilde{P}$, one needs to invert the model

[40]This is analogous to the situation encountered in §7.1 where the spectral parameter λ does not affect the $\hat{X}$-model operator since this model operator is obtained by first multiplying by ϵ^2, here $\mathfrak{m}^2$, before taking the limit $\epsilon \searrow 0$ or $\mathfrak{m} \searrow 0$.

[41]We ignore here the question of how to deal with the singularity of the coefficients of $P_{\mathfrak{m}}$ at $r_+(\mathfrak{m})$ and $r_{\mathcal{C}}(\mathfrak{m})$; these turn out to removable if one expresses the SdS metric in better coordinates.

operators $P_\circ$ and $\hat{P}$ on appropriate function spaces; in the rough discussion below, we ignore the issues at $r_{\mathcal{C}}(\mathfrak{m})$ and $r_+(\mathfrak{m})$, and instead focus on the b-behavior at $r = 0$ in $X_\circ$ and $\hat{r} = \infty$ in $\hat{X}$, respectively.

The inversion of $P_\circ$ has two facets. The first one is that $P_\circ$ is defined on $[0, r_{\mathrm{dS}})$, or if one puts back the spherical dependence on $[0, r_{\mathrm{dS}}) \times \mathbb{S}^2$, i.e. the blow-up of the r_{dS}-ball in $\mathbb{R}^3$ at 0. Thus, one needs to show that the possibly singular behavior of elements of $\ker P_\circ$ at $r = 0$ is actually a removable singularity (cf. Step 2 in the proof of Theorem 7.1). The second one is that $P_\circ$ is invertible only if $\sigma \notin \mathrm{QNM}_{\mathrm{dS}}$. The invertibility of $\hat{P}$ on a b-Sobolev space on $\hat{X}$ with appropriate weights at $\partial\hat{X}$ is not difficult to show.

Combining the three ingredients implies the injectivity of $P_{\mathfrak{m}}$ for all small $\mathfrak{m} > 0$ and thus shows that $\sigma \notin \mathrm{QNM}(\mathfrak{m})$ when $\sigma \notin \mathrm{QNM}_{\mathrm{dS}}$ and $\mathfrak{m} > 0$ is small. The full result is proved in [HX22] (for fixed spherical harmonics) and in [Hin21] (without this restriction, and globally for σ in any half space $\operatorname{Im} \sigma > -C$). $\square$

7.3. Problems.

Problem 7.1 (Spectrum under singular perturbations: higher dimensions)**.** Prove Theorem 7.1 when X has dimension $n \geq 3$.

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Department of Mathematics, ETH Zürich, Rämistrasse 101, 8092 Zürich, Switzerland
Email address: peter.hintz@math.ethz.ch