Lecture 1: rectifiable sets. June 24, 2024

(Geometric Measure Theory)

Geometric measure theory is by now a well-developed branch of geometric analysis, collecting techniques to have a *calculus of variations* for the *area* functional, as well as other important geometric functionals. Today it is also used in many other situations: for instance, rectifiable sets appear while looking at energy concentration, singular set stratifications, properties of singular kernels, and so on.

1.1. **The quest for compactness.** When we look for a minimizer or a critical point of a functional $F : \mathcal{S} \to \mathbb{R}$, initially defined on a class $\mathcal{S}$ of smooth objects, we often follow a seemingly roundabout approach. Namely, we first look for such a point in a wider class $\mathcal{W} \supset \mathcal{S}$, where the functional F can still be defined in a meaningful way. Only *after* we located a critical point $p_0 \in \mathcal{W}$, using its criticality we can show that actually we have $p_0 \in \mathcal{S}$.

For instance, given a smooth domain $\Omega \subset \mathbb{R}^n$, the set $\mathcal{S}$ could be the space of smooth functions $\bar{\Omega} \to \mathbb{R}$, with a given boundary condition $u|_{\partial\Omega} = g$, and we could be interested in minimizing the Dirichlet energy

$$F(u) := \int_\Omega |du|^2.$$

Rather than finding a minimizer in $\mathcal{S}$, we look for it in $\mathcal{W} := W^{1,2}(\Omega)$, the Sobolev space of functions admitting just a *weak* notion of square-summable differential. Why is it easier to work on this space? The reason is that it is *compact*, in a suitable sense: given a bounded sequence of functions $(u_k) \subset \mathcal{W}$, we can extract a subsequence converging to a function $u \in \mathcal{W}$ (while this does not hold in $\mathcal{S}$: for instance, in dimension one, the functions $\sqrt{x^2 + 2^{-k}}$ are smooth but their natural limit $|x|$ is not).

Thus, in this example, we can consider a sequence such that $\lim_{k\to\infty} F(u_k) = \inf_{\mathcal{S}} F$ and, after extracting a (weak) limit $u \in \mathcal{W}$, check that $F(u) = \inf_{\mathcal{S}} F = \inf_{\mathcal{W}} F$. Hence, this particular u minimizes F on $\mathcal{W}$; this minimality property is the fact which allows to conclude that u must in fact be smooth (intuitively, too much irregularity, i.e., oscillation of u would result in a large Dirichlet energy, contradicting the minimality of u), and so actually $u \in \mathcal{S}$.

Roughly speaking, for us $\mathcal{S}$ will consist of the set of compact smooth k-dimensional submanifolds of a given Euclidean space $\mathbb{R}^n$ (or a Riemannian manifold (M^n, g)), whose boundary ∂S is either empty or assigned. We will often focus on the simpler case $M^n = \mathbb{R}^n$, since the theory is very similar in a curved ambient (M^n, g).

The main question that we will deal with is the following: what is an appropriate space $\mathcal{W}$ in this case? As in the previous example, it should be a superset of $\mathcal{S}$ and its elements should correspond to a weak notion of submanifold. Generally speaking, there are three main approaches:

- we view submanifolds as images of parametrizations;
- we view them as level sets of $\mathbb{R}^{n-k}$-valued maps;
- we view them in an intrinsic way, merely as subsets of M^n.

Each of these approaches has its own advantages. In the first two, the obvious thing to do is to enlarge the class of maps (e.g., from smooth ones to Sobolev ones). In this course, we will mostly focus on the third approach.

In the next two lectures we will see two possible answers to the previous question in the intrinsic viewpoint: the first one (*currents*) is designed for minimization, while the second one (*varifolds*) is designed to find general critical points.

Currents and varifolds are fundamentally different objects, and indeed their general definitions look completely different from each other. On the other hand, for simplicity, we will look only at *rectifiable* currents and varifolds, with *integer multiplicity*. In this special setting, it would seem that they only differ in that currents come with an orientation, while varifolds do not have one.

Even if we look at rectifiable currents and varifolds, the fundamental difference is the *topology*, i.e., the meaning of convergence $\Sigma_j \to \Sigma$, as well as the assumptions guaranteeing precompactness of a sequence, i.e., the conditions on a sequence (Σ_j) guaranteeing that a subsequence $\Sigma_{j_\ell} \to \Sigma$ converges to some limit Σ.

Rectifiable currents and varifolds are objects whose backbone is a special type of subset of $\mathbb{R}^n$, namely a *k-rectifiable set*. In this lecture, we will focus on the study of such sets.

1.2. **Hausdorff measures.** Before introducing k-rectifiable sets, it is useful to recall that there is a very robust definition of k-dimensional area, as well as dimension, of an arbitrary set. In fact, it is so general that it makes sense in any metric space (X, d).

Given a subset $S \subseteq X$ and a number $k \geq 0$ (for this definition, it does not need to be an integer), we let

$$\mathcal{H}^k_\delta(S) := \frac{\omega_k}{2^k} \inf \Big\{ \sum_j \operatorname{diam}(E_j)^k \mid S \subseteq \bigcup_j E_j,\ \operatorname{diam}(E_j) \leq \delta \Big\},$$

where the infimum is taken over all finite or countable covers $S \subseteq \bigcup_j E_j$ with the constraint that each set E_j has diameter at most δ, and we define the k-dimensional *Hausdorff measure* of S to be

$$\mathcal{H}^k(S) := \lim_{\delta\to 0} \mathcal{H}^k_\delta(S)$$

(the limit exists since clearly $\mathcal{H}^k_\delta(S)$ increases as δ decreases). We adopt the usual convention that $0^0 = 1$, while $\omega_k := \frac{2^k \pi^{k/2}}{\Gamma(\frac{k}{2}+1)}$ is a dimensional constant, which agrees with the volume of the unit k-dimensional ball when $k \in \mathbb{N}$.

Remark 1.1. If E_j is a k-dimensional ball in $\mathbb{R}^k$ (or in an affine k-plane of $\mathbb{R}^n$), then $\frac{\omega_k}{2^k}\operatorname{diam}(E_j)$ is precisely its area. This suggests taking the sets E_j to be balls; however, in general this would give another measure, called *spherical Hausdorff measure* (although it is comparable to the previous one and actually matches with it on sufficiently nice sets).

Example 1.2. For $k = 0$, we have $\mathcal{H}^0(S) = \#S$ if S is finite, while $\mathcal{H}^0(S) = +\infty$ otherwise.

Example 1.3. If S is an infinite spiral, then we see why we need to consider smaller and smaller values of δ, and constrain our cover to be made of sets of diameter $\le \delta$. Roughly speaking, δ is the "resolution" that we are using while looking at our set S. For any fixed $\delta > 0$ the quantity $\mathcal{H}^1_\delta(S)$ is finite in this case, since S is covered by finitely many small balls, while we get the correct result $\mathcal{H}^1(S) = \infty$ (reflecting the fact that S has infinite length) only once we let $\delta \to 0$.

When $k \in \mathbb{N}$, one can show the following fact (which is the reason why we have a carefully chosen normalization constant $\frac{\omega_k}{2^k}$).

Proposition 1.4. *Assume that (X, d) is the Euclidean space, or a Riemannian manifold with the induced geodesic distance. If S is a k-dimensional embedded submanifold, then $\mathcal{H}^k(S)$ coincides with the k-dimensional area of S.*

In fact, since a submanifold S is almost flat at small scales, the previous proposition reduces without much effort to the verification that $\mathcal{H}^k$ agrees with the Lebesgue measure on $\mathbb{R}^k$. However, this fact is surprisingly nontrivial. It can be deduced from the *isodiametric inequality*, asserting that the round ball $B_R(0) \subset \mathbb{R}^k$ maximizes the Lebesgue measure among sets of diameter $2R$.

Remark 1.5. Strictly speaking, $\mathcal{H}^k$ is an outer measure defined on all the subsets of X (warning: many people write "measure" to mean "outer measure"). It becomes a genuine measure, satisfying σ-additivity, if we restrict to Borel subsets of X.

Exercise 1.6. In the definition of Hausdorff measure, check that we get the same result if we require that the sets E_j are closed. Also, check that we can require them to be open (but not necessarily closed).

Exercise 1.7. If $X \subseteq Y$ are metric spaces, with d_X given by the restriction of d_Y, then for any $S \subseteq X$ the quantity $\mathcal{H}^k(S)$ is the same if we compute it in X or in Y.

Exercise 1.8. If $f : X \to Y$ is an L-Lipschitz map between metric spaces, then $\mathcal{H}^k(f(S)) \leq L^k \mathcal{H}^k(S)$.

Using the Hausdorff measure, we can define a notion of dimension: the *Hausorff dimension* of a set $S \subseteq X$ is

$$\mathcal{H}\text{-dim}\, S := \sup\{k \geq 0 \,:\, \mathcal{H}^k(S) < \infty\} = \inf\{k \geq 0 \,:\, \mathcal{H}^k(S) = \infty\}.$$

(By convention, the sup is 0 if the first set is empty, while the inf is ∞ if the second set is empty.) As expected, a nonempty k-dimensional submanifold of $\mathbb{R}^n$ or (M^n, g) has Hausdorff dimension k.

Sometimes, other variants are more appropriate, such as the Minkowski and packing dimensions (based on different ways to measure the k-dimensional area), although we will not look at them.

Exercise 1.9. Check that there is a (unique) value $k_S \in [0, \infty]$ such that $\mathcal{H}^k(S) = \infty$ for all $k < k_S$ and $\mathcal{H}^k(S) = 0$ for all $k > k_S$, showing the previous equality in the definition of Hausdorff dimension.

Exercise 1.10. If $f : X \to Y$ is α-Hölder for some $\alpha \geq 0$, then $\mathcal{H}\text{-dim}\, f(S) \leq \alpha \mathcal{H}\text{-dim}\, S$.

Exercise 1.11. The Cantor set $C \subset \mathbb{R}$ has Hausdorff dimension $\frac{\log 2}{\log 3}$.

1.3. **Covering theorems and density of measures.** Given a Borel set $S \subseteq X$ with finite or σ-finite Hausdorff measure $\mathcal{H}^k$, we can consider the measure

$$\mathcal{H}^k \llcorner S$$

on X, which is then a finite or σ-finite measure on Borel sets. The same holds if we consider a Borel function $f : S \to (0, \infty)$ and we let

$$d\mu := f \, d(\mathcal{H}^k \llcorner S),$$

i.e., for any Borel set E, we let

$$\mu(E) := \int_{E \cap S} f(x) \, d\mathcal{H}^k(x).$$

It is a very useful fact that measures of this kind can be detected by looking at their *density*, provided that they are finite on bounded sets.

Theorem 1.12. *A measure μ on Borel subsets of X, finite on bounded sets, can be written as above if and only if the* upper k-density

$$\Theta_k^*(\mu, x) := \limsup_{r\to 0} \frac{\mu(B_r(x))}{\omega_k r^k} \in (0,\infty) \quad \mu\text{-a.e.}$$

The proof is based on the following tool, called the *Vitali covering lemma*, which we will not prove. Given a collection of balls covering a given set, it gives a more efficient (i.e., less redundant) subcollection.

Lemma 1.13 (Vitali covering lemma). *Given a (possibly uncountable) collection of open balls $(B_{r_j}(x_j))_{j\in J} \subseteq X$ with $\sup_j r_j < \infty$, we can find a subset J' of indices such that the subcollection $(B_{r_j}(x_j))_{j\in J'}$ consists of* disjoint *balls and*

$$\bigcup_{j\in J} B_{r_j}(x_j) \subseteq \bigcup_{j\in J'} B_{5r_j}(x_j).$$

The same holds for collections of closed balls.

Remark 1.14. The constant 5 is definitely not sharp: the proof shows that we can replace it with any constant greater than 3. Also, if we wish to cover only the set of centers, any constant greater than 2 will be enough (this applies if, for any point x of a set S, we can find a ball $B_r(x)$ with some particular property, and we wish to find a disjoint subcollection of balls such that the dilated balls still cover S).

Proof of Theorem 1.12. ($\Rightarrow$) Let S_0 be the set of points in S where the upper density is 0 and S_∞ the set where it is ∞. We need to show that

$$\mu(S_0) = \mu(S_\infty) = 0.$$

To check that $\mu(S_\infty) = 0$, we note that for any fixed $\Lambda > 0$ and any $x \in S_0$ there exists $r > 0$ such that $\mu(B_r(x)) \geq \Lambda r^k$. We can also find such an $r \in (0,\delta)$ for an arbitrary $\delta > 0$ fixed in advance. Applying Vitali's covering lemma, we can find a disjoint subcollection $(B_{r_j}(x_j))_{j\in J'}$ such that the dilated balls $B_{5r_j}(x_j)$ cover the original union of balls, and in particular

$$S \subseteq \bigcup_{j\in J'} B_{5r_j}(x_j).$$

Note that J' is at most countable (since μ is σ-finite and $\mu(B_{r_j}(x_j)) > 0$ for each of these disjoint balls). Thus,

$$\mathcal{H}^k_{10\delta}(S_\infty) \leq \frac{\omega_k}{2^k} \sum_{j\in J'} (5r_j)^k \leq \frac{5^k\omega_k}{2^k}\Lambda^{-1} \sum_{j\in J'} \mu(B_{r_j}(x_j)) \leq \frac{5^k\omega_k}{2^k}\Lambda^{-1}\mu(X),$$

where we used the fact that the balls in the subcollection are disjoint in the last inequality. If $\mu(X) < \infty$ then we are done by letting $\Lambda \to \infty$ and $\delta \to 0$ (the order does not matter), deducing $\mathcal{H}^k(S_0) = 0$. Otherwise we repeat the argument locally, replacing S_0 with a bounded set $S_0 \cap B_R(x_0)$.

To see that $\mu(S_0) = 0$, we can assume without loss of generality that f is bounded and $\mathcal{H}^k(S)$ is finite (as S_0 increases when we replace μ with a measure $\mu' \leq \mu$). Given $\varepsilon, \rho > 0$, we consider the set

$$S_0' := \{x \in S_0 \,:\, \mu(B_r(x)) < \varepsilon r^k \text{ for all } 0 < r < \rho\}.$$

Given any finite or countable cover of S_0' by closed sets E_j of diameter $< \rho$ intersecting S_0', we can bound

$$\mu(S_0') \leq \sum \mu(E_j) \leq \varepsilon \sum \operatorname{diam}(E_j)^k.$$

Taking the infimum over such covers, we get

$$\mu(S_0') \leq \frac{2^k \varepsilon}{\omega_k} \mathcal{H}^k_\delta(S_0') \leq \frac{2^k \varepsilon}{\omega_k} \mathcal{H}^k(S_0').$$

Since ρ was arbitrary (and the increasing union of the previous sets as $\rho \to 0$ includes S_0), we deduce

$$\mu(S_0) \leq \frac{2^k \varepsilon}{\omega_k} \mathcal{H}^k(S_0).$$

Finally, letting $\varepsilon \to 0$ we obtain $\mu(S_0) = 0$.

($\Leftarrow$) Consider the Borel set S consisting of points where the upper density is in $(0, \infty)$. By assumption, μ is concentrated on this set. We will show that $\mathcal{H}^k \llcorner S$ is σ-finite and that $\mu \ll \mathcal{H}^k \llcorner S$, giving $d\mu = f\, d(\mathcal{H}^k \llcorner S)$ for some $f \geq 0$, from which the claim follows (by possibly replacing S with $S \setminus \{f = 0\}$; actually, one can show that $f > 0$ $\mathcal{H}^k$-a.e. on S, and hence this replacement is unnecessary).

We assume for simplicity that $\mu(X) < \infty$ (otherwise we perform the following argument locally). Given $t > 0$, let

$$S_t := \{x \in X \,:\, \Theta^*_k(\mu, x) > t\}.$$

The same argument used at the beginning of the proof gives

$$\mathcal{H}^k(S_t) \leq 5^k t^{-1} \mu(X).$$

Since $S \subseteq \bigcup_{t>0} S_t$, we obtain the σ-finiteness of $\mathcal{H}^k \llcorner S$.

Finally, assume that a Borel set E satisfies $\mathcal{H}^k(E \cap S) = 0$. We claim that $\mu(E) = 0$. To check this, we let $E_t := \{x \in E \cap S \,:\, \Theta^*_k(\mu, x) < t\}$. Again, arguing as above (with $\omega_k t$ in place of ε) we obtain

$$\mu(E_t) \leq 2^k t \mathcal{H}^k(E_t) = 0,$$

and hence $\mu(E) = \mu(E \cap S) = 0$ (as $\bigcup_{t>0} E_t = E \cap S$). $\square$

A slightly more refined proof gives something more in $\mathbb{R}^n$ or in a Riemannian manifold (M^n, g), as illustrated by the next exercises. This will follow from another powerful covering lemma.

Lemma 1.15 (Besicovitch covering lemma). *Assume that we have a collection $\mathcal{F}$ of open balls and a set $S \subseteq \mathbb{R}^n$ such that for all $x \in S$ there exists $r \leq R$ with $B_r(x) \in \mathcal{F}$, for some $R > 0$ independent of x. Then there exist $C(n)$ subcollections $\mathcal{F}_j \subseteq \mathcal{F}$, each consisting of disjoint balls, such that*

$$S \subseteq \bigcup_{j=1}^{C(n)} \bigcup_{B \in \mathcal{F}_j} B.$$

The same statement holds for closed balls. The same holds on a Riemannian manifold (M^n, g), provided that R is small enough (depending on the particular manifold).

Exercise 1.16. Assume that a collection $\mathcal{F}$ of closed balls is a *fine cover* of a Borel set $S \subseteq M^n$, i.e., for every $x \in S$ there exists a ball $\bar{B}_r(x)$ in the collection with arbitrarily small $r > 0$. Deduce from the previous covering lemma that given a (positive) measure μ, finite on bounded sets, we can find a disjoint subcollection $\mathcal{F}' \subseteq \mathcal{F}$ such that

$$\mu\Big(S \setminus \bigcup_{B \in \mathcal{F}'} B\Big) = 0.$$

Using Vitali, deduce the same on an arbitrary separable metric space assuming that μ is *doubling* (so that $\mu(B_{5r}(x)) \leq C\mu(B_r(x))$).

Exercise 1.17. Given a (positive) measure μ on $\mathbb{R}^n$, finite on bounded sets, and a Borel set $T \subseteq \mathbb{R}^n$, assume that $\Theta^*_k(\mu, x) \geq t$ for all $x \in T$ and show that $\mu \geq t\mathcal{H}^k \llcorner T$. Similarly, assuming that $\Theta^*_k(\mu, x) \leq t$ for all $x \in T$, show that $\mu \leq 2^k t\mathcal{H}^k \llcorner T$.

1.4. **Rectifiable sets.** Given a metric space (X, d), a set $S \subseteq X$ is *k-rectifiable* if we can write

$$S \subseteq E_0 \cup \bigcup_{j=1}^{\infty} E_j, \quad E_j = f_j(A_j),$$

where $\mathcal{H}^k(E_0) = 0$ and $f_j : A_j \to M$ is Lipschitz for some $A_j \subseteq \mathbb{R}^k$. (Some people call sets as above *countably $\mathcal{H}^k$-rectifiable.*)

We will restrict to the case of $\mathbb{R}^n$ for simplicity, even if all of the results will also hold on any Riemannian manifold.

Remark 1.18. In the definition, we could equivalently use C^1 functions f_j defined on an open set $U_j \supset A_j$. Essentially, the reason is that we can always extend f_j to be defined on $\mathbb{R}^k$ and the differential df_j of a Lipschitz function exists a.e. and is measurable. Thus, we can write $\mathbb{R}^k = A_{j0} \cup \bigcup_{\ell=1}^\infty A_{j\ell}$ with A_{j0} negligible and $A_{j\ell}$ compact with $df_j|_{A_{j\ell}}$ continuous, hence by Whitney's extension theorem we can find a C^1 function $f_{j\ell}$ defined in a neighborhood of $A_{j\ell}$ such that $f_{j\ell}|_{A_{j\ell}} = f_j|_{A_j}$, showing the claim. On the other hand, one can show that C^1 cannot be replaced with C^2.

Remark 1.19. As we will see later, one can further require that each f_j is a C^1 embedding. Hence, in $\mathbb{R}^n$ a set is rectifiable if and only if it can be covered with countably many C^1 graphs, plus an $\mathcal{H}^k$-negligible set.

Given that rectifiable sets provide a suitable framework for our weak notions of submanifold, it is important to be able to recognize them among general sets. In order to formulate criteria to do so, it is again convenient to look at measures instead.

A positive Radon measure μ is said to be *k-rectifiable* if it is of the form

$$d\mu = f\, d(\mathcal{H}^k \llcorner S)$$

for some Borel k-rectifiable set S.

Exercise 1.20. If a set S is k-rectifiable and its projection on each of the $\binom{n}{k}$ coordinate k-planes is negligible, then $\mathcal{H}^k(S) = 0$.

Exercise 1.21. Take a variant of Cantor's set where we start from $[0,1]$ and each time, from each segment, we remove the middle segment of half the width. So from $[0,1]$ we pass to $[0,\frac{1}{4}] \cup [\frac{3}{4},1]$, and so on. Calling C the resulting set, check that $\mathcal{H}^1(C \times C) \in (0,\infty)$ but $C \times C$ is not rectifiable. [Hints: to show a lower bound on $\mathcal{H}^1$, project the set on the line $\{y = 2x\}$; to show that it is not rectifiable, check that the projection on the two axes is negligible.]

We will now state one of the most well-known rectifiability criteria.

Theorem 1.22 (classical rectifiability criterion). *A (positive) measure μ, finite on bounded sets, is rectifiable if and only if*

$$0 < \Theta_{*,k}(\mu,x) \le \Theta_k^*(\mu,x) < \infty \quad \text{for } \mu\text{-a.e. } x$$

and, for μ-a.e. x, there exists a k-plane P_x such that each limit of the form

$$\lim_{j\to\infty} r_j^{-k} (\delta_{x,r_j})_* \mu$$

is a multiple of P_x, i.e., equals $c\mathcal{H}^k \llcorner P_x$ for a constant $c > 0$ (depending on the sequence).

Here $\Theta_{*,k}(\mu, x) := \liminf_{r\to 0} \frac{\mu(B_r(x))}{\omega_k r^k}$ and $\delta_{x,r_j}(y) := \frac{y-x}{r_j}$ is the map making $B_{r_j}(x)$ correspond to $B_1(0)$, allowing us to blow-up at the point x. Also, the previous limit is in the usual weak-* topology on measures.

Proof. We will just sketch the proof of the useful implication ($\Leftarrow$). By the previous results, we already know that $d\mu = f\, d(\mathcal{H}^k \llcorner S)$ for some Borel set S of σ-finite $\mathcal{H}^k$ measure.

Let us fix a k-plane P_0 and a constant $\Lambda > 0$. We consider the set S' of points x such that

$$\Lambda^{-1} \leq \Theta_{*,k}(\mu, x) \leq \Theta^*_k(\mu, x) < \infty$$

and such that any limit of the form $\lim_{j\to\infty} r_j^{-k}(\delta_{x,r_j})_*\mu$ equals $c\mathcal{H}^k \llcorner P_x$, for some plane P_x such that $\|P_x - P_0\| \leq \delta$; here δ is a small constant depending only on k and n. Clearly, it suffices to see that S' is rectifiable, since μ-a.e. point of S belongs to a countable union of such sets.

Given a radius $\rho > 0$, we further restrict attention to the set $S'' \subseteq S'$ of points x such that

$$d\Big(\frac{(\delta_{x,r})_*\mu}{\mu(B_r(x))}, \frac{\mathcal{H}^k \llcorner P_x}{\omega_k}\Big) \leq \delta \quad \text{for all } r < \rho.$$

Here we restrict the two measures on the ball $B_1(0)$ and we use a distance d metrizing the weak-* topology on probability measures on $B_1(0)$. Since the latter is compact, it is again easy to see (by a compactness-and-contradiction argument) that S' is the union of these sets S'' as $\rho > 0$ varies.

The simple geometric observation now is the following: given $x, x' \in S''$ with $|x - x'| \leq \frac{\rho}{2}$, let

$$r := 2|x - x'|.$$

Then $\delta_{x,r}$ makes x' correspond to a point y' at distance $\frac{1}{2}$ from the origin. Also, $\frac{(\delta_{x,r})_*\mu}{\mu(B_r(x))}$ is close to $\frac{\mathcal{H}^k \llcorner P_x}{\omega_k}$, but the image $\delta_{x,r}(B_{sr}(x')) = B_s(y')$ has mass at least $c(\Lambda)s^n > 0$ with respect to this rescaled measure, for any $s \in (0, \frac{1}{4})$: this holds thanks to the definition of S', which ensures that $\mu(B_{sr}(x')) \simeq (sr)^k$ and $\mu(B_r(x)) \simeq r^k$.

Thus, y' must be close to P_x, and hence to P_0. This tells us that $\frac{x-x'}{|x-x'|}$ is almost parallel to P_0, concluding the proof that S'' is rectifiable thanks to the next exercise below. $\square$

Remark 1.23. The previous result is still true if the plane P_x is also allowed to depend on the sequence $r_j \to 0$. In this case rectifiability is much harder to prove and the result is due to Marstrand–Mattila.

Exercise 1.24. Conclude the previous sketch by showing the following fact: if $\frac{x-x'}{|x-x'|}$ is almost parallel to a fixed k-plane, for all pairs of points in a given set, then this set is included in a Lipschitz graph.

The following is a very deep (and very difficult to prove) theorem due to Preiss.

Theorem 1.25 (Preiss rectifiability criterion). *A Radon measure $\mu \geq 0$ is k-rectifiable if and only if the limit*

$$\lim_{r\to 0} \frac{\mu(B_r(x))}{r^k}$$

exists and belongs to $(0,\infty)$ for μ-a.e. x.

Exercise 1.26. Show the implication ($\Rightarrow$).

Remark 1.27. We already know that the assumption in the difficult implication ($\Leftarrow$) guarantees that the measure has the form $d\mu = f\, d(\mathcal{H}^k \llcorner S)$. If $f = 1$ and we assume that the previous limit is equal to 1 a.e., then the rectifiability of S is easier to prove (this was proved earlier by Besicovitch–Marstrand–Mattila).

As we will see, in many situations, the previous limit exists because μ enjoys a special *monotonicity* property, namely the function $\frac{\mu(B_r(x))}{r^k}$ is increasing in r for any given x. However, in such situations the same facts leading to this monotonicity can typically be used to obtain rectifiability in a simpler way.

1.5. **Area and coarea formulas.** Given a locally Lipschitz map $f : \Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^k$ open, we have the following formula if f is injective:

$$\mathcal{H}^k(f(S)) = \int_S J_f\, d\mathcal{L}^k,$$

where $S \subseteq \Omega$ is a Borel set and $J_f(x) = \sqrt{\det(df(x)^T df(x))}$ is often called the *Jacobian* of f at x. Recall that locally Lipschitz functions are differentiable a.e., making the right-hand side well-defined (also, it is part of the statement that $f(S)$ belongs to the $\mathcal{H}^k$-completion of Borel sets).

While we will not prove this in detail, we can easily get some intuition why this formula should hold. Near a differentiability point x_0, f behaves like the linear map $x \mapsto Ax$, with $A := df(x_0)$. Since we can write $A = RDR'$, for two linear isometries $R \in O(n)$, $R' \in O(k)$ and a "diagonal" matrix (i.e., rectangular but with $d_{ij} = 0$ if $i \neq j$), it is intuitive that the image of a set E contained in a small neighborhood of x_0 has area

$$\mathcal{H}^k(f(E)) \approx \mathcal{H}^k(A(E)) = \prod_{j=1}^k |d_{jj}| \cdot \mathcal{L}^k(E)$$

and we have

$$J_f(x_0)^2 = \det(A^T A) = \det(D^T R^T R D) = \det(D^T D) = \prod_{j=1}^k |d_{jj}|^2.$$

On the other hand, the set N of points where f is not differentiable is negligible, and hence $\mathcal{H}^k(N) = \mathcal{L}^k(N) = 0$, so that also $\mathcal{H}^k(f(N)) = 0$ (since Lipschitz functions map $\mathcal{H}^k$-negligible sets to $\mathcal{H}^k$-negligible sets). The following is a slightly more general fact.

Theorem 1.28 (area formula). *Given f as above, not necessarily injective, and given $h : S \to [0, \infty)$ measurable, we have*

$$\int_{f(S)} \Big(\sum_{x \in f^{-1}(y)} h(x) \Big) \, d\mathcal{H}^k(y) = \int_S h(x) J_f(x) \, dx$$

(again, the measurability of the integrand in the left-hand side is part of the statement).

Note that the previous result is only interesting when $k \leq n$, since otherwise both sides vanish.

Exercise 1.29. Taking for granted the area formula and Whitney's extension theorem, give a rigorous proof of the fact that a k-rectifiable set in $\mathbb{R}^n$ is included in a countable union of C^1 graphs, plus an $\mathcal{H}^k$-negligible set.

We also have the following *coarea formula*, which deals with situations where the dimension of the domain is larger than the dimension of the codomain.

Theorem 1.30 (coarea formula). *Assume that $m \geq n$ and let $k := m - n$. Given $f : \Omega \to \mathbb{R}^n$ locally Lipschitz, with $\Omega \subseteq \mathbb{R}^m$ open, and given $h : S \to [0, \infty)$ measurable, the set $f^{-1}(y)$ is k-rectifiable for a.e. y and*

$$\int_{f(S)} \Big(\int_{f^{-1}(y)} h \, d\mathcal{H}^k \Big) \, dy = \int_S h J_f,$$

where now $J_f(x) := \sqrt{\det(df(x) df(x)^T)}$.

Although the coarea formula reduces to the area formula when $m = n$, it is considerably harder to prove in general, since (after a reduction to the case $h \equiv 1$) it involves integrating the Hausdorff measure $\mathcal{H}^{m-n}(f^{-1}(y))$ over y, whose measurability is not obvious at all. Ultimately, the proof reduces again to the linear case, although this reduction is really the difficult part. Moreover, although not necessary to state the formula, the previous statement also gives the rectifiability of a.e. level set.

Exercise 1.31. Given a Borel set $S \subseteq \mathbb{R}^m$ with $\mathcal{L}^m(S) < \infty$ and given $0 < \sigma < \tau < \infty$, show that there exists $r \in (\sigma, \tau)$ such that

$$\mathcal{H}^{m-1}(S \cap \partial B_r(0)) \leq \frac{1}{\tau - \sigma} \mathcal{L}^m(S).$$

Remark 1.32. Similar formulas hold if we replace Euclidean spaces with Riemannian manifolds M and N. In this case, we need to compute the Jacobian $J_f(x)$ with respect to (arbitrary) orthonormal bases of T_xM and $T_{f(x)}N$.

Remark 1.33. The last theorem is a more robust version of Sard's theorem: the latter says that if f is sufficiently regular (C^1 is not enough when $m > n$) then almost every level set is a smooth submanifold. The last theorem says that, assuming merely f Lipschitz, a.e. level set is certainly rectifiable (even if perhaps it is not a submanifold).

1.6. **Tangent space to a rectifiable set.** For many purposes, a rectifiable set is a good enough surrogate of a smooth submanifold. For instance, we can define the tangent space at a.e. point, as follows: given a k-rectifiable Borel set $S \subseteq \mathbb{R}^n$, we write it as a disjoint union

$$S = E_0 \cup \bigcup_{j=1}^{\infty} E_j,$$

as usual, where E_j is included in a C^1 graph M_j for $j \geq 1$ and $\mathcal{H}^k(E_0) = 0$; in particular, M_j is a smooth submanifold. If $x \in E_j$ for some $j \geq 1$, we let

$$T_xS := T_xM_j.$$

It can be shown that this is well-defined in the following sense: given another decomposition $S = E_0' \cup \bigcup_j E_j'$, the definition of tangent space T_xS given by this new decomposition matches with the previous one for $\mathcal{H}^k$-a.e. $x \in S$. Ultimately, this boils down to the classical fact that if $f, g : \mathbb{R}^k \to \mathbb{R}^n$ are two C^1 maps then

$$df(x) = dg(x) \quad \text{a.e. on } \{x \,:\, f(x) = g(x)\}.$$

Example 1.34. If $k = 1$ and S is the union of the x-axis and the y-axis, we could let $E_0 := \emptyset$, $E_1 := \{(s, 0) \mid s \in \mathbb{R}\}$, and $E_2 := \{(0, t) \mid t \in \mathbb{R} \setminus \{0\}\}$. With this decomposition, T_0S is the x-axis; on the other hand, with a similar decomposition (with the roles of the axes interchanged), T_0S would be the y-axis.

Remark 1.35. If we replace $\mathbb{R}^n$ with a manifold M, the tangent space of a k-rectifiable set S at a given $x \in S$ can be defined by reducing to the Euclidean case; it will be a k-plane in T_xM.

The area and coarea formulas can be generalized by taking S to be a rectifiable set in their statements (of dimension k and m, respectively). For instance, we have the following.

Theorem 1.36 (generalized area formula). *Given $S \subseteq \Omega \subseteq \mathbb{R}^{k'}$ a k-rectifiable Borel set and a locally Lipschitz map $f : \Omega \to \mathbb{R}^n$, we have*

$$\int_{f(S)} \Big(\sum_{x \in f^{-1}(y)} h(x) \Big) \, d\mathcal{H}^k(y) = \int_S h(x) J_f^S(x) \, d\mathcal{H}^k(x)$$

for any measurable $h : S \to [0, \infty)$.

In this situation, writing $S = E_0 \cup \bigcup_j E_j$ as above (with $E_j \subseteq M_j$), the restricted Jacobian $J_f^S(x)$ is defined as the one of the locally Lipschitz map $f|_{M_j} : M_j \to \mathbb{R}^n$ at x (if $x \in M_j$), thus computed using an orthonormal basis of $T_x M_j = T_x S$. Alternatively, one can show that $f|_{T_x S}$ is differentiable at x for $\mathcal{H}^k$-a.e. $x \in S$ (viewing $T_x S$ as an affine plane through x) and define $J_f^S(x)$ accordingly.

Remark 1.37. For a k-rectifiable measure μ, we have seen that at μ-a.e. x its blow-ups are all supported on the same k-plane. In fact, this coincides with the tangent space $T_x S$ of the underlying rectifiable set S. Thus, the tangent space has a more satisfactory definition for rectifiable measures rather than rectifiable sets (since for sets it depends on the specific decomposition used). Note that if S is k-rectifiable with locally finite $\mathcal{H}^k$ measure then $\mathcal{H}^k \llcorner S$ is a rectifiable measure; hence, in this case we have a better definition of $T_x S$: it is the support of all blow-ups of $\mathcal{H}^k \llcorner S$ at x (provided these are all supported on the same k-plane).

Useful references:

- Sections 2.3, 2.8, 2.9 of Ambrosio–Fusco–Pallara, *Functions of bounded variation and free discontinuity problems*;
- Chapters 5, 6, 7, 10 of Maggi, *Sets of finite perimeter and geometric variational problems*;
- Chapters 1, 3 of Simon, *Geometric measure theory*;
- for the more advanced rectifiability results: Chapters 4, 5, 6 of De Lellis, *Rectifiable sets, densities and tangent measures*;
- for the area and coarea formulas: Sections 2.10, 2.11, 2.12 of Ambrosio–Fusco–Pallara, *Functions of bounded variation and free discontinuity problems.*