# Lecture 2: currents. June 25, 2024

## (Geometric Measure Theory)

We now turn to the first weak notion of submanifold, namely *currents*, which are particularly well suited to minimize area under a given homological condition (assigned boundary or homology class).

2.1. **General currents, normal currents, and operations with them.** While we will soon focus on rectifiable currents (and actually on *integral* ones), it is useful to clarify what a current is in general.

Let $(M^n, g)$ be a Riemannian manifold. Similar to distributions, $k$-dimensional currents are elements in the dual of $\Omega_c^k(M)$, the space of smooth real-valued $k$-forms with compact support. In principle, this definition does not require $M$ to come with an assigned metric $g$.

Note that an *oriented* embedded $k$-dimensional submanifold $\Sigma \subseteq M$ (of locally finite area) gives automatically a current, denoted $[\![\Sigma]\!]$, by the formula

$$\langle [\![\Sigma]\!], \omega \rangle := \int_\Sigma \omega, \quad \text{for } \omega \in \Omega_c^k(M).$$

In the previous formula, it is important that $\Sigma$ is oriented in order to give a meaning to the right-hand side.

Currents can be used to define a homology theory. Indeed, given a $k$-dimensional current $T$, we can define its *boundary* $\partial T$, which is a $(k-1)$-dimensional current, by the formula

$$\langle \partial T, \eta \rangle := \langle T, \partial \eta \rangle, \quad \text{for } \eta \in \Omega_c^{k-1}(M).$$

If $\Sigma$ is a compact oriented *manifold with boundary*, then the current $\partial[\![\Sigma]\!]$ that we just defined coincides with the current $[\![\partial\Sigma]\!]$ arising from the boundary $\partial\Sigma$, since by Stokes

$$\langle [\![\partial\Sigma]\!], \eta \rangle = \int_{\partial\Sigma} \eta = \int_\Sigma d\eta = \langle [\![\Sigma]\!], d\eta \rangle = \langle \partial[\![\Sigma]\!], \eta \rangle.$$

Thus, the definition of $\partial T$ can be seen as a weak notion of boundary, generalizing the obvious one for manifolds with boundary.

Moreover, since $d \circ d = 0$ on forms, we immediately deduce that any current $T$ satisfies

$$\partial(\partial T) = 0.$$

In particular, given any class of currents (i.e., a choice of an additive subgroup $G_k$ of the space of $k$-dimensional currents on $M$, for each $k = 0, \dots, n$), assuming that it is preserved by the boundary operator (i.e., $\partial T \in G_{k-1}$ for all $T \in G_k$), we can define

$$H_k(M) := \frac{\ker \partial_k}{\text{image } \partial_{k+1}},$$

where $\partial_k : G_k \to G_{k-1}$ is the boundary operator (and $G_k := \{0\}$, $\partial_k := 0$ for $k \notin \{0, \dots, n\}$).

**Theorem 2.1.** *For the class of integral currents (defined below), this homology is naturally isomorphic with the usual singular homology $H_*(M, \mathbb{Z})$.*

While we will soon specialize to integral currents, there is a slightly larger class which is equally important, given by *normal currents* (although they are not really relevant to our story, concerning the calculus of variations for the area). This is the space of currents $T$ such that both $T$ and $\partial T$ have finite *mass*, where

$$\mathbb{M}(T) := \sup\{\langle T, \omega\rangle \mid \omega \in \Omega^k_c(M) \text{ such that } |\omega| \leq 1 \text{ pointwise}\}$$

is the definition of the mass of $T$. For this to make sense, it is now important to have an assigned metric $g$ on $M$ (so that $|\omega|$ is defined).

*Exercise* 2.2. Check that $\mathbb{M}([\![\Sigma]\!])$ is precisely the area of $\Sigma$.

*Remark* 2.3. The correct definition of mass is slightly more complicated and involves the *comass norm* on covectors, but the result is comparable with the one from the previous definition. In fact, the previous definition agrees with the correct one for rectifiable currents.

**Example 2.4.** Consider the set $\mathcal{C}$ of nonconstant, injective, constant-speed loops $\gamma : S^1 \to M$ and the set $\mathcal{A}$ of arcs $\gamma : [0,1] \to M$ (with the same properties). Given finite (positive) measures $\sigma, \tau$ on these sets, we can define a normal 1-current $T$ by

$$\langle T, \omega\rangle := \int_{\mathcal{C}} \langle [\![\gamma]\!], \omega\rangle \, d\sigma(\gamma) + \int_{\mathcal{A}} \langle [\![\gamma]\!], \omega\rangle \, d\tau(\gamma),$$

where $\langle [\![\gamma]\!], \omega\rangle := \int_\gamma \omega$. If $M$ is complete, a theorem due to Smirnov says that a normal 1-current $T$ always arises as a superposition of curves in this way (to be precise, one has to allow for something more general than loops, namely suitably normalized "infinite loops": consider the classical curves of irrational slope on $\mathbb{T}^2$). Moreover, the decomposition can be chosen to be irredundant, in the sense that

$$\mathbb{M}(T) = \int_{\mathcal{C}} \mathbb{M}([\![\gamma]\!]) \, d\sigma(\gamma) + \int_{\mathcal{A}} \mathbb{M}([\![\gamma]\!]) \, d\tau(\gamma)$$

and

$$\mathbb{M}(\partial T) = \int_{\mathcal{A}} \mathbb{M}(\partial [\![\gamma]\!]) \, d\tau(\gamma) = 2\tau(\mathcal{A}).$$

*Remark* 2.5. Even when $T$ has an irredundant representation involving only arcs, it does not need to be unique. For instance, the union of the two segments in the plane joining $(0,0)$ to $(1,1)$ and $(0,1)$ to $(1,0)$, respectively, can also be viewed as a union of two "wedges," joining $(0,0)$ to $(1,0)$ and $(0,1)$ to $(1,1)$, respectively.

By the Riesz representation theorem, if $T$ has finite mass then in fact we can think it as a finite $\Lambda^k(TM)$-valued measure, in the sense that there exists a finite (positive) measure $\mu$ and a Borel unit section $\tau : M \to \Lambda^k(TM)$ such that

$$\langle T, \omega \rangle = \int_M \langle \omega(x), \tau(x) \rangle \, d\mu(x),$$

where $\langle \omega(x), \tau(x) \rangle$ is the usual pairing between (multi)covectors and (multi)vectors on $T_x M$. We will often write $|T|$ to indicate this positive measure $\mu$ and refer to it as the *weight* of $T$.

**Example 2.6.** Note that for the special case $T = [\![\Sigma]\!]$ seen above we have $\mu = \mathcal{H}^k \llcorner \Sigma$ and $\tau(x)$ is obtained by wedging an oriented basis of $T_x\Sigma$, for all $x \in \Sigma$ (while we can define it in an arbitrary way on $M \setminus \Sigma$, which is $\mu$-negligible).

Assume now that $M, M'$ are two manifolds of possibly different dimension. Given a $C^\infty$-smooth proper map $\varphi : M \to M'$ (meaning that the preimage of a compact set is compact) and a $k$-dimensional current $T$ on $M$, we can push it to a $k$-dimensional current on $M'$, called the *pushforward* of $T$ via $\varphi$ and denoted $\varphi_* T$. It is given by

$$\langle \varphi_* T, \omega \rangle := \langle T, \varphi^* \omega \rangle$$

for any $\omega \in \Omega_c^k(M')$, where $\varphi^*\omega$ is the pullback of $\omega$ via $\varphi$.

*Exercise* 2.7. Check that if $T = [\![\Sigma]\!]$ then $\varphi_* T = [\![\varphi(\Sigma)]\!]$. Thus, this is a weak notion of image of a submanifold.

*Exercise* 2.8. Check that the boundary operator commutes with the pullback, i.e., $\varphi_*(\partial T) = \partial(\varphi_* T)$.

*Exercise* 2.9. Check that $\mathbb{M}(\varphi_* T) \le L^k \mathbb{M}(T)$ if $\varphi$ is $L$-Lipschitz.

For a normal current, we can merely assume that $\varphi$ is locally Lipschitz and proper. Indeed, we can let

$$\langle \varphi_* T, \omega \rangle := \lim_{\varepsilon \to 0} \langle (\varphi_\varepsilon)_* T, \omega \rangle,$$

where $(\varphi_\varepsilon)_{\varepsilon > 0}$ is a regularization of $\varphi$ (namely, a family of smooth maps converging pointwise to $\varphi$ such that $\varphi_\varepsilon$ is locally Lipschitz and proper, uniformly with respect to $\varepsilon$).

**Proposition 2.10.** *The previous limit exists and defines a current. In fact, it defines a normal current if $\varphi$ is Lipschitz and $M$ is complete (in which case we can also drop the properness assumption).*

While the proof is not important for the purposes of this course, we give a sketch of proof since a similar idea will appear later on.

*Proof.* We will just show the existence of the limit, assuming for simplicity that $M = M' = \mathbb{R}^n$ and $T$ has compact support $K$. Given $0 < \delta < \varepsilon$, we define

$$\psi(t,x) := (1-t)\varphi_\delta + t\varphi_\varepsilon,$$

a smooth homotopy between $\varphi_\delta$ and $\varphi_\varepsilon$. Letting $I := [0,1]$, we consider the current

$$\hat{T} := [\![I]\!] \times T$$

on $\mathbb{R}^{n+1}$ (the Cartesian product of two currents is defined in the natural way). Moreover, we have

$$\partial\hat{T} = [\![\{1\}]\!] \times T - [\![\{0\}]\!] \times T.$$

Hence,

$$\partial(\psi_*\hat{T}) = \psi_*(\partial\hat{T}) = (\varphi_\varepsilon)_*T - (\varphi_\delta)_*T.$$

Moreover, given a $(k+1)$-form $\omega$ we have

$$|\langle \psi_*\hat{T}, \omega\rangle| \le \mathbb{M}(\hat{T})\|\psi^*\omega\|_{C^0(K)} \le \mathbb{M}(\hat{T})\|\omega\|_{C^0} \cdot C(K)\sup_K |\varphi_\delta - \varphi_\varepsilon| \to 0$$

as $\delta, \varepsilon \to 0$ since, calling the coordinates $(t, x_1, \dots, x_n)$, in the pullback $\psi^*\omega$ only the terms of the form $f\,dt \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}$ matter (as $\hat{T}$ vanishes against the remaining ones) and for these $|f| \le C(K)\sup_K |\partial_t\psi| = C(K)\sup_K |\varphi_\delta - \varphi_\varepsilon|$ on $K$.

Thus, while $(\varphi_\varepsilon)_*T - (\varphi_\delta)_*T$ might have a large mass, it is the boundary of a current of vanishing mass (as we see from the previous bound by taking the supremum over the forms $\omega$ with $\|\omega\|_{C^0} \le 1$). This proves the claim, since

$$|\langle(\varphi_\varepsilon)_*T, \omega\rangle - \langle(\varphi_\delta)_*T, \omega\rangle| = |\langle\partial(\psi_*\hat{T}), \omega\rangle| = |\langle\psi_*\hat{T}, d\omega\rangle| \le \mathbb{M}(\psi_*\hat{T})\|d\omega\|_{C^0}$$

converges to 0 as $\delta, \varepsilon \to 0$. $\square$

*Exercise* 2.11. Give the definition of Cartesian product of two currents, assuming for simplicity that we are working on Euclidean spaces.

*Exercise* 2.12. Check that indeed $\partial\hat{T} = [\![\{1\}]\!] \times T - [\![\{0\}]\!] \times T$ and $\psi_*([\![\{0\}]\!] \times T) = (\varphi_\delta)_*T$ (and similarly for the other endpoint).

Another operation that can be done with normal $k$-currents with $k \ge 1$ is *slicing*. To simplify, assume that we are dealing with a cycle $T$, i.e., a $k$-current with zero boundary, and assume that $\phi : M \to \mathbb{R}$ is a Lipschitz map. We would like to "intersect" $T$ with a generic level set $\{\phi = t\}$ in order to obtain a $(k-1)$-current. Assuming that $T$ is normal with $\partial T = 0$, this can be defined for any $t \in \mathbb{R}$ by means of the formula

$$\langle T, \phi, t\rangle := \partial(\mathbf{1}_{\{\phi < t\}}T).$$

This makes sense since, thanks to the representation of normal currents as measures, it makes sense to multiply them by an indicator function, obtaining another $k$-current $\mathbf{1}_{\{\phi<t\}}T$ of which we can take the boundary. Actually, one can prove the following.

**Proposition 2.13.** *The $t$-slice $\langle T, \phi, t\rangle$ has zero boundary and finite mass for a.e. $t$ and we have*

$$\int_{\mathbb{R}} \mathbb{M}(\langle T, \phi, t\rangle)\, dt \le \mathrm{Lip}(\phi)\mathbb{M}(T).$$

We can even define slicing via $\mathbb{R}^p$-valued Lipschitz maps $\phi : M \to \mathbb{R}^p$. A possible definition is to iterate the previous slicing for each component, $p$ times. Otherwise, given a standard family of smooth approximations $\chi_\varepsilon$ of the Dirac mass $\delta_0$, we can consider the function $\chi_{t,\varepsilon} := \chi_\varepsilon(\cdot - t)$ and let

$$\langle\langle T, \phi, t\rangle, \omega\rangle := \lim_{\varepsilon\to 0}\langle T, (\chi_{t,\varepsilon} \circ \phi)\, d\phi_1 \wedge \cdots \wedge d\phi_p \wedge \omega\rangle.$$

It can be shown that the previous limit defines a normal $(k-p)$-current for a.e. $t$, with the same bound as above (note that, since $T$ has finite mass, it makes sense to pair it with any bounded measurable $k$-form).

2.2. **Integral currents.** Integral currents are normal currents $T$ such that both $T$ and $\partial T$ are given by rectifiable sets with integer multiplicity. More precisely, we assume that there exist a $k$-rectifiable Borel set $S$ and Borel (or just $\mathcal{H}^k$-measurable) maps

$$\nu : S \to \mathbb{N} \setminus \{0\}, \quad \tau : S \to \Lambda^k(TM)$$

such that $\tau(x)$ is of the form $\tau = v_1 \wedge \cdots \wedge v_k$ for an orthonormal basis $\{v_j\}_{j=1}^k$ of $T_xS$ (at a.e. $x \in S$) and $\int_S \nu\, d\mathcal{H}^k < \infty$, as well as

$$\langle T, \omega\rangle = \int_S \nu(x)\langle\omega(x), \tau(x)\rangle\, d\mathcal{H}^k(x),$$

and we assume that $\partial T$ has a similar structure. In other words, in terms of the objects $\mu$ and $\tau$ obtained above for any finite-mass current, we require that $\mu$ is rectifiable with integer density ($\mu$-a.e.) and that $\tau(x)$ is an orientation of $T_xS$ ($\mu$-a.e.), and similarly for $\partial T$.

*Remark* 2.14. When $k = 0$, an integral 0-current is just a measure of the form $\sum_j \alpha_j\delta_{p_j}$ supported on a discrete set of points, with $\alpha_j \in \mathbb{Z}\setminus\{0\}$ (encoding the multiplicity $\nu(p_j) = |\alpha_j|$ and the orientation $\tau(p_j) = \mathrm{sgn}(\alpha_j)$).

The following is a rather surprising fact, which we just state.

**Theorem 2.15 (boundary rectifiability).** *In fact, for a normal current, the corresponding requirements for $\partial T$, as well as the fact that $\tau(x) \in \Lambda^k(T_xM)$ is an orientation of $T_xS$, are automatically satisfied. Thus, if the weight $|T|$ of a normal current $T$ is rectifiable with integer multiplicity then $T$ is an integral current.*

*Exercise* 2.16. The $(k-1)$-rectifiability of $|\partial T|$ could fail if we replace positive integers with positive real numbers $(0, \infty)$: find a counterexample when $k = 1$, by building an infinite tree in which the multiplicity of each edge is divided equally among its children (so that intermediate nodes do not contribute to $\partial T$).

**Example 2.17.** If $E \subseteq M$ is a set of finite perimeter, then it is an important result of De Giorgi that its reduced boundary $\partial^* E$ is rectifiable and comes with a natural orientation. The induced $(n-1)$-current is precisely the boundary of $[\![E]\!]$. In fact, if $M$ is simply connected and $T$ is an integral $(n-1)$-current, then we can always find nested sets

$$(E_j)_{j\in\mathbb{Z}}, \quad E_{j+1} \subseteq E_j$$

of finite perimeter, with $\sum_j \mathcal{H}^{n-1}(\partial^* E_j) < \infty$, such that "$T = \sum_{j\in\mathbb{Z}} [\![E_j]\!]$," or more precisely

$$T = \sum_{j>0} [\![E_j]\!] - \sum_{j\leq 0} [\![M \setminus E_j]\!]$$

(with convergence of partial sums in the mass distance $\mathbb{M}(T' - T'')$).

*Exercise* 2.18. Find a counterexample when $M$ is not simply connected.

For integral currents, the pushforward and slicing have explicit expressions, which can be proved using the area and coarea formulas. For instance, assuming for simplicity that $\varphi : M \to M'$ is smooth, the pushforward is obtained with $S' := \varphi(S)$ and $\tau'(y)$ an arbitrary orientation of each tangent plane $T_y S'$, while

$$\nu'(y) := \sum_{x\in\varphi^{-1}(y)} \varepsilon(x)\nu(x),$$

where $\varepsilon(x)$ is 1 if $d\varphi(x)$ maps $\tau(x)$ to $\tau'(y)$ and $-1$ if it maps $\tau(x)$ to $-\tau'(y)$ (for a.e. $y$, the sum is finite and consists of points $x$ where $d\varphi(x)$ is injective, and actually $d\varphi[\tau(x)] = \pm\tau'(y)$ for all of them). If we insist on having multiplicity in $\mathbb{N} \setminus \{0\}$, we should remove from $S'$ the points $y$ with $\nu'(y) = 0$ and we should replace $\tau'(y)$ with $-\tau'(y)$ at the points $y$ with $\nu'(y) < 0$ (so that the multiplicity becomes positive).

**Example 2.19.** If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\varphi(x_1, x_2) := (|x_1|, x_2)$ and $T := [\![[-1, 1] \times \{0\}]\!]$ (oriented from left to right), then $\varphi_* T = 0$. Intuitively, $\varphi$ folds the segment into two segments with opposite orientation, which cancel each other.

Similarly, assuming that $\phi : M \to \mathbb{R}$ is smooth and real-valued, for a.e. $t \in \mathbb{R}$ the $t$-slice is obtained with $S' := S \cap \phi^{-1}(t)$ and $\nu' := \nu|_{S'}$, while $\tau'$ is such that $\frac{\nabla\varphi}{|\nabla\varphi|}(x) \wedge \tau'(x) = \tau(x)$.

2.3. **Polyhedral deformation, closure, and compactness.** The proof that pushforward via a Lipschitz map can be defined for normal currents essentially showed convergence of $(\varphi_\varepsilon)_* T$ in the *flat topology*. This has a different definition depending on the context.

On the class of integral $k$-currents, it is the topology induced by the following *flat metric*:

$$d(T, T') = \mathcal{F}(T - T') := \inf\{\mathbb{M}(R) + \mathbb{M}(S) \mid T - T' = R + \partial S\},$$

where $R$ ranges among integral $k$-currents and $S$ among integral $(k+1)$-currents such that $T - T' = R + \partial S$ (for normal currents, one just takes $R$ and $S$ normal, and so on). In ambients where $H_k(M, \mathbb{Z}) = 0$, on the class of integral $k$-cycles one can also use the similar (not equivalent) definition $\inf\{\mathbb{M}(S) \mid T - T' = \partial S\}$.

Clearly, if a sequence of currents converges in the flat metric then it converges in the natural weak topology (of the dual of $k$-forms).

*Remark* 2.20. The name "flat metric" is an unfortunate one: it does not have anything to do with the curvature of the ambient manifold $M$.

**Example 2.21.** As a simple example, let $T_k$ be the segment $[0,1] \times \{2^{-k}\}$ with multiplicity 1 and the orientation from left to right. Then $T_k$ converges to the 1-current $T$ given by the segment $[0,1] \times \{0\}$, with respect to the flat metric. Indeed, we can take $S_k$ to be the rectangle $[0,1] \times [0, 2^{-k}]$ with the standard orientation and $R_k$ the sum of the vertical segments $\{0\} \times [0, 2^{-k}]$ and $\{1\} \times [0, 2^{-k}]$ (with the orientation from bottom to top for the first one and the opposite for the second one). Then we have

$$T - T_k = R_k + \partial S_k$$

and $\mathbb{M}(R_k) = 2 \cdot 2^{-k} \to 0$, as well as $\mathbb{M}(S_k) = 2^{-k} \to 0$. On the other hand, with respect to the much stronger mass distance $(T, T') \mapsto \mathbb{M}(T - T')$ we do not have convergence, since $\mathbb{M}(T - T_k) = 2$ for all $k$.

It turns out that, with respect to this metric, one can approximate any integral current with particularly nice ones, the *polyhedral currents*, which are finite sums of $k$-faces of a smooth simplicial complex (so that these faces are smoothly embedded and any two of them meet in a transversal way along a lower dimensional face).

**Theorem 2.22 (polyhedral deformation theorem).** *Given $k \geq 1$ and an integral $k$-current $T$, in $\mathbb{R}^n$ or in a compact manifold $(M^n, g)$, and given $\rho > 0$ (arbitrary in $\mathbb{R}^n$*

*and small enough in a manifold $(M^n, g)$, depending only on the latter), we can write*

$$T = P + R + \partial S$$

*for integral currents $P, R, S$ (of dimension $k$, $k$, and $k+1$, respectively) supported in a $C(n)\rho$-neighborhood of $\operatorname{spt}(T)$, with $P$ a polyhedral current whose faces have diameter comparable with $\rho$ and*

$$\mathbb{M}(P) \leq C(n)\mathbb{M}(T), \quad \mathbb{M}(\partial P) \leq C(n)\mathbb{M}(\partial T),$$

*as well as*

$$\mathbb{M}(R) \leq C(n)\rho\mathbb{M}(\partial T), \quad \mathbb{M}(S) \leq C(n)\rho\mathbb{M}(T)$$

*(so that clearly $\mathbb{M}(\partial R) \leq C(n)\mathbb{M}(\partial T)$ and $\mathbb{M}(\partial S) \leq C(n)\mathbb{M}(T) + C(n)\rho\mathbb{M}(\partial T)$).*

*Proof.* We work in $\mathbb{R}^n$ for simplicity but we give a proof that can be easily adapted to manifolds. By scaling, we can assume that $\sigma = 1$. We decompose $\mathbb{R}^n$ into a simplicial complex in which each simplex has diameter comparable with 1.

For simplicity, we also assume that $\partial T = 0$. Assume that $T$ is supported on the $\ell$-skeleton (the union of $\ell$-dimensional faces of the complex) for some $\ell > k$. We want to construct a new $T'$ supported on the $(\ell-1)$-skeleton by suitably projecting $T$ onto it.

The proof uses a similar idea to the one used in the proof that pushforward via Lipschitz maps makes sense for normal currents. Given an $\ell$-simplex $\Delta$, we consider a smaller simplex $\Delta'$ included in its interior. For any fixed $p \in \Delta'$, we consider the (locally Lipschitz) radial projection map

$$\pi_p : \Delta \setminus \{p\} \to \partial\Delta.$$

If we take a point $p = p_\Delta$ in each $\ell$-simplex, these maps glue to a map $\pi$ from the $\ell$-skeleton to the $(\ell-1)$-skeleton. However, $\pi$ is not defined on the finite set of such center points $p_\Delta$. The idea is that, taking these points $p_\Delta$ randomly, we can still perform the pushforward through $\pi$, with the desired estimates.

Indeed, note that $|d\pi_p(x)| \leq C(n)|x-p|^{-1}$. Thus, considering the current

$$\tilde{T} := \mathbf{1}_{\bigcup_\Delta (\Delta \setminus B_r(p_\Delta))} T$$

(this is a well-defined normal current for a.e. $r$), since it is supported far from each $p_\Delta$ we can define $\pi_* \tilde{T}$ and, using the area formula, we see that

$$\mathbb{M}(\pi_* \tilde{T}) \leq \sum_{\Delta\ \ell\text{-simplex}} \int_\Delta |x - p_\Delta|^{-k}\, d|\tilde{T}|(x) \leq \sum_{\Delta\ \ell\text{-simplex}} \int_\Delta |x - p_\Delta|^{-k}\, d|T|(x),$$

where $|\tilde{T}|$ is the weight of $\tilde{T}$. The same computation shows that these currents form a Cauchy sequence (in the mass distance) as $r \to 0$ if the last sum of integrals is finite, so we can define the pushforward $T' := \pi_* T$ as their limit. It is easy to check that this limit is rectifiable, with integer multiplicity.

Now we have

$$\int_{\Delta'} \int_{\Delta} |x-p|^{-k}\, d|T|(x)\, d\mathcal{H}^{\ell}(p) = \int_{\Delta} \int_{\Delta'} |x-p|^{-k}\, d\mathcal{H}^{\ell}(p)\, d|T|(x) \le C(n)|T|(\Delta),$$

where we crucially used the fact that $\ell > k$ in the last inequality. Since $\Delta'$ has measure comparable with 1, in average (as $p$ ranges) the previous integral over $\Delta$ is bounded by $C(n)|T|(\Delta)$. We can then find $p = p_\Delta$ such that

$$\int_{\Delta} |x-p_\Delta|^{-k}\, d|T|(x) \le C(n)|T|(\Delta) < \infty.$$

Moreover, by constructing a homotopy on each $\Delta \setminus \{p_\Delta\}$ between the identity and $\pi_{p_\Delta}$ (and arguing with the truncated currents for each $r > 0$), we can construct an integral $(k+1)$-current $S$ such that

$$T - T' = \partial S,$$

with the desired bound on $S$. The theorem follows by repeating this projection $n-k$ times, obtaining finally a current on the $k$-skeleton. The latter is a sum of $k$-dimensional faces by the constancy theorem (see the exercise below). □

*Exercise* 2.23. Prove the general case where $\partial T \neq 0$, by first applying the special case to $\partial T$ in place of $T$ and reducing to a $k$-current whose boundary is already a sum of $(k-1)$-simplices.

*Exercise* 2.24. Prove the *constancy theorem*: given a connected, embedded $C^1$-smooth $k$-dimensional submanifold $\Sigma^k \subset U$ and an integral $k$-current with zero boundary supported on $\Sigma$ in an open set $U$, prove that this current is a constant multiple of $[\![\Sigma]\!]$ in $U$ (reduce first to the case where $\Sigma$ is a subset of a $k$-plane in $\mathbb{R}^n$).

*Exercise* 2.25. If $T$ is an integral $k$-cycle (i.e., $\partial T = 0$) in $\mathbb{R}^n$, show that there exists an integral $(k+1)$-current $S$ such that

$$T = \partial S, \quad \mathbb{M}(S) \le C(n)\mathbb{M}(T)^{(k+1)/k}$$

(assuming $k \geq 1$). Show that the same holds in a compact manifold, by embedding it in a Euclidean space and using the nearest point projection, provided that $\mathbb{M}(T)$ is small enough.

A consequence of the previous exercise is that, in a compact manifold, an integral cycle with small enough mass is a boundary.

We also state the following fundamental result without proof.

**Theorem 2.26 (closure of integral currents).** *Assume that $T_j \rightharpoonup T$, for a sequence of integral currents $T_j$ such that*

$$\sup_j \mathbb{M}(T_j) < \infty, \quad \sup_j \mathbb{M}(\partial T_j) < \infty.$$

*Then $T$ is also an integral current.*

Note that it is not enough to assume $T_j$ rectifiable (take $T_j$ to be $j$ horizontal segments of the form $[0,1] \times \{\frac{\ell}{j}\}$, for $\ell = 1, \ldots, j$, with multiplicity $\frac{1}{j}$; the limit is a diffuse current).

This theorem was initially proved by relying on another difficult result, asserting that a purely unrectifiable set (a Borel set with finite $\mathcal{H}^k$ measure such that no Borel subset of positive measure is rectifiable) has negligible projection on a.e. $k$-plane. Later, a much simpler proof was found (partly independently) by Ambrosio–Kirchheim, Jerrard, and White, using a characterization of integral $k$-currents saying essentially that, given a projection $\pi$ on a $k$-plane $P$, the slices (obtained via $\pi$) give a sort of $BV$ function from $P$ to the set of integral 0-currents in $\mathbb{R}^n$.

Let us rather see how to deduce the fundamental compactness property.

**Theorem 2.27 (compactness of integral currents).** *Given a sequence $(T_j)$ of integral $k$-currents with $\sup_j \mathbb{M}(T_j) < \infty$ and $\sup_j \mathbb{M}(\partial T_j)) < \infty$, we can extract a subsequence converging to a limit integral $k$-current.*

*Proof.* We assume for simplicity that the supports are all contained in a fixed compact set $K$ (otherwise the following argument applies locally). Given $\rho > 0$, we apply the polyhedral deformation theorem to write

$$T_j = P_j + R_j + \partial S_j.$$

As we saw in the proof, $P_j$ is a linear combination with integer coefficients of the faces of a fixed simplicial complex covering $K$. These coefficients are bounded by the masses of $T_j$ and $\partial T_j$ (for fixed $\rho$), and hence are bounded sequences of integers. Thus, a subsequence converges:

$$P_j \rightharpoonup P_\infty.$$

Now we are done by letting $\rho \to 0$ and using a diagonal argument, since given a $k$-form $\omega$ we have

$$|\langle T_j - P_j, \omega\rangle| \le |\langle R_j, \omega\rangle| + |\langle \partial S_j, \omega\rangle| \le C(n)\rho[\mathbb{M}(T_j) + \mathbb{M}(\partial T_j)][\|\omega\|_{C^0} + \|d\omega\|_{C^0}]. \quad \square$$

The previous proof shows the following: given a compact set $K \subseteq M$ and $C \ge 0$, the set of integral $k$-currents

$$\{T : \mathrm{spt}(T) \subseteq K,\ \mathbb{M}(T) + \mathbb{M}(\partial T) \le C\}$$

is compact with respect to the flat distance: indeed, it is complete by the closure theorem and it is totally bounded since, for any $\varepsilon > 0$, it is covered by finitely many balls $B_\varepsilon^{\mathcal{F}}(T_j)$ in the flat metric (the polyhedral ones associated to a fixed, sufficiently fine simplicial complex; these $T_j$ do not necessarily belong to the previous set, but of course any such ball intersecting the set is included in a ball $B_{2\varepsilon}^{\mathcal{F}}(T_j')$ with another center $T_j'$ in the set).

In particular, since the flat and the weak topologies are Hausdorff (and clearly the flat one is finer), this compactness shows that they coincide on any set as above.

*Remark* 2.28. Using a blow-up argument and similar ideas as in the proof of the constancy theorem, one can show the following rectifiability criterion: if the weight $|T|$ of a normal $k$-current $T$ has

$$\Theta_k^*(|T|, x) \in (0, \infty) \quad \text{for } |T|\text{-a.e. } x$$

then $T$ is a rectifiable $k$-current, and actually an integral one if $\Theta_k^*(|T|, x) \in \mathbb{N}$ a.e. However, this is not so useful in practice: since the mass of currents is only lower semicontinuous, if $T$ arises as a limit of currents it is not easy to control its density. On the contrary, for *varifolds* $V$, mass is continuous and thus rectifiability criteria of this kind, based on density, are useful since they can often be checked in practice (and in fact the density automatically exists and is finite if the varifold is *stationary*).

2.4. **A partial solution to the Plateau problem.** We can apply the previous compactness result to minimize the area in various settings. For instance, we might want to minimize area among submanifolds with an assigned boundary, or among those without boundary belonging to a given homology class in $H_k(M, \mathbb{Z})$.

Let us focus on the first setting, when $M = \mathbb{R}^n$ with the Euclidean metric. Given a smooth embedded $(k-1)$-submanifold $\Gamma^{k-1}$ without boundary (with $k \geq 2$), we want to find a $k$-dimensional submanifold $\Sigma^k$ of least area with $\partial\Sigma = \Gamma$.

To do this, we replace the set of smooth submanifolds with the wider set of integral currents. We let

$$\Lambda := \inf\{\mathbb{M}(T) \mid T \text{ integral } k\text{-current with } \partial T = [\![\Gamma]\!]\}.$$

We claim that this infimum is achieved, or in other words it is a minimum. First of all, note that this set is not empty. The reason is that one possible competitor $T$ is the so-called *cone* over $\Gamma$: fixing any $x_0 \in \mathbb{R}^n$, we let

$$\psi(t, x) := (1-t)x_0 + tx, \quad T := \psi_*([\![I]\!] \times [\![\Gamma]\!]).$$

This pushforward is defined even if $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is not proper nor Lipschitz, since $[\![I]\!] \times [\![\Gamma]\!]$ has compact support.

*Exercise* 2.29. Check that, for this cone $T$, we indeed have $\partial T = [\![\Gamma]\!]$. What goes wrong for $k = 1$?

We then take a minimizing sequence, i.e., a sequence $T_j$ of currents in the previous set such that $\mathbb{M}(T_j) \to \Lambda$. In fact, we can take them supported in a common compact set: calling $B$ a large closed ball including $\Gamma$, we can consider the nearest point projection $\pi_B : \mathbb{R}^n \to B$. Viewing it as a 1-Lipschitz map $\pi_B : \mathbb{R}^n \to \mathbb{R}^n$, we can replace $T_j$ with $(\pi_B)_* T_j$, which has mass $\leq \mathbb{M}(T_j)$ and boundary $(\pi_B)_*(\partial T_j) = (\pi_B)_*[\![\Gamma]\!] = [\![\Gamma]\!]$.

By the previous compactness theorem, up to extracting a subsequence we have $T_j \rightharpoonup T$ for an integral current $T$. Because of the weak definition of boundary, we have

$$\partial T = \lim_{j\to\infty} \partial T_j = [\![\Gamma]\!].$$

Finally, we also have

$$\langle T, \omega \rangle = \lim_{j\to\infty} \langle T_j, \omega \rangle \leq \lim_{j\to\infty} \mathbb{M}(T_j) = \Lambda$$

for any $k$-form $\omega$ with $|\omega| \leq 1$ pointwise. Taking the supremum over $\omega$ gives $\mathbb{M}(T) \leq \Lambda$, and hence (since $\Lambda$ was the infimum)

$$\mathbb{M}(T) = \Lambda.$$

This shows that $T$ minimizes the mass in the previous set. Out of this property, we would like to obtain some regularity of $T$. In the best case scenario, we would like to say that $T$ is actually a smooth submanifold with multiplicity 1, thus belonging to the initial class where we considered our minimization problem.

Even if regularity of $T$ looks quite intuitive (too much wiggling would cost too much area), this is actually the hardest part of the story. We will hint at some ideas in the regularity theory in codimension one (when $k = n - 1$), which is the most successful situation.

*Remark* 2.30. The previous computation shows that, in general, the mass of currents is lower semicontinuous with respect to weak convergence. In fact, it can really drop in the limit: for instance this happens when we consider two parallel segments with opposite orientation coming closer and closer to each other, giving the trivial current in the limit.

**Useful references:**

- Chapter 6 of Simon, *Geometric measure theory*;
- Chapter 7 of Krantz–Parks, *Geometric integration theory*;
- for the modern proof of the closure theorem of integral currents: Chapter 7 of Krantz–Parks, *Geometric integration theory*;

- for sets of finite perimeter: Chapter 12 of Maggi, *Sets of finite perimeter and geometric variational problems*;
- again for sets of finite perimeter: Sections 3.3 and 3.5 of Ambrosio–Fusco–Pallara, *Functions of bounded variation and free discontinuity problems.*