Lecture 3: varifolds. June 27, 2024

(Geometric Measure Theory)

When we want to construct general critical points of the area rather than minimizing it, we are often in the following situation: we have a sequence of "almost critical" k-dimensional submanifolds $\Sigma_1, \Sigma_2, \dots$ (or diffuse versions thereof) with area bounds

$$0 < \inf_j \mathcal{H}^k(\Sigma_j) \le \sup_j \mathcal{H}^k(\Sigma_j) < \infty$$

and we want to find a limit Σ critical for the area. Working with currents is not appropriate since the mass (i.e., "area") is only lower semicontinuous and we could end up with the trivial limit $\Sigma = 0$ (in the minimization setting, this is not possible since Σ has an assigned nontrivial boundary or homology class).

To circumvent this issue, we use another weak definition of submanifold. Naively, one could just take a limit of $\mathcal{H}^k \llcorner \Sigma_j$ in the weak-* topology (or a limit of the sets Σ_j in the Hausdorff topology). However, we would like to study the regularity of the limit and exploit the fact that it is critical for the k-dimensional area, and the problem arises that we cannot easily define what criticality means for the limit measure (or set).

We could say that this limit is "critical" if its area is preserved at first order by ambient diffeomorphisms. More precisely, taking any vector field $X \in C^1_c(M, TM)$ and considering its flow $(\Phi^X_t)_{t\in\mathbb{R}}$, if we differentiate the area of the "image" $\Phi^X_t(\Sigma)$ in t (at time $t = 0$) we should get zero. But for a measure the obvious definition of "image" does not work, since the pushforward of a measure has the same mass, and hence this definition of criticality is not meaningful. A limit in the sense of sets does not work either, since in general one must allow for multiplicity and, even once one devises a way to do this, checking criticality of the limit becomes difficult.

Rather, we consider a sort of limit in the sense of *Young measures*, namely one which remembers the direction of the tangent planes of Σ_j as well. This will allow us to define a meaningful notion of "image" and "criticality."

3.1. **General varifolds and first variation.** A *varifold* V on (M^n, g) is a (positive) Radon measure on the Grassmannian bundle $\mathrm{Gr}_k(M)$ of unoriented k-planes on M. This is the fiber bundle $\mathrm{Gr}_k(M) \to M$ whose fiber above each $x \in M$ is the set of all k-planes $P \subseteq T_xM$. If $M = \mathbb{R}^n$, this bundle is trivial: we can simply think it as $\mathbb{R}^n \times \mathrm{Gr}^n_k$, where the second factor is the set of all k-planes in $\mathbb{R}^n$ (passing through the origin). We can make Gr^n_k into a compact metric space with the metric

$$d(P, P') := \|\pi_P - \pi_{P'}\|,$$

where $\pi_P : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection onto P and we use the Hilbert–Schmidt norm for $n \times n$ matrices.

The *weight* of V is $|V| := (\pi_M)_* V$, the pushforward of V via the projection $\pi_M : \mathrm{Gr}_k(M) \to M$ to the base manifold M, and is thus a measure on M. Hence, $|V|(M)$ is the "mass" or "area" of V.

Recall that, given a fiber bundle $\pi : X \to B$ with typical fiber F and a measure μ on X, in a local trivialization $X \supseteq \pi^{-1}(U) \cong U \times F$ (with coordinates $(b, f) \in U \times F$) we can always disintegrate μ as $d\mu(b, f) = d\pi_*\mu(b) \otimes \alpha_b(f)$ for a suitable probability measure α_b on F, depending on $b \in U$.

Example 3.1. Given a k-rectifiable set S and a multiplicity $f : S \to (0, \infty)$ (such that $f\, d(\mathcal{H}^k \llcorner S)$ is locally finite), these give rise to the varifold V whose disintegration with respect to π_M is

$$dV(x, P) := d(\mathcal{H}^k \llcorner S)(x) \otimes \delta_{T_x S}(P).$$

This means that at a.e. $x \in S$ only the tangent plane $T_x S$ is "important." Hence, given a sequence of k-dimensional manifolds Σ_j of bounded area, we can always take the limit of the associated varifolds up to a subsequences, which is just a limit of measures on $\mathrm{Gr}_k(M)$ in the weak-* topology.

We can now give a meaningful definition of "image," in the following way: given a diffeomorphism $\Phi : M \to M$, it induces a diffeomorphism

$$\hat{\Phi} : \mathrm{Gr}_k(M) \to \mathrm{Gr}_k(M)$$

which maps a k-plane $P \subseteq T_x M$ to the plane $d\Phi(x)[P] \subseteq T_{\Phi(x)} M$. We also have the *Jacobian* $J_\Phi(P)$ (given by $\sqrt{\det(A^T A)}$, where A represents the linear map $d\Phi(x)|_P$ with respect to orthonormal bases of P and $T_{\Phi(x)} M$, or equivalently $|\det(B)|$, where the square matrix B represents $d\Phi(x)|_P$ with respect to orthonormal bases of P and $\hat{\Phi}(P)$). We define the *varifold pushforward*

$$\Phi_* V := \hat{\Phi}_* \tilde{V}, \quad \text{where } d\tilde{V}(P) := J_\Phi(P)\, dV(P).$$

Here we omit the point $x = \pi_M(P)$. In other words, we perform the usual measure pushforward but only after multiplying by the correction factor J_Φ. In particular, the mass of $\Phi_* V$ can be different from the one of V.

Exercise 3.2. This is consistent with the usual image of submanifolds: if V is the varifold associated to a rectifiable set S, then $\Phi_* V$ is the one associated to $\Phi(S)$.

We say that V is *stationary* if it is a "critical point for the area" in the following sense: for any vector field $X \in C^1_c(M,TM)$ we have

$$\langle \delta V, X\rangle := \frac{d}{dt}|(\Phi^X_t)_* V|(K)\Big|_{t=0} = 0,$$

where K is an arbitrary compact set including the support of X (outside of $\mathrm{spt}(X)$, the flow Φ^X_t is just the identity; hence, two different K's will give the same result). If $|V|(M) < \infty$, we can simply take $K := M$.

Exercise 3.3. Show that the previous derivative always exists and is actually given by

$$\int_{\mathrm{Gr}_k(M)} \mathrm{div}^P X(x)\, dV(x,P),$$

where $\mathrm{div}^P X(x)$ is the divergence of X along P, given by $\sum_\ell \langle e_\ell, \nabla_{e_\ell} X\rangle$, where $\{e_\ell\}_{\ell=1}^k$ is an arbitrary orthonormal basis of $P \subseteq T_x M$.

Exercise 3.4. Show that if $V_j \rightharpoonup V$ and each V_j is stationary, then also the limit V is stationary.

More generally, if there exists $C(K) \geq 0$ such that $|\langle \delta V, X\rangle| \leq C(K)\|X\|_{C^0}$ for any vector field $X \in C^1_c$ supported in K (for all compact K), we say that V has *locally bounded first variation* and we can find a vector-valued measure H on M (with values in TM) such that

$$\langle \delta V, X\rangle = -\int_M \langle X, \nu\rangle\, d|H|, \quad \text{where } \nu := \frac{dH}{d|H|}.$$

Exercise 3.5. If Σ is a smooth, compact k-dimensional submanifold without boundary, then the associated varifold has $dH = H^\Sigma\, d(\mathcal{H}^k \llcorner \Sigma)$, where H^Σ is the mean curvature of Σ.

Exercise 3.6. This example shows how the notion of convergence for currents and varifolds is fundamentally different. On the plane $\mathbb{R}^2$, for each $j \geq 1$ consider the 1-dimensional manifold

$$\Sigma_j := \bigcup_{\ell=1}^j \partial B_{1/(2\pi j)}(p_{j,\ell}), \quad p_{j,\ell} := \Big(\frac{2\ell-1}{2j}, 0\Big)$$

consisting of j disjoint circles of radius $\frac{1}{2\pi j}$, whose centers are equally spaced along the segment $L := [0,1]\times\{0\}$. Consider the associated current T_j obtained by orienting each circle counterclockwise (and assigning multiplicity 1) and the associated varifold V_j (again with multiplicity 1). Show that

$$\lim_{j\to\infty} T_j = 0, \quad \lim_{j\to\infty} V_j = (\mathcal{H}^1 \llcorner L) \otimes \alpha,$$

where α is the uniform measure on $\mathrm{Gr}_1^2 \cong \mathbb{RP}^1$. Thus, even if $\mathcal{H}^1(\Sigma_j) = 1$ is constant, the limit as currents vanishes, while as expected the limit varifold is nontrivial (but is not rectifiable: it assigns "equal importance" to all lines). [Hint: to show that T_j converges to zero, note that it bounds a current of infinitesimal mass, given by j disks.]

3.2. **Monotonicity formula for stationary varifolds.** A fundamental tool in the theory of stationary varifolds (and minimal submanifolds) is the existence of a monotone quantity. To illustrate the heuristic reason behind it, we begin with a sloppy computation in the smooth setting. Assume that $\Sigma^k \subset \mathbb{R}^n$ is a (properly) embedded submanifold minimizing the area locally, meaning that for any $\tilde{\Sigma}^k$ such that $K := \Sigma \Delta \tilde{\Sigma}$ is compact we have

$$\mathcal{H}^k(\Sigma \cap K) \leq \mathcal{H}^k(\Sigma' \cap K).$$

This must be assumed to hold locally, since one can show that the total area of Σ must be infinite (precisely as a consequence of the monotonicity that we are going to show).

Given a point $p \in \mathbb{R}^n$, not necessarily belonging to Σ, and given $r > 0$, we can consider the intersection $\Gamma_{p,r} := \Sigma \cap \partial B_r(p)$ of Σ with the sphere of center p and radius r. This is an embedded $(k-1)$-dimensional submanifold by Sard, for a.e. $r > 0$. Moreover, using the coarea formula, we can check that

$$\mathcal{H}^{k-1}(\Gamma_{p,r}) \leq f'(r), \quad \text{where } f(r) := \mathcal{H}^k(\Sigma \cap B_r(p)),$$

and equality holds precisely when Σ meets the sphere orthogonally.

Now, inside the ball, we can replace Σ with the cone over $\Gamma_{p,r}$ (with tip p). This is the place where the argument is sloppy, since the new object is no longer a smooth submanifold, although it becomes rigorous when Σ is an area-minimizing integral current.

The cone has area $\frac{r}{k}\mathcal{H}^{k-1}(\Gamma_{p,r})$. By minimality, we get

$$f(r) \leq \frac{r}{k}\mathcal{H}^{k-1}(\Gamma_{p,r}) \leq \frac{r}{k} f'(r),$$

which means that

$$\frac{d}{dr}\frac{f(r)}{r^k} \geq 0 \quad \text{for a.e. } r,$$

and hence $r \mapsto \frac{f(r)}{r^k}$ is increasing.

In general, at least for small $r > 0$, we can approximate the characteristic function of the ball $B_r(p)$ by taking a smooth decreasing function $\phi : [0,\infty) \to \mathbb{R}$ with $0 \leq \phi \leq \mathbf{1}_{[0,1]}$, equal to 1 near the origin, and by looking at $\phi(\frac{d_p(x)}{r})$, where d_p is the distance function from p.

The previous comparison argument suggests testing the stationarity $\delta V = 0$ by using radial vector fields, which would leave V intact precisely when it is already conical. Namely, we take

$$X := (\phi \circ d_p)\nabla\frac{d_p^2}{2}.$$

In $\mathbb{R}^n$ we simply have $\nabla \frac{d_p^2}{2}(x) = x - p$; integrating the resulting equation between $r = a$ and $r = b$, and letting $\phi \to \mathbf{1}_{[0,1]}$, we arrive at the following identity, called *monotonicity formula*:

$$\frac{|V|(B_b(p))}{b^k} = \frac{|V|(B_a(p))}{a^k} + \int_{\mathrm{Gr}_k(B_b(p)\setminus B_a(p))} \frac{|\nabla^{P^\perp} d_p|^2}{d_p^k}\, dr,$$

where $\nabla^{P^\perp} d_p$ is the projection of ∇d_p onto $P^\perp$. As the name suggests, an immediate corollary is the fact that

$$r \mapsto \frac{|V|(B_r(p))}{r^k}$$

is an increasing function of $r > 0$.

Exercise 3.7. Show the previous identity in $\mathbb{R}^n$.

On a Riemannian manifold, error terms appear because of the ambient curvature. However, we can reach a similar monotonicity formula of the form

$$e^{Cb}\frac{|V|(B_b(p))}{b^k} \geq e^{Ca}\frac{|V|(B_a(p))}{a^k} + \int_{\mathrm{Gr}_k(B_b(p)\setminus B_a(p))} \frac{|\nabla^{P^\perp} d_p|^2}{d_p^k}\, dr,$$

and thus the quantity $e^{Cr}\frac{|V|(B_r(p))}{r^k}$ is increasing.

In particular, the density

$$\Theta_k(|V|, p) = \lim_{r\to 0} \frac{|V|(B_r(p))}{\omega_k r^k}$$

exists at all points $p \in M$. In $\mathbb{R}^n$, for the rescaled varifold $(\delta_{p,r})_* V$, we then have

$$\lim_{r\to 0} |(\delta_{p,r})_* V|(B_R(0)) = \lim_{r\to 0} \frac{|V|(B_{Rr}(p))}{r^k} = \Theta_k(|V|, p)\cdot \omega_k R^k < \infty$$

for any given $R > 0$. Note that we do not need to divide by r^k in the left-hand side, since this normalization factor is already given by the Jacobian in the definition of varifold pushforward. Thus, for any sequence $r_j \to 0$, a limit varifold $\lim_{j\to\infty}(\delta_{p,r_j})_* V$ always exists along a subsequence, by the usual compactness of Radon measures. Such a limit is called a *blow-up* at p; it might depend on the sequence $r_j \to 0$ in general.

We can do the same in a Riemannian manifold (M^n, g), using normal coordinates at p. Such a blow-up W is now a stationary varifold in $T_x M \cong \mathbb{R}^n$, even if V is initially defined on M.

The following exercises contain observations which will be crucially important in the sequel.

Exercise 3.8. For such a blow-up W, show that $\frac{|W|(B_R(0))}{\omega_k R^k} = \Theta_k(|V|, p)$ for all $R > 0$, so that in particular $\Theta_k(|W|, 0) = \Theta_k(|V|, p)$. Show also that

$$\Theta_k(|W|, x) \leq \frac{|W|(B_R(x))}{\omega_k R^k} \leq \Theta_k(|W|, 0)$$

for all $x \in \mathbb{R}^n$ and $R > 0$. [Hint: consider a very large ball $B_{R'}(x)$ and the ball $B_{R'+|x|}(0)$ including it, and apply monotonicity with center x.]

In particular, for a blow-up W, the monotonicity formula with center 0 is saturated, in the sense that the inequality $\frac{|W|(B_b(0))}{b^k} \geq \frac{|W|(B_a(0))}{a^k}$ becomes an equality, for all radii $0 < a < b$. The monotonicity formula then tells us that $\nabla^{P^\perp} d_0(x) = 0$ for W-a.e. (x, P) with $x \neq 0$. Since $d_0(x) = |x|$, we obtain that $\nabla d_0(x) = \frac{x}{|x|} \perp P^\perp$, or in other words $x \in P$, for W-a.e. (x, P). Since this is a closed condition, we deduce the following.

Corollary 3.9. *For a blow-up W we have $x \in P$ for all $(x, P) \in \operatorname{spt}(W)$.*

This suggests that W is a cone, matching the equality case in the heuristic proof of monotonicity seen before. We will confirm this later on, for stationary varifolds with density ≥ 1 on the support.

Exercise 3.10. Using the monotonicity formula, show that for a stationary varifold V the density is *upper semicontinuous*, namely

$$\Theta_k(|V|, p) \geq \limsup_{j\to\infty} \Theta_k(|V|, p_j)$$

whenever $p_j \to p$ (in other words, sublevel sets of the form $\{p \in M : \Theta_k(|V|, p) < \lambda\}$ are open).

Example 3.11. A simple example where continuity of the density fails is given by two crossing lines in $\mathbb{R}^2$, with multiplicity one. This is a stationary varifold whose density is 2 at the intersection of the two lines and 1 at all other points (in the union of the two lines).

Exercise 3.12. Assume that $V_j \rightharpoonup V$ for a sequence of stationary varifolds with density ≥ 1 on their supports. Show that we have a sort of "upper semicontinuity of the support": if $p_j \to p$ and $p_j \in \operatorname{spt}|V_j|$, then $p \in \operatorname{spt}|V|$ and in fact $\Theta_k(|V|, p) \geq 1$. As a consequence, $\operatorname{spt}|V_j| \to \operatorname{spt}|V|$ locally in the sense of Hausdorff convergence (i.e., given any compact set K and $\varepsilon > 0$, eventually $\operatorname{spt}|V_j| \cap K$ is included in the ε-neighborhood of $\operatorname{spt}|V|$ and $\operatorname{spt}|V| \cap K$ is included in the ε-neighborhood of $\operatorname{spt}|V_j|$).

3.3. **Rectifiable varifolds and rectifiability criterion for stationary varifolds.** As we did for currents, we can single out two important classes of varifolds: rectifiable and integral ones. We say that a k-dimensional varifold V is *rectifiable* if its weight $|V|$ is a k-rectifiable measure and moreover we have the disintegration

$$dV(x,P) = d|V|(x) \otimes \delta_{T_x|V|}(P),$$

where $T_x|V|$ is the tangent space of the rectifiable measure, as discussed in the first lecture (it is defined for $|V|$-a.e. x). Thus, besides rectifiability of the projected measure $\pi_* V = |V|$ on the base M, the definition of rectifiable varifold also requires that at any point $x \in M$ "only the k-plane $T_x|V|$ is important." Note that there is a one-to-one correspondence between k-rectifiable measures and k-dimensional rectifiable varifolds.

Exercise 3.13. Working on $\mathbb{R}^n$ for simplicity and letting $\delta_{x,r}(y) := \frac{y-x}{r}$, as usual, the second condition above is equivalent to the fact that, for $|V|$-a.e. x, we have

$$\lim_{r\to 0} (\delta_{x,r})_* V = \Theta_k(|V|, x) \cdot T_x|V|$$

as Radon measures, where in the right-hand side we identify the plane $T_x|V|$ with the associated k-dimensional varifold of multiplicity 1 (this can be seen using the fact that, in a general disintegration as above, $b \mapsto \alpha_b$ is approximately weak-$*$ continuous at a.e. b).

Example 3.14. Given a k-dimensional rectifiable current T, associated to a k-rectifiable (Borel) set S, a multiplicity $\nu : S \to (0,\infty)$, and an orientation τ, we can always form a k-dimensional rectifiable varifold V simply by forgetting the orientation: we let

$$dV(x,P) = d(\mathcal{H}^k \llcorner S)(x) \otimes \delta_x(T_x V).$$

Conversely, to pass from a rectifiable varifold to a rectifiable current, we have to choose an orientation at every point (obtaining the same current if the two choices agree a.e.).

Example 3.15. In the previous example concerning the different ideas of convergence for currents and varifolds, the limit is not stationary: even if its weight is 1-rectifiable, at every point x the probability measure α_x is the uniform measure on the Grassmannian factor, thus giving "equal importance" to all lines.

The following is an important rectifiability criterion for varifolds, due to Allard. As already mentioned, differently from the case of currents (where we should rather hope that our current is a limit of *integral* currents), the density assumption can often be checked in practice.

Theorem 3.16 (rectifiability criterion for varifolds). *If V is a k-dimensional stationary varifold such that $\Theta^*_k(|V|, x) > 0$ for $|V|$-a.e. $x \in M$, then V is rectifiable. The same holds more generally if V is not stationary but has locally bounded first variation.*

Remark 3.17. In the stationary case, the upper density $\Theta^*_k(|V|, x)$ is in fact the density $\Theta_k(|V|, x)$, which exists at every point as a consequence of the monotonicity formula.

Exercise 3.18. Show with a counterexample that the analogous statement for currents is false: in $\mathbb{R}^2$ there exists a sequence of 1-dimensional rectifiable currents $T_j \rightharpoonup T_\infty$ with $\partial T_j = 0$ and $\Theta_1(|T_j|, x) \geq 1$ at every point $x \in \operatorname{spt}|T_j|$, but such that T_∞ is not rectifiable.

Exercise 3.19. Assume that $V_j \rightharpoonup V$, for a sequence of stationary rectifiable varifolds V_j satisfying $\Theta_k(|V_j|, x) \geq \theta_0$ at all $x \in \operatorname{spt}|V_j|$, for some constant $\theta_0 > 0$ independent of j. Show that the limit V is stationary and rectifiable, as well.

We are going to give two different proofs, since both of them are quite instructive (in some more general situations, a mix of both could be needed).

First proof of Theorem 3.16. We just sketch a proof in the stationary case, in $\mathbb{R}^n$. Because of the more general rectifiability criterion for measures discussed in the first lecture (due to Marstrand–Mattila), it suffices to show the following: for $|V|$-a.e. point x, for any limit of the form

$$\tilde{V} = \lim_{j \to \infty} (\delta_{x, r_j})_* V$$

as Radon measures on the Grassmannian bundle, we have $\tilde{V} = \Theta_k(|V|, x) \cdot P$ for some plane P, depending on x and possibly also on the sequence $r_j \to 0$. Here $\Theta_k(|V|, x) \cdot P$ denotes the varifold associated to P, with constant multiplicity $\Theta_k(|V|, x)$.

This claim will be obtained by looking at those points x satisfying a carefully chosen list of requirements. Namely, we consider a point $x_0 \in \mathbb{R}^n$ such that:

- $\Theta_k(|V|, x_0) > 0$;
- x_0 is an approximate continuity point for the function $x \mapsto \Theta_k(|V|, x)$;
- the limit $\frac{|V|(B_r(x))}{\omega_k r^k} \to \Theta_k(|V|, x)$ is approximately uniform near x_0.

It is important to clarify the precise meaning of each requirement:

- we have $\Theta_k(|V|, x_0) = \lim_{r \to 0} \frac{|V|(B_r(x_0))}{\omega_k r^k} > 0$;
- given any $\varepsilon, \varepsilon' > 0$, the set $\{x \in B_r(x_0) : |\Theta_k(|V|, x) - \Theta_k(|V|, x_0)| > \varepsilon\}$ has $|V|$-measure at most $\varepsilon' |V|(B_r(x_0))$ for $r > 0$ small enough, depending on $\varepsilon, \varepsilon'$ (because of the previous assumption, we could equivalently require that the measure of this bad set is at most $\varepsilon' r^k$);

- given any $\varepsilon, \varepsilon' > 0$, there exists $\rho > 0$ (depending on ε) such that, letting

$$G := \Big\{x \,:\, \Big|\frac{|V|(B_s(x))}{\omega_k s^k} - \Theta_k(|V|, x)\Big| \le \varepsilon \text{ for all } 0 < s \le \rho\Big\},$$

  we have

$$|V|(B_r(x_0) \setminus G) \le \varepsilon' |V|(B_r(x_0)).$$

Even if it looks rather restrictive at first glance, the previous list of requirements is still generic, in the sense that $|V|$-a.e. point x_0 satisfies it. Let us focus on the last one, which is the only one whose genericity is not a priori clear: working on a bounded set K, note that (by Egorov's lemma) we can find a set $K_\varepsilon \subseteq K$ and a radius $\rho_\varepsilon > 0$ such that $|\frac{|V|(B_r(x))}{\omega_k r^k} - \Theta_k(|V|, x)| \le \varepsilon$ for all $x \in K_\varepsilon$ and $0 < r \le \rho_\varepsilon$, with $|V|(K \setminus K_\varepsilon) \le \varepsilon$. Then, for any $\ell \in \mathbb{N}$, any Lebesgue density point of the set $\bigcap_{j \ge \ell} K_{2^{-j}}$ (with respect to the measure $|V|$) will work, and moreover as $\ell \in \mathbb{N}$ varies these sets cover K except for a $|V|$-negligible set.

Having established that a.e. point x_0 satisfies the previous assumptions, we now fix any such point x_0. Recall that, for any sequence $r_j \to 0$, a limit $\lim_{j\to\infty} (\delta_{x,r_j})_* V$ always exists along a subsequence. Thus, considering a limit

$$\tilde{V} = \lim_{j\to\infty} (\delta_{x_0,r_j})_* V$$

along a fixed sequence $r_j \to 0$, it suffices to show that $\tilde{V} = \theta_0 \cdot P$ for a suitable k-plane P, where $\theta_0 := \Theta_k(|V|, x_0)$.

As we saw before, since $\tilde{V}$ arises as a blow-up of a stationary varifold, we have

$$y \in P \quad \text{for all } (y, P) \in \operatorname{spt}(\tilde{V}).$$

Also, from the second and third items above, it is easy to check that

$$\Theta_k(|\tilde{V}|, \tilde{x}) = \theta_0 \quad \text{for all } \tilde{x} \in \operatorname{spt}|\tilde{V}|.$$

As we saw in a previous exercise, this implies that the ratio $\frac{|\tilde{V}|(B_R(\tilde{x}))}{R^k}$ is constant in $R > 0$, and this tells us that $\tilde{V}$ resembles a cone, with respect to any point $\tilde{x} \in \operatorname{spt}|\tilde{V}| =: S$: namely, as above, we have

$$y - \tilde{x} \in P \quad \text{for all } (y, P) \in \operatorname{spt}(\tilde{V}).$$

Since $0 \in \operatorname{spt}|\tilde{V}|$, we can find a plane P_0 such that $(0, P_0) \in \operatorname{spt}(\tilde{V})$. Taking $y := 0$ and letting $\tilde{x}$ vary, we obtain $-\tilde{x} \in P_0$ for all $\tilde{x} \in S$, and hence

$$S \subseteq P_0.$$

Thus, $|\tilde{V}|$ is supported on P_0. By the constancy theorem (see the exercise below), it is a constant multiple of this plane. $\square$

Exercise 3.20. Complete the previous sketch.

Remark 3.21. The third requirement in the list could be omitted when we deal with the simpler case of a stationary varifold, since the second requirement already implies the inequality $\Theta_k(|\tilde{V}|,\cdot) \geq \theta_0$ on $\operatorname{spt}|\tilde{V}|$, by upper semicontinuity of the density along sequences of stationary varifolds, while the reverse inequality follows from the fact that $\tilde{V}$ is a blow-up. However, it is useful to treat the general case (where none of the rescalings $(\delta_{x,r})_*V$ is stationary).

Remark 3.22. In the general case, a more general version of the monotonicity formula can be used to obtain $\Theta_k^*(|V|,x) < \infty$ for $|V|$-a.e. x. Moreover, we can still guarantee that the blow-up $\tilde{V}$ is stationary: in fact, a covering argument shows that $\limsup_{r\to 0} \frac{\eta(B_r(x))}{|V|(B_r(x))} < \infty$ for $|V|$-a.e. x, where η is the absolute value of the first variation of V, and it suffices to observe that the first variation of $(\delta_{x,r})_*V$ is given by $r^{1-k}(\delta_{x,r})_*\eta$, whose mass is then infinitesimal on any given compact set (we do not use the notation δV for the first variation in order not to confuse it with the rescaling).

Exercise 3.23. Show the *constancy theorem*: given a connected, embedded C^2-smooth k-dimensional submanifold $\Sigma^k \subset U$ and a k-dimensional stationary varifold whose weight is supported on Σ in the open set U, prove that this varifold is a constant multiple of Σ in U. [Hint: test the stationarity with a vector field of the form ∇f^2, where f is a C^2 function vanishing on M.]

Second proof of Theorem 3.16. Again, we just sketch a proof in the stationary case, in $\mathbb{R}^n$. We want to apply the first, more classical rectifiability criterion for measures discussed in the first lecture. It suffices to show the following: for $|V|$-a.e. point x there exists a plane P_x such that

$$(\delta_{x,r})_*V \rightharpoonup \Theta_k(|V|,x)\cdot P_x$$

as Radon measures on the Grassmannian bundle, where the right-hand side is the varifold associated to the plane P_x, with constant multiplicity $\Theta_k(|V|,x)$.

This claim will be obtained by looking at those points x satisfying the following requirements. Calling $dV(x,P) = d|V|(x)\otimes\alpha_x(P)$ the disintegration of V, we consider a point $x_0 \in \mathbb{R}^n$ such that:

- $\Theta_k(|V|,x_0) > 0$;
- x_0 is an approximate continuity point for the function $x \mapsto \alpha_x$.

The second requirement means that, for any continuous function $\varphi : \mathrm{Gr}_k^n \to \mathbb{R}$, the point x_0 is an approximate continuity point (as in the previous proof) for the function $x \mapsto \int_{\mathrm{Gr}_k^n} \varphi\, d\alpha_x$.

It is satisfied by a.e. point: it suffices to take x_0 to be an approximate continuity point for the previous function, for each φ in a countable, dense family of functions.

We now fix any such point x_0 and a limit

$$\tilde{V} = \lim_{j\to\infty} (\delta_{x_0,r_j})_* V$$

along a fixed sequence $r_j \to 0$. We claim that $\tilde{V} = \theta_0 \cdot P_{x_0}$ for a plane P_{x_0} independent of the sequence, where $\theta_0 := \Theta_k(|V|, x_0)$. By the second assumption, it is easy to see that we can write

$$d\tilde{V}(x, P) = d|\tilde{V}|(x) \otimes \alpha(P),$$

where α is actually equal to the probability measure α_{x_0} from the disintegration of V, at the point x_0. In other words, $\tilde{V}$ has "constant Grassmannian part."

Moreover, as we saw before, since $\tilde{V}$ arises as a blow-up of a stationary varifold, we have

$$y \in P \quad \text{for } \tilde{V}\text{-a.e. pair } (y, P).$$

Since the condition $y \in P$ is a closed one, it then holds for all pairs $(y, P) \in \operatorname{spt}(V)$. In particular, recalling that $V = |V| \otimes \alpha$, we obtain that $y \in P$ for all $y \in \operatorname{spt}|V| =: S$ and all $P \in \operatorname{spt}(\alpha)$.

Thus, fixing $P_0 \in \operatorname{spt}(\alpha)$, we obtain that

$$S \subseteq P_0.$$

The constancy theorem tells us that S is a constant multiple of P_0. This multiple is θ_0, since $\Theta_k(|\tilde{V}|, 0) = \Theta_k(|V|, x_0) = \theta_0$.

In particular, we obtain $P_0 = S$. Thus, the support of α is the singleton $\{S\}$, or equivalently $\alpha = \delta_S$. Since $\alpha = \nu_{x_0}$ is independent of the particular sequence $r_j \to 0$, we obtain that the plane S is also independent of the sequence. □

3.4. **Compactness of rectifiable and integral varifolds.** As seen in a previous exercise, from the previous result, it follows immediately that if we have a sequence of k-dimensional, rectifiable, stationary varifolds $V_j \rightharpoonup V$ with $\Theta_k(|V_j|, \cdot) \geq \theta_0$ on the support of $|V_j|$ for some $\theta_0 > 0$ independent of j, then V is also rectifiable and stationary, since the same density lower bound holds for this limit varifold (in fact, rectifiability of each V_j is automatic for the same reason). In fact, we could state the following compactness result.

Theorem 3.24 (compactness of rectifiable varifolds). *Given a sequence (V_j) of k-dimensional rectifiable varifolds with locally bounded first variation, obeying uniform local* *bounds of the form*

$$\sup_j |V_j|(K) < \infty, \quad \sup_j |\delta V_j|(K) < \infty$$

for any compact set K, as well as $\Theta^*_k(|V_j|, x) \geq \theta_0$ for $|V_j|$-a.e. point x (with $\theta_0 > 0$ independent of j), we can extract a subsequence converging to a rectifiable varifold V with locally bounded first variation and the same lower density bound $|V|$-a.e.

The fact that a limit exists along a subsequence and has locally finite first variation follows from the usual compactness property of Radon measures. The rectifiability of V follows from the criterion shown before, once we obtain the density lower bound (for the same reason, it holds automatically for each V_j and does not need to be assumed). The actual content of this statement is that we still have $\Theta^*_k(|V|, x) \geq \theta_0$ for $|V|$-a.e. x. Essentially, the proof is similar to the stationary case, but more technical.

Example 3.25. In the previous example where each V_j is a union of j circles of radius $\simeq \frac{1}{j}$, the mean curvature is of order $\simeq j$ (the inverse of the radius). In order to get the total mass of the first variation, we need to multiply this by the total length, which is $\simeq j \cdot \frac{1}{j} = 1$. Thus, the first variation δV_j has total mass $\simeq j \cdot 1$, which blows up (and, indeed, the limit varifold is not rectifiable).

More interestingly, we also have a compactness theorem for integral varifolds, again due to Allard. A k-dimensional varifold V is *integral* if $\Theta^*_k(|V|, x) \in \mathbb{N} \setminus \{0\}$ for $|V|$-a.e. x.

Theorem 3.26 (compactness of integral varifolds). *Given a sequence (V_j) of k-dimensional integral varifolds with locally bounded first variation, obeying uniform local bounds of the form*

$$\sup_j |V_j|(K) < \infty, \quad \sup_j |\delta V_j|(K) < \infty$$

for any compact set K, we can extract a subsequence converging to an integral varifold V with locally bounded first variation.

We can assume that we are in an open set of $\mathbb{R}^n$, up to embedding M isometrically in a Euclidean space (this can be shown to preserve the assumptions). Since $\Theta_k(|V_j|, x) \geq 1$ for $|V_j|$-a.e. x, the fact that V is rectifiable follows already from the compactness of rectifiable varifolds. To show that the limit has integer density a.e., we can look at a (generic) point where V blows up to a plane. By a diagonal argument (suitably rescaling each V_j), we can in fact assume that

$$V = \theta \cdot P$$

for a constant $\theta \geq 1$ and a plane P. We denote by $\pi_P : \mathbb{R}^n \to P$ the orthogonal projection onto P. The task now is to show that $\theta \in \mathbb{N}$.

Assuming each V_j to be stationary for simplicity, the original proof by Allard is based on the following observation: for $y \in P$, one can show that the typical fiber $\operatorname{spt}|V_j| \cap \pi_P^{-1}(y)$ consists of finitely many points $x_1, \dots, x_\ell$. Assuming, for the sake of illustration, that $\ell = 2$, for radii $r \lesssim |x_1 - x_2|$ one can show that eventually

$$|V_j|(B_r(x_1) \cup B_r(x_2)) \geq (\theta^{(1)} + \theta^{(2)} - \varepsilon)\omega_k r^k,$$

where $\theta^{(i)} := \Theta_k(|V_j|, x_i) \in \mathbb{N}$ (for $r \leq \frac{|x_1 - x_2|}{2}$ as a consequence of monotonicity and for $r \simeq |x_1 - x_2|$ as a consequence of the fact that, by the convergence $V_j \rightharpoonup \theta \cdot P$ and a covering argument, for a typical choice of y the varifold $V_j \llcorner \pi_P^{-1}(B_r(y))$ looks "almost flat" for all $r \in (0,1)$). At a larger scale $r \gg |x_1 - x_2|$, the two points x_1 and x_2 essentially look like a single point, and one can conclude that the previous lower bound holds eventually, for any $r > 0$. Thus, we obtain

$$\theta = \frac{|V|(\pi_P^{-1}(B_r(y)))}{\omega_k r^k} = \lim_{j \to \infty} \frac{|V_j|(\pi_P^{-1}(B_r(y)))}{\omega_k r^k} \geq \lim_{j \to \infty} (\theta^{(1)} + \theta^{(2)}) \in \mathbb{N}.$$

The latter must be almost an equality at the typical point y, since letting $\theta_j(y) := \sum_{x \in \pi_P^{-1}(y)} \Theta_k(|V_j|, x)$ we have

$$\frac{1}{\omega_k} \int_{P \cap B_1(0)} \theta_j(y) \, d\mathcal{H}^k(y) \to \theta,$$

giving $\theta \approx \theta_j(y) \in \mathbb{N}$.

Another proof, again due to Allard, appeared later on. It is based on the observation that the density $\theta_j(y)$ of the projected measure $(\pi_P)_*|V_j|$ is close to an integer at many points y and, since V_j has locally bounded first variation, its differential $d\theta_j : P \to \mathbb{R}^k$ can be shown to satisfy

$$d\theta_j \to 0 \quad \text{in } W^{-1,1}_{loc},$$

i.e., goes to zero in a rather weak sense (namely, in a *negative* Sobolev space). Then Allard proved a result which became known as the *strong constancy lemma*, asserting that this guarantees that θ_j is (locally) L^1-close to its average. Since this average converges to θ, we obtain again $\theta \approx \theta_j(y)$ for many points y, and thus $\theta \in \mathbb{N}$. This idea, as well as the constancy lemma itself, proved useful in many other situations.

3.5. **Tangent cones.** We now make a few more fundamental observations about blow-ups of stationary integral varifolds, starting from a simple exercise.

Exercise 3.27. Show that for an integral varifold V we have $\Theta_k(|V|, x) \geq 1$ at each point $x \in \operatorname{spt}|V|$.

Recall that, given $x_0 \in \operatorname{spt}|V|$, we can consider a blow-up W, which is now a stationary varifold in $T_{x_0}M \cong \mathbb{R}^n$ and moreover satisfies

$$\frac{|W|(B_R(0))}{\omega_k R^k} = \Theta_k(|V|, x_0) \quad \text{for all } R > 0.$$

Hence, it saturates the monotonicity property of the area ratio, giving

$$x \in P \quad \text{for all } (x, P) \in \operatorname{spt}(W).$$

We now show rigorously that any blow-up W is a cone.

Proposition 3.28 (conicality of blow-ups). *A blow-up W of a stationary integral varifold is a* cone, *in the sense that it is dilation-invariant: we have $(\delta_{0,\rho})_* W = W$ for all $\rho > 0$.*

Proof. Since W is a limit of rescalings of V, which are stationary integral varifolds, it is itself stationary and integral by the compactness theorem; in fact, we will just need rectifiability of W here, which follows from the more elementary compactness theorem for rectifiable varifolds.

Since a rectifiable varifold is uniquely determined by its weight, it suffices to see that

$$|(\delta_{0,\rho})_* W| = |W| \quad \text{for all } \rho > 0.$$

In other words, given any $\chi \in C^1_c(\mathbb{R}^n)$, we claim that

$$\int_{\mathbb{R}^n} \chi \, d|(\delta_{0,\rho})_* W| = \int_{\mathbb{R}^n} \chi \, d|W|.$$

To show this, consider the position vector field $X(x) := x$. Its flow $(\Phi^X_t)_{t\in\mathbb{R}}$ is given by dilations, namely $\Phi^X_t(x) = e^t x$. Hence,

$$\frac{d}{d\rho} \int_{\mathbb{R}^n} \chi \, d|(\delta_{0,\rho})_* W| \Big|_{\rho = e^t} = e^{-t} \frac{d}{dt} \int_{\mathbb{R}^n} \chi \, d|(\Phi^X_t)_* W|.$$

For $t = 0$ (corresponding to $\rho = 1$), we easily compute that the previous expression equals

$$\int_{\mathbb{R}^n} [\langle \nabla\chi, X\rangle(x) + \operatorname{div}^{P_x} X(x)] \, d|W|(x),$$

where P_x denotes the tangent plane to W (or, equivalently, $|W|$) at $|W|$-a.e. x. We claim that this vanishes. Since the next argument can be applied also to the rescalings $(\Phi^X_t)_* W = (\delta_{0,e^t})_* W$ in place of W, we conclude that $\int_{\mathbb{R}^n} \chi \, d|(\delta_{0,\rho})_* W|$ is constant in ρ, as we wanted.

As seen earlier, we have

$$x \in P_x \quad \text{for } |W|\text{-a.e. } x.$$

Now the vector field $\tilde{X} := \chi X$ has

$$\operatorname{div}^P \tilde{X} = \langle \pi_P(\nabla\chi), \pi_P(X)\rangle + \chi \operatorname{div}^P X = \langle \nabla\chi, \pi_P(X)\rangle + \chi \operatorname{div}^P X$$

for any k-plane P. Since W is stationary, taking $P := P_x$, the latter integrates to zero against the weight $|W|$. On the other hand, since $X(x) = x \in P_x$, we have $\pi_{P_x}(X) = X$ at a.e. x. This proves our claim. $\square$

For this reason, blow-ups of a stationary integral varifold are called *tangent cones*. We conclude this part with two more observations. As already observed while proving the rectifiability criterion, if we have the equality $\Theta_k(|W|, \hat{x}) = \Theta_k(|W|, 0)$ for another point $\hat{x}$, then in fact W is a cone also with respect to the new center $\hat{x}$ (i.e., if we translate W by the vector $-\hat{x}$, we obtain again a cone). Indeed, recall that in this case we have the constant ratio $\frac{|W|(B_R(\hat{x}))}{\omega_k R^k} = \Theta_k(|W|, 0)$ for all $R > 0$.

As a consequence, we have conical invariance with respect to the two different points 0 and $\hat{x}$. Intuitively, this gives us a *translation invariance* along the direction $\hat{x}$. And indeed, for $|W|$-a.e. x we have both

$$x \in P_x, \quad x - \hat{x} \in P_x.$$

Thus, $\hat{x} \in P_x$ for a.e. x. By the same technique used in the previous proof, this implies that W is invariant under translations by multiples of $\hat{x}$. Thus, assuming that $\hat{x} = e_1$ up to rotations, we can write

$$W = \mathbb{R} \times W',$$

where W' is an integral varifold in $\mathbb{R}^{n-1}$.

Exercise 3.29. Check that W' is still stationary.

While this observation is very useful in many situations, we will actually need a similar one, easier to prove.

Exercise 3.30. Given any $\hat{x} \in \operatorname{spt}|W| \setminus \{0\}$, with a possibly different density compared to 0, any blow-up of W at $\hat{x}$ is translation invariant in the direction $\hat{x}$.

3.6. **Regularity of area-minimizing currents in codimension one by dimension reduction.** We have seen at the end of the previous lecture how the Plateau problem admits a solution in the class of integral currents. We now give some hints about how one obtains regularity of area-minimizing currents T *in codimension one*, at least far from the (assigned) boundary ∂T, namely how one reaches the following theorem, which comes from the subsequent efforts of many mathematicians and is one of the cornerstones of geometric measure theory.

Theorem 3.31 (regularity theorem for area-minimizing currents in codimension one). *Given a Riemannian manifold (M^n, g) and an $(n-1)$-dimensional integral current T in M, assume that T is area-minimizing in an open set $U \subset M$, meaning that*

$$\mathbb{M}(T) \leq \mathbb{M}(T') \quad \textit{whenever } T' - T = \partial S$$

for some integral current S compactly supported in U. Then T is a smooth hypersurface, with locally constant multiplicity, away from a closed set $\mathcal{S}(T) \subseteq \operatorname{spt}|T|$ of Hausdorff dimension at most $n-8$.

We assume $n \geq 3$ in the sequel, since the case $n = 2$ is quite elementary. The set $\mathcal{S}(T)$ is called the *singular set*. The previous theorem essentially asserts that $\mathcal{S}(T)$ is very small, and actually $\mathcal{S}(T) = \emptyset$ when $n \leq 7$. This result is optimal, as the next example shows. It is called the *Simons cone*, since it was initially proposed by Simons.

Example 3.32. Let $\Gamma := \frac{1}{\sqrt{2}} \cdot (S^3 \times S^3) \subset \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8$, which is a 6-dimensional manifold sitting inside the sphere S^7. Taking T to be the cone over Γ, it was proved by Bombieri–De Giorgi–Giusti that T is area-minimizing in the unit ball. In this case $\mathcal{S}(T) = \{0\}$ has Hausdorff dimension 0, matching what the previous statement predicts.

To obtain the regularity theorem, the starting point is to realize that we can assume that T is the boundary of a finite perimeter set, and thus has multiplicity 1 at almost every point. Indeed, recall that in codimension one any integral current can locally be written as a superposition of boundaries of *nested* finite perimeter sets; it is then easy to see that if T is area-minimizing then each of these boundaries must be area-minimizing as well. Once the regularity theorem is proved for each of these, we notice that two such boundaries either coincide or must be disjoint, since locally they are graphs solving a scalar elliptic PDE (and the maximum principle prevents two solutions $u_1 \leq u_2$ "touching" each other tangentially).

In the sequel, we can then assume that T has multiplicity 1 a.e. We can immediately define Σ to be the smallest closed set on whose complement regularity holds, so that our task is now to show that Σ has Hausdorff dimension at most $n-8$.

One of the fundamental tools in the proof is a result giving an effective criterion guaranteeing that a point $x \in \operatorname{spt}|T|$ belongs to the regular set $\operatorname{spt}|T| \setminus \mathcal{S}(T)$. Essentially, it requires that T is almost flat around x. Since T is rectifiable, this result applies at a.e. x and immediately shows that $\mathcal{S}(T)$ is negligible, i.e.,

$$\mathcal{H}^{n-1}(\mathcal{S}(T)) = 0.$$

While this still looks very far from what we want to show, the same tool can be applied in a more clever way to reach the stronger conclusion that $\mathcal{S}(T)$ has Hausdorff dimension $\leq n-8$.

Theorem 3.33 (Allard's regularity theorem). *For every $n \geq 2$ there exists a small constant $c(n)$ such that the following holds. If V is a k-dimensional integral stationary varifold in a ball $B_r(x) \subset \mathbb{R}^n$, with $x \in \operatorname{spt}|V|$ and mass*

$$|V|(B_r(x)) \leq (1+c(n))\omega_k r^k,$$

then on the smaller ball $B_{r/2}(x)$ the varifold V is given by a smooth graph of multiplicity 1.

The conclusion says that we can find a k-dimensional plane P and a smooth function $f : P \to P^{\perp}$ such that V agrees with the graph of f (a subset of $\mathbb{R}^n$), on the ball $B_{r/2}(x)$. In fact, given any $\ell \geq 0$ and $\varepsilon > 0$, we can find $\delta(n,\ell,\varepsilon) > 0$ such that if $|V|(B_r(x)) \leq (1+\delta)\omega_k r^k$ then $\|f\|_{C^\ell} \leq \varepsilon$.

Roughly speaking, the proof is based on an *improvement of flatness*: if V looks almost flat at a given scale, then it looks even flatter at a smaller scale. This is essentially due to the fact that, if V is already a smooth graph, it satisfies the minimal submanifold equation, which in codimension one (for functions $u : \mathbb{R}^{n-1} \to \mathbb{R}$) reads

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0.$$

Thus, when ∇u is small, we expect u to behave like a solution to the linearized equation (linearized around $u_0 = 0$), which is just the Laplace equation $\Delta u = 0$, whose solutions do enjoy such improvement of flatness.

Assuming $x = 0$, the proof consists of the following steps:

- the assumption implies that V is close to saturating the monotonicity formula, with $\Theta_k(|V|, 0) \approx 1$; thus, by a soft compactness argument, using upper semicontinuity of the support along limits of stationary varifolds, $|V|$ is close to the measure $\mathcal{H}^k \llcorner P$, for a certain k-plane P, on a smaller ball $B_{3r/4}(0)$, and here $\operatorname{spt}|V|$ is close to P in the Hausdorff topology (this is a sort of C^0 bound);
- we approximate V with the graph of a suitable Lipschitz function f, by looking at the "good set" G of points x' such that the *excess*

$$\mathbf{E}(B_s(x'), P) := s^{-k} \int_{B_s(x')} \|\pi_{T_y|V|} - \pi_P\|^2 \, d|V|(y)$$

  is small at all scales $s \in (0,r)$ and showing that these indeed belong to a graph with a small Lipschitz constant (there are plenty of such points thanks for a covering argument);

- we transfer stationarity from V to f, showing that f "almost satisfies" the minimal submanifold equation, in a weak sense, and actually that the "vertically rescaled" function
$$\tilde{f} := \frac{f-a}{\sqrt{\int |df|^2}}$$
(where a is the average of f) "almost satisfies" the Laplace equation, again in a weak sense (essentially, $\Delta \tilde{f}$ is small in a negative Sobolev space);
- we deduce an improvement of flatness for $\tilde{f}$: there exists a linear map $\ell : \mathbb{R}^k \to \mathbb{R}^{n-k}$ such that
$$\frac{1}{(\rho s)^k} \int_{B_{\rho s}} |d\tilde{f} - \ell|^2 \leq \frac{1}{2} \cdot \frac{1}{s^k} \int_{B_s} |d\tilde{f}|^2$$
for some $\rho = \rho(n) \in (0,1)$ (we identify the domain of $\tilde{f}$ with the ball $B_{3r/4}(0) \subset \mathbb{R}^k$), by a simple compactness argument, exploiting the fact that $\tilde{f}$ is normalized and that $W^{1,2}$ compactly embeds in L^2, and thus $\tilde{f}$ is L^2-close to a harmonic function, for which the previous inequality is easily obtained (more precisely, for $\tilde{f}$ we get an L^2 version of this, after which a Caccioppoli-type bound is used);
- we transfer this information back to V, obtaining that the excess decays in an analogous way: for a new plane P' we have
$$\mathbf{E}(B_{\rho s}(0), P') \leq \frac{1}{2}\mathbf{E}(B_s(0), P);$$
- replacing $x = 0$ with the other points $y \in \operatorname{spt}|V| \cap B_{r/2}(0)$ and applying this result, we see that in fact $G \supseteq \operatorname{spt}|V| \cap B_{r/2}(0)$;
- the previous excess decay says that $f \in C^{1,\alpha}$ on $B_{r/2}(0)$ for some $\alpha = \alpha(n)$, and now it is easy to conclude using standard bootstrap techniques for elliptic PDEs.

In the sequel, assume that T is an integral area-minimizing $(n-1)$-current in the unit ball $B_1(0) \subset \mathbb{R}^n$, with multiplicity 1 and $\partial T = 0$ (in $B_1(0)$). It is easy to check that the associated varifold is stationary and integral. Hence, as already mentioned before, we deduce the following.

Corollary 3.34 (regularity at density one points). *We have $\Theta_{n-1}(|T|, x) \geq 1$ for all $x \in \operatorname{spt}|T|$ and if $\Theta_{n-1}(|T|, x) < 1 + c(n)$ then $x \notin \mathcal{S}(T)$. Thus,*
$$\mathcal{S}(T) = \{x \in B_1(0) : \Theta_{n-1}(|T|, x) > 1\} = \{x \in B_1(0) : \Theta_{n-1}(|T|, x) \geq 1 + c(n)\}.$$
In particular, $\mathcal{S}(T)$ is $|T|$-negligible, and thus $\mathcal{H}^{n-1}$-negligible.

As discussed before, we can also assume that T is the boundary of a finite perimeter set. By compactness results for BV functions (similar to Rellich–Kondrachov), this is easily seen to imply that multiplicity 1 is preserved by taking limits: essentially, this holds because the limit current is the boundary of a finite perimeter set, as well. We immediately deduce the following fundamental fact, which is peculiar of the codimension-one setting.

Proposition 3.35 (persistence of singularities). *If (T_j) is a sequence of area-minimizing currents of multiplicity 1 in $B_1(0)$, with $\sup_j \mathbb{M}(T_j) < \infty$ and $0 \in \mathcal{S}(T_j)$ for all j, then we can extract a limit area-minimizing current T of multiplicity 1 (along a subsequence) with $0 \in \mathcal{S}(T)$.*

Proof. The existence of a limit of multiplicity 1 is granted by the compactness theorem of integral currents and the previous observation (actually, it follows in a much more elementary way from compactness results for BV functions). The fact that $0 \in \operatorname{spt}|T|$ requires some care, since a priori we could have cancellations. It will follow from the monotonicity formula (as the upper semicontinuity of the support for converging stationary integral varifolds) once we show that $|T_j| \rightharpoonup |T|$.

Assume that the measures $|T_j|$ converge to a limit μ, up to extracting a subsequence. It is clear that $|T| \le \mu$, by lower semicontinuity of mass for currents. Assume that eventually

$$|T|(B_\sigma(x)) \le |T_j|(B_\rho(x)) - \delta,$$

for some $x \in B_1(0)$ and two fixed radii $0 < \rho < \sigma < 1 - |x|$ and $\delta > 0$. Since weak convergence is equivalent to convergence in the flat metric, we could find a fixed radius $r \in (\rho, \sigma)$ such that the $(n-2)$-cycle $S_j := (T_j - T) \cap \partial B_r(0)$ (more rigorously, the r-slice of T_j with respect to the function $x \mapsto |x|$) has mass converging to zero, along a subsequence. Thus, eventually $S_j = \partial R_j$ for an integral current R_j on the sphere $\partial B_r(x)$, of vanishing mass $\mathbb{M}(R_j) \to 0$. We can then replace T_j with the $(n-1)$-cycle

$$T_j' := \mathbf{1}_{B_r(x)} T + R_j + \mathbf{1}_{B_1(0) \setminus B_r(x)} T_j.$$

Since it differs from T_j by a cycle compactly supported in $B_1(0)$, it differs by a boundary. Hence, by the minimality assumption, $\mathbb{M}(T_j') \ge \mathbb{M}(T_j)$. Thus, $|T|(B_r(x)) + \mathbb{M}(R_j) \ge |T_j|(B_r(x))$. However, this contradicts the fact that $|T|(B_r(x)) \le |T_j|(B_r(x)) - \delta$ for j large.

This implies that $|T|(B_\sigma(x)) \ge \mu(B_\rho(x))$ for all radii as above, which easily gives $|T| \ge \mu$ and thus $|T| = \mu$, as claimed.

If we had $\Theta_{n-1}(|T|, 0) < 1 + c(n)$, then $|T|(\bar{B}_r(0)) < (1 + c(n))\omega_{n-1} r^{n-1}$ for $r > 0$ small and thus, by the previous convergence $|T_j| \rightharpoonup |T|$, the same would hold for T_j, for j large enough, implying that $0 \notin \mathcal{S}(T_j)$, a contradiction. □

Exercise 3.36. Complete the previous proof by showing the existence of the radius $r \in (\rho, \sigma)$ in the contradiction argument.

Exercise 3.37. Show that the limit current is still area-minimizing.

Example 3.38. The previous persistence of singularities is the core tool used in the dimension reduction argument explained below. Unluckily, this fails in codimension higher than one. As an example, the complex submanifold

$$T := \{(z, w) \in \mathbb{C} \times \mathbb{C} = \mathbb{R}^4 \,:\, z^2 = w^3\}$$

(with multiplicity 1) is an area-minimizing 2-dimensional current in $\mathbb{R}^4$; in fact, it can be shown that any complex submanifold of $\mathbb{C}^n$ is area-minimizing. Its singular set is the origin, but when we consider a sequence of rescalings $(\delta_{0,r_j})_* T$ with $r_j \to 0$ we obtain the plane $\{z = 0\}$ with multiplicity 2 in the limit. Thus, multiplicity can increase and we can obtain a limit with empty singular set. This is one of the reasons why the regularity theory in codimension higher than one is way more difficult and leads to weaker results (the dimension of the singular set of an area-minimizing k-current is at most $k-2$, which is sharp as this example shows).

The idea of the so-called *dimension reduction* is to exploit this persistence of singularities under limits, in conjunction with the previous observation that when we blow-up at a given point we get a cone, and if we further blow-up at a point $\hat{x} \neq 0$ we gain a symmetry by translations. Thus, we obtain an area-minimizing cone T' which splits a line: we can write $T' = \mathbb{R} \times C$ up to rotations, with C an area-minimizing cone in $\mathbb{R}^{n-1}$. We can repeat this procedure, lowering the dimension of C until the singular set is just the origin.

Let us make this rigorous. We aim at showing the following result.

Theorem 3.39 (dimension reduction). *If $\mathcal{S}(T)$ has Hausdorff dimension $> \ell \in \mathbb{N}$, then there exists an area-minimizing cone T' of the form $T' = \mathbb{R}^{\ell+1} \times C$, where C is an area-minimizing cone in $\mathbb{R}^{n-\ell-1}$ with $\mathcal{S}(C) \neq \emptyset$. Moreover, up to increasing ℓ in this conclusion, we can assume that C is smooth outside of the origin.*

In the previous sketch, the main concern is being able to retain a substantial singular set after each blow-up. Indeed, even if the rescalings of $\mathcal{S}(T)$ converge with respect to the Hausdorff convergence of closed sets, the Hausdorff dimension could drop in the limit. To prevent this, we use the following tool.

Proposition 3.40. *If $\mathcal{H}^a(S) > 0$ for a set $S \subseteq B_1(0)$, then there exists a point $x \in S$ such that*

$$\limsup_{r \to 0} \frac{\mathcal{H}^a_\infty(S \cap \bar{B}_r(x))}{r^a} \geq 2^{-a}.$$

The reason why we use the outer measure $\mathcal{H}^a_\infty$ (introduced before defining the Hausdorff measure) instead of $\mathcal{H}^a$ is explained in the following exercise.

Exercise 3.41. Assume that we have the Hausdorff convergence of (relatively) closed sets $S_j \subset B_1(0)$ to $S_\infty \subset B_1(0)$. Then we have $\mathcal{H}^a_\infty(S_\infty) \geq \limsup_{j\to\infty} \mathcal{H}^a_\infty(S_j \cap K)$ for any compact set $K \subset B_1(0)$. Give a counterexample showing that this fails if we use instead $\mathcal{H}^a$.

We will apply the previous result with the relatively closed set $S := \mathcal{S}(T) \subset B_1(0)$, for a fixed $a > \ell$ such that $\mathcal{H}^a(S) > 0$, obtaining a good center $x \in S$. Taking a sequence $r_j \to 0$ such that

$$\lim_{j\to\infty} \frac{\mathcal{H}^a_\infty(S \cap B_{r_j}(x))}{r_j^a} \geq 2^{-a},$$

the rescaled currents $T_j := (\delta_{x,r_j})_* T$ have singular set $S_j := \delta_{x,r_j}(\mathcal{S}(T)) \cap B_1(0)$ in the unit ball. Since

$$\mathcal{H}^a_\infty(S_j \cap \bar{B}_{1/2}(0)) = \frac{\mathcal{H}^a_\infty(S \cap \bar{B}_{r_j/2}(x))}{r_j^a} \geq 4^{-a},$$

the singular set of the blow-up $T_\infty = \lim_{j\to\infty} T_j$ (in $\mathbb{R}^n$, along a subsequence), which is locally the Hausdorff limit $S_\infty = \lim_{j\to\infty} S_j$ as seen before, has $\mathcal{H}^a_\infty(S_\infty \cap B_1(0)) \geq 4^{-a}$. In particular, we have $\mathcal{H}^a(S_\infty \cap B_1(0)) > 0$ and thus $\mathcal{S}(T_\infty) \cap B_1(0) = S_\infty \cap B_1(0)$ has still Hausdorff dimension $> \ell$.

Proof of Proposition 3.40. Given $\eta \in (0, 2^{-a})$ and $\delta > 0$, consider the set $S' \subseteq S$ of all points $x \in S$ such that

$$\mathcal{H}^a_\infty(S \cap \bar{B}_r(x)) < \eta \cdot \omega_a r^a \quad \text{for all } 0 < r \leq \delta.$$

It suffices to show that $\mathcal{H}^a(S') = 0$; from the definition of Hausdorff measure, it is easy to see that this is equivalent to $\mathcal{H}^a_\delta(S') = 0$ (this equivalence holds for any set S'). To show that $\mathcal{H}^a_\delta(S') = 0$, we can assume that $S = S'$ (up to replacing S with S').

Since $S \subseteq B_1(0)$, we can cover S with finitely many sets of diameter $\leq \delta$; in particular, we have $\mathcal{H}^a_\delta(S) < \infty$. Given any $\varepsilon > 0$, we can then find a finite or countable cover $S \subseteq \bigcup_j E_j$ with sets of diameter $r_j := \operatorname{diam}(E_j) \leq \delta$ and

$$\mathcal{H}^a_\delta(S) \leq \sum_j \omega_a \frac{\operatorname{diam}(E_j)^a}{2^a} < \mathcal{H}^a_\delta(S) + \varepsilon.$$

Without loss of generality, each E_j intersects S at a point x_j.

Since $x_j \in S = S'$, we have $\mathcal{H}^a_\infty(E_j) \leq \mathcal{H}^a_\infty(\bar{B}_{r_j}(x_j)) < \eta \cdot \omega_a r_j^a$. In particular, for each j we can find another cover $E_j \subseteq \bigcup_m E_{j,m}$ such that

$$\sum_m \omega_a \frac{\operatorname{diam}(E_{j,m})^a}{2^a} < \eta \cdot \omega_a r_j^a.$$

Up to intersecting each $E_{j,m}$ with E_j, we can also assume that $\operatorname{diam}(E_{j,m}) \le \operatorname{diam}(E_j) \le \delta$. Thus, taking the cover $E \subseteq \bigcup_{j,m} E_{j,m}$, we obtain

$$\mathcal{H}^a_\delta(S) \le \sum_{j,m} \omega_a \frac{\operatorname{diam}(E_{j,m})^a}{2^a} \le \sum_j \eta \cdot \omega_a r_j^a = 2^a \eta \sum_j \omega_a \frac{\operatorname{diam}(E_j)^a}{2^a} < 2^a \eta (\mathcal{H}^a_\delta(S) + \varepsilon).$$

Since $2^a \eta < 1$ and $\varepsilon > 0$ was arbitrary, we arrive at $\mathcal{H}^a_\delta(S) = 0$, as we wanted. □

We are now ready to prove Theorem 3.39.

Proof of Theorem 3.39. Since $\mathcal{S}(T)$ has Hausdorff dimension $> \ell$, we can fix $a > \ell$ such that $\mathcal{H}^a(\mathcal{S}(T)) > 0$. As discussed before, we can then find $x_0 \in B_1(0)$ where a certain blow-up T_0 of T has singular set $\mathcal{S}(T_0) \cap B_1(0)$ of dimension $\ge a > \ell$. Moreover, since T_0 is a blow-up, it is an area-minimizing cone of multiplicity 1 (the varifolds associated to the rescaled currents converge to the varifold associated to T_0, by the convergence of the weight measures established before, and it follows that $(\delta_{0,\rho})_* T_0 = T_0$ also in the sense of currents).

Assume inductively that there exists an area-minimizing cone T_j of multiplicity 1 in $\mathbb{R}^n$, of the form $T_j = [\![\mathbb{R}^j]\!] \times C_j$, with singular set (in $B_1(0)$) of Hausdorff dimension $> \ell$, for some $j \in \{0, \dots, \ell\}$. Note that the $(n-j-1)$-current C_j is necessarily a cone of multiplicity 1, since this is true for T_j. It is easy to check that C_j must also be area-minimizing. We claim that an area-minimizing cone T_{j+1} exists with the same properties (with $j+1$ in place of j).

Indeed, the singular set of C_j necessarily contains $\{0\}$, since otherwise C_j would be smooth near the origin and thus a plane by conical symmetry, contradicting the assumption $\mathcal{S}(T_j) \ne \emptyset$. Moreover, we must have $\mathcal{S}(C_j) \supsetneq \{0\}$, since otherwise we would have $\mathcal{S}(T_j) = \mathbb{R}^j \times \{0\}$, whose dimension is $j \le \ell$, a contradiction. We can then blow-up at a point $\hat{x} \in \mathcal{S}(T_j) = \mathbb{R}^j \times \mathcal{S}(C_j)$ of the form $\hat{x} = (0, y_j) \in \mathbb{R}^j \times \mathbb{R}^{n-j}$, with $y_j \ne 0$. We obtain a blow-up of the form $T_{j+1} = [\![\mathbb{R}^j]\!] \times C_j'$, where C_j' is a blow-up of C_j at $y_j \ne 0$. As we saw in a previous exercise (stated for varifolds), C_j' is translation-invariant along the direction y_j. Thus, T_{j+1} is also translation invariant along $(0, y_j)$, besides $\mathbb{R}^j \times \{0\}$. The new current is then translation invariant along a subspace of dimension $j+1$. We can then write it as $T_{j+1} = [\![\mathbb{R}^{j+1}]\!] \times C_{j+1}$, up to a rotation. □

While it is hard to classify smooth minimal cones $C^{m-1} \subset \mathbb{R}^m \setminus \{0\}$ (in fact, by looking at the cross-section, these correspond to minimal hypersurfaces in the sphere S^{m-1}), luckily we are able to say that area-minimizing ones are flat in dimension $m \le 7$. In fact, *stability* of C is enough. Recall that a critical point x for a smooth function $E : \mathbb{R}^N \to \mathbb{R}$ is said to be *stable* if the Hessian $\nabla^2 E(u)$ is positive semidefinite (a necessary condition for u to be a local minimum point). In our setting, we say that C is stable if, whenever $(C_t)_{t \in (-\varepsilon, \varepsilon)}$ is a smooth family of perturbations of C (obtained by means of ambient isotopies, given by flows of vector fields $X \in C_c^\infty(\mathbb{R}^m \setminus \{0\}, \mathbb{R}^m)$), we have

$$\frac{d^2}{dt^2}\mathcal{H}^{m-1}(C_t \cap K)\Big|_{t=0} \geq 0,$$

where $K \subset \mathbb{R}^m \setminus \{0\}$ is any compact set such that $C_t \setminus K = C \setminus K$ for all t.

Theorem 3.42. *Assume that the cone $C \subset \mathbb{R}^m \setminus \{0\}$ is a smooth minimal hypersurface and that C is stable. If $m \leq 7$ then $C = P \setminus \{0\}$ for some hyperplane P.*

The statement applies in particular to area-minimizing cones. It is false when $m \geq 8$, as the example of the Simons cone shows. The proof of this result is quite short and exploits a certain elliptic PDE satisfied by the squared norm of the second fundamental form, called the *Simons identity*. Together with the previous observations, this last ingredient concludes the proof of the regularity theorem: if we had an area-minimizing current whose singular set has Hausdorff dimension $> n-8$, we would find an area minimizing $(m-1)$-dimensional cone in $\mathbb{R}^m$, with $m \leq n-(n-8)-1 = 7$, smooth outside of the origin. This is impossible by the previous theorem.

Useful references:

- Chapters 4, 8 of Simon, *Geometric measure theory*;
- for the compactness of integral varifolds: Section 6 of Allard, *On the first variation of a varifold*;
- for the alternative proof and the strong constancy lemma: Allard, *An integrality theorem and a regularity theorem for surfaces whose first variation with respect to a parametric elliptic integrand is controlled*, in the proceedings book *Geometric measure theory and the calculus of variations*;
- for Allard's regularity theorem: De Lellis, *An invitation to stationary varifolds*;
- for the same in the setting of finite perimeter sets: Chapters 22, 23, 24, 25, 26 of Maggi, *Sets of finite perimeter and geometric variational problems*;
- for the regularity of area-minimizing currents in codimension one: Chapter 7 and Appendices A, B of Simon, *Geometric measure theory.*