Geometric Measure Theory

SLMath 2024 Graduate School

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Contents

Introduction

These notes are transcript of Dr. Alessandro Pigati's lectures at SLMath Graduate School in 2024 at St. Mary's College of California, Moraga, CA. The knowledge of measure theory is assumed for these lectures.

The lecture notes and problem/supplement sheets written by the lecturer will be available via the following URL.

https://www.slmath.org/summer-schools/1066#schedule_notes

However, I decided to write these notes so that for recording what the actual lectures were in context and for my understanding on the topics.

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1 Day 1: June 24th

Plan for lectures

- Day 1: Rectifiable sets
- Day 2: Currents
- Day 3: Varifolds
- Day 4: Diffuse approximation of currents

What is Geometric measure theory (GMT)? The answer is that it is a collection of tools to do calculus of variations with area. It also actually has more applications beyond that. The usual framework will be the following. Fix an ambient n-dimensional Riemannian manifold (M, g), and let $1 \leq k \leq n-1$. The goal is to find a k-dimensional submanifiold $\Sigma \subset M$ which is a critical point for the area functional.

There are two issues for this. First, how to describe or think or the space Σ? Three ways can be possible.

- As a parameterized subset,
- As a level set, or
- as an intrinsic object.

The second issue is to identify the *weak notion* of submanifold that makes the space Σ compact.

Recall the case of harmonic functions, i.e. soving the boundary-value problem

$$\begin{aligned} \triangle u &= 0 \quad \text{on } \Omega \\ u &= h \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded domain. This can be regarded as a variational problem: if we define the Dirichlet energy functional

$$F(u) := \int_\Omega |du|^2,$$

then $\triangle u = 0$ if and only if u is a critical point of F. In fact, this crticial point u becomes a minimizer in this case.

In this situation, we can consider a minimizing sequence (u_j) such that $F(u_j) \to \inf_{u|_{\partial\Omega}=h} F(u)$. We want to set the space of u to extract a convergent subsequence of (u_j). This can be taken as the Sobolev space $W^{1,2}$. Then, using the weak convergence, we can obtain a weakly convergent subsequence of (u_j).

From the intrinsic point of view, we can consider two types of such weak notions.

- Currents. This notion is useful for minimization.
- Varifolds. This is useful for general critical points including saddle points.

Why should we consider saddle or unstable critical points? For example, consider the equator of the sphere S^2. Then, it is a critical point for the length functional because it is a geodesics. However, it is unstable since by pushing a part of the equator a bit towards the north, we can make it shorter.

In practice, we only focus on *rectifiable* currents and varifolds. Note that the notions of convergence will be different. However, the idea is that they are *wrappers* whose underlying objest is a *rectifiable set* with multiplicity.

Hausdorff measure We now want to see a way to measure k-dimensional area of any set S in a metric space (X, d). First, we consider a *resolution* $\delta > 0$, and define the following.

$$\mathcal{H}^k_\delta(S) := \inf\left\{ \omega_k \sum_{j=1}^{\infty} \operatorname{diam}(E_j)^k \,\middle|\, S \subset \bigcup_{j=1}^{\infty} E_j \text{ with } \operatorname{diam}(E_j) < \delta \right\}.$$

Here, $\omega_k = \alpha_k / 2^k$, where α_k is the volume of the k-dimensional unit ball. This is not yet a measure since it does not see below the resolution $\delta > 0$.

Definition 1.1. Define

$$\mathcal{H}^k(S) := \lim_{\delta \to 0} \mathcal{H}^k_\delta(S).$$

This is the *k-dimensional hausdorff measure* of S. This limit exists since $\mathcal{H}^k(S)$ is monotone decreasing.

Theorem 1.2. *On the σ-algebra of Borel sets, $\mathcal{H}^k$ is a measure.*

However, However, it is *not* σ-finite in general. For example, consider $\mathcal{H}^1$ on $\mathbb{R}^2$. Intuitively, $\mathbb{R}^2$ cannot be a countable union of 1-dimensional subsets, so it is expected to be $\mathcal{H}^1$ on $\mathbb{R}^2$ is not σ-finite.

On a Riemannian manifold, we have the following theorem.

Theorem 1.3. *Let X be a Riemannian manifold with the geodesic distance d. Then, for any k-dimensional embedded submanifold S, $\mathcal{H}^k(S)$ is the usual k-dimensional are of S.*

Exercise 1.4. Prove that

$$\mathcal{H}^0(S) := \begin{cases} \#S & \text{if } S \text{ is a finite set} \\ \infty & \text{if } S \text{ is an infinite set.} \end{cases}$$

Exercise 1.5. In the definition of $\mathcal{H}^k_\delta$, the sets E_j can be taken as closed sets. Similarly, those can be taken as open sets.

Exercise 1.6. Let X be a subspace of the metric space Y with the inherited distance. Suppose that $S \subset X$. Show that

$$\mathcal{H}^k_X(S) = \mathcal{H}^k_Y(S).$$

Exercise 1.7. Let $f : X \to Y$ be a Lipschitz map with Lipschitz constant L. Then,

$$\mathcal{H}^k_Y(f(S) \leq L^k \mathcal{H}^k_X(S).$$

Hausdorff dimension Using the Hausdroff measures we can define the notion of dimension for subsets.

Definition 1.8. Given a metric space (X, d), let $S \subset X$. Define

$$\mathcal{H}\text{-}\dim(S) := \{k \in [0, \infty) \mid \mathcal{H}^k(S) = \infty\} = \inf\{k \in [0, \infty) \mid \mathcal{H}^k(S) = 0\}.$$

Exercise 1.9. Show that these two definitions are actually the same.

Therefore, this is the threshold to distinguish the regions of k such that $\mathcal{H}^k(S) = \infty$ or $\mathcal{H}^k(S) = 0$.

Exercise 1.10. Find the Hausdorff dimension of the ternary Cantor set $\mathcal{C}$. This must be less than 1! *Hint:* Try to write

$$\mathcal{C} = \frac{1}{3}\mathcal{C} \cup \left(\frac{1}{3}\mathcal{C} + \frac{2}{3}\right).$$

Exercise 1.11. Let $f : X \to Y$ be an α-Hölder continuous map, where $\alpha > 0$. Then,

$$\mathcal{H}\text{-}\dim_Y(f(S)) \leq \alpha \cdot \mathcal{H}\text{-}\dim_X(S)$$

for any subset S of X.

Sets and measures Often, it is better to work on measures instead of sets.

Definition 1.12. Let (X, d) be a metric space, and let $S \subset X$ with $\mathcal{H}^k(S) < \infty$ (or σ-finite). Then, we can define a new measure by

$$(\mathcal{H}^k \llcorner S)(E) := \mathcal{H}^k(S \cap E).$$

Note that if S is a Borel set, then the measure $\mathcal{H}^k \llcorner S$ is also a Borel measure.

Moreover, for a Borel measurable function $f : S \to (0, \infty)$, we can define the measure

$$f(\mathcal{H}^k \llcorner S) \text{ (or } f d(\mathcal{H}^k \llcorner S)).$$

Basically, we can regard these types of measures as *sets*. The question is how to detect these measures.

Definition 1.13. Given nonnegative σ-finite Borel measure μ on X, we define the *k-dimensional upper density* by

$$\Theta^*_k(\mu, x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k}.$$

Theorem 1.14. *A measure μ can be realized by a set S and a function f if and only if for μ-almost all x, $\Theta^*_k(\mu, x) \in (0, \infty)$.*[1]

[1] Author's comment: Here, the feature of this density is to extract the growth relative to the k-dimensional ball. If it is ∞, then μ can have some higher-dimensional content, and if it becomes 0, then it may have some lower-dimensional content. Therefore, the theorem makes sense, i.e. $(0, \infty)$ is *not* a typo. Then, a question: is this related to the notion of Hausdorff dimension?

Remark 1.15. Note that $\Theta^*_k(\mu, x)$ is *not* f in general. However, these functions are comparable in some sense, and if S is a k-rectifiable set in $\mathbb{R}^n$, which will be defined later, then indeed those functions are the same.

We only sketch the proof for the the implication ($\Longleftarrow$). For this, we need a covering lemma.

Lemma 1.16 (Vitali covering)**.** *Suppose that μ is a finite measure on a metric space (X, d). Given a (possibly uncountable) collection of open balls $(B_{r_j}(x_j))_{j\in J}$ with $\sup_j r_j < \infty$, there exists a disjoint subcollection J' such that*

$$\bigcup_{j\in J} B_{r_j}(x_j) \subset \bigcup_{j\in J'} B_{5r_j}(x_j).$$

Sketch of the proof of Theorem. We may assume that $\mu(X) < \infty$. Our candidate for the set S is given by

$$S := \{x \in X \mid \Theta^*_x(\mu, x) \in (0, \infty)\}.$$

One can show that this set is indeed Borel. We claim that the measure $\mathcal{H}^k \llcorner S$ is σ-finite and $\mu \ll \mathcal{H}^k \llcorner S$. Then, this absolute continuity gives the function f, so we complete the proof.

We decompose the set S as follows: given $t \in (0, \infty)$ and $\delta > 0$, define

$$S_{t,\delta} := \{x \in X \mid \mu(B_r(x)) \geq t\omega_k r^k \text{ for some } r \in (0, \delta/10)\}.$$

Then,

$$S \subset \bigcup_{t\in(0,\infty)} S_{t,\delta}$$

for all $\delta > 0$. Now, given $x \in S_{t,\delta}$, take $r_x > 0$ such that $\mu(B_r(x)) \geq t\omega_k r^k$. Now, we can apply the Vitali covering lemma, then we obtain a countable subcollection $B_{r_j}(x_j)$ satisfying the result of the lemma. Then, first

$$S_{t,\delta} \subset \bigcup_{j=1}^{\infty} B_{5r_j}(x_j),$$

and we estimate

$$\begin{aligned}
\mu(X) &\geq \sum \mu(B_{r_j}(x_j)) \\
&\geq t \sum \omega_k r_j^k \\
&\geq \frac{t}{5^k} \mathcal{H}^k_{\delta/5}(S_{t,\delta}).
\end{aligned}$$

For $t > 0$, define

$$S_t := \{x \in X \mid \Theta^*_k(\mu, x) \in (0, t)\}.$$

Then, from the previous estimate, we can show that

$$\mathcal{H}^k(S_t) \leq \frac{\mu(X)}{5^k t}.$$

Letting $t \longrightarrow \infty$, we have that

$$\mathcal{H}^k(S) \leq \frac{\mu(X)}{5^k} < \infty.$$

This argument shows the σ-finiteness. Moreover, by a similar argument, we can show that μ is absolutely continuous with respect to $\mathcal{H}^k \llcorner S$. $\square$

Rectifiable sets Let (X, d) be a metric space. Fix $k \subset \mathbb{Z}_{\geq 0}$, and let $S \subset X$. S is called *k-rectifiable* if

$$S \subset E_0 \cup \bigcup_{j=1}^{\infty} E_j,$$

where E_0 is $\mathcal{H}^k$-negligible, and $E_j \subset f_j(K_j)$ for some Lipschitz function $f_j : K_j \to X$ from a compact set $K_j \subset \mathbb{R}^k$.

Remark 1.17. Consider the case when $X = \mathbb{R}^n$ and S is a Borel set. Note that by Kirszbraun theorem, we may assume that f_j is defined on the whole $\mathbb{R}^k$. Moreover, in this case, if S is k-rectifiable, then

$$S = E_0' \cup \bigsqcup_{j=1}^{\infty} E_j',$$

where $E_j' \subset M_j$ is a compact subset of an embedded k-dimensional C^1-submanifold M_j in $\mathbb{R}^n$. We will see the reason below.

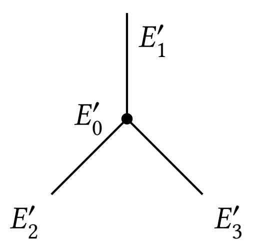


Figure 1: 1-rectifiable set

Example 1.18. Consider Figure 1. This is a 1-rectifiable set contained in $\mathbb{R}^2$.

In the definition of rectifiable sets, why do we consider the images of Lipschitz functions? Note that by Rademacher's theorem, a Lipschitz function on a Euclidean space is differentiable almost everywhere with respect to the Lebesgue measure $\mathcal{L}$. Note that since $\mathcal{H}^k = \mathcal{L}^k$, if we define $S_j \subset \mathbb{R}^k$ as the set of points $x \in \mathbb{R}^k$ such that either

- f_j is not differentiable at x, or
- f_j is differentiable at x but not injective at x.

Then, the measure of the image $\mathcal{H}^k(S_j)$ is 0.[2] Thus, we can include the set $f_j(S_j)$ in E_0. Moreover, we can cover $\mathbb{R}^k \setminus S_j$ with compact sets $(K_{jl})_{l=1}^{\infty}$ such that df is C^1 on K_{jl}. Then, by Whitney's extension theorem, there are open sets $U_{jl} \supset K_{jl}$ and functions $\widetilde{f_{jl}} : U_{jl} \to \mathbb{R}^n$ such that $\widetilde{f_{jl}}$ extends $f_j|_{K_{jl}}$ with $d\widetilde{f_{jl}}$ injective at every point.

[2]Author's comment: Why? Especially, for the second condition. Possibly, due to Sard's theorem?

Exercise 1.19. Let $S \subset \mathbb{R}^n$ be a k-rectifiable Borel set, and suppose that $\mathcal{H}^k(S) < \infty$. Then, there exists $1 \le l_1 < \cdots < l_k \le n$ such that the projection

$$\pi = \pi_{l_1 \dots l_k} : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$

has the image $\pi(S)$ is *not* negligible.

Exercise 1.20. Show that there exists a subset $S \subset \mathbb{R}^n$ such that $\mathcal{H}^k(S) \in (0, \infty)$, but it is not recifiable.

Hint: Consider the quaternary Cantor set $\mathcal{C}'$. Then, consider the set $S = \mathcal{C}' \times \mathcal{C}' \subset \mathbb{R}^2$. You can show that this set has a nonzero finite $\mathcal{H}^1$ measure but it is not 1-rectifiable. You can also consider the fact that $\pi_x(S) = \pi_y(S) = \mathcal{C}'$ is negligible.

Rectifiability criteria We now consider criteria of rectifiability. Note that the definition itself is not useful for this task. The first criterion is the following.

Theorem 1.21 (Besicovitch–Marstrand–Mattila)**.** *Let $S \subset \mathbb{R}^k$ be a Borel set with $\mathcal{H}^k(S) \in (0, \infty)$. Then, S is rectifiable if and only if*

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{r^k} = 1$$

μ-a.e., where $\mu = \mathcal{H}^k \llcorner S$.

In general, it is better to work on measures instead of sets.

Definition 1.22. Let μ be a Borel measure. Then, μ is said to be k*-rectifiable* if we can write

$$d\mu = f d(\mathcal{H}^k \llcorner S)$$

for some k-rectifiable set S.

Theorem 1.23. *Let μ is a finite measure on $\mathbb{R}^n$. This is k-rectifiable if the following hold.*

- *We have*

$$0 < \Theta_{k,*}(\mu, x) \le \Theta_k^*(\mu, x) < \infty$$

for μ-almost all x. Here, $\Theta_{k,*}$ is the k-lower density defined by

$$\Theta_{k,*}(\mu, x) := \liminf_{r \to 0} \frac{\mu(B_r(x))}{\omega_k r^k}.$$

- *For μ-almost all x, there exists a k-plane P_x such that any* blow-up *of μ at x is equal to $c\mathcal{H}^k \llcorner P_x$ for a constant $c > 0$.*

Here, a blow-up[3] *means a limit of the form*

$$\lim_{j \to \infty} \frac{(\delta_{x,r_j})_* \mu}{r_j^k},$$

where

$$\delta_{x,r}(y) = \frac{y - x}{r}.$$

[3] Author's note: I feel that the idea of blow-up is to consider a tangent object in a GMT sense.

Together with the previous criterion[4], the following are more advanced recifiability criteria.

Theorem 1.24 (Marstrand-Mattila). *In the previous theorem, we can allow each plane P_x to depend on the sequence (r_j).*

Theorem 1.25 (Preiss). *A measure μ is k-rectifiable if and only if for μ-almost all x,*

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{r^k} \in (0, \infty).$$

Idea of the proof of Theorem 1.23. We provide very rough idea for a proof. Given $\eta > 0$ and for μ-a.e. x, we can take $\rho(\eta, x) > 0$ such that

$$d\left(\frac{(\delta_{x,r})_* \mu}{r^k}, \{\text{measures on } P_x\}\right) \leq \eta.$$

Then, for a very small $\eta 0$, without loss of generality, we may assume that from measure theory knowledge[5], ρ is uniform and P_x's are close to the same plane P_0. Then, for each pair of x and x', x' is closed to $x + P_0$, and actually, $x' - x$ is almost parallel to P_0. □

[4] Author's comment: I guess the first condition is for a set S we should consider, and for the second condition, basically, that says at almost all points in S, the tangent object is well-defined in a GMT sense.

[5] I guess from Egorov's theorem.

2 Day 2: June 25th

Area and coarea formulas Before introducing the notion of currents, we first see some tools for rectifiable sets.

Theorem 2.1 (Area formula)**.** *Let $f : \mathbb{R}^k \to \mathbb{R}^m$ be a locally Lipschitz function. Suppose that $h : \mathbb{R}^k \to [0, \infty)$ is measurable and that $S \subset \mathbb{R}^k$ is a Borel set. Then,*

$$\int_S h J_f dx = \int_{f(S)} \sum_{x \in f^{-1}(y)} h(x) d\mathcal{H}^k(y).$$

Here, $J_f(x)$ is the Jacobian *of f which is defined by*

$$J_f(x) := \sqrt{\det(df(x)^T df(x)}.$$

Remark 2.2. Here, clearly $f(S)$ is a k-rectifiable set, and the measurability of the integrand of the right-hand side is also a part of the statement. Moreover, the Jacobian is well-defined since f is differentiable a.e.

Moreover, the formula is only useful when $m \geq k$. Otherwise, the formula becomes just $0 = 0$.

Idea of proof. First, one can reduce to the case that $h \equiv 1$. Moreover, we may assume that f is differentiable everywhere. Now, let x_0 be a point, then

$$f(x) = f(x_0) + df(x_0)(x - x_0) + o(|x - x_0|).$$

In this regard, somehow we reduce to the case of linear maps. Therefore, we assume $f = L$, where $L : \mathbb{R}^k \to \mathbb{R}^m$ is a liner map. Note that $L = dL$, and by the singular value decomposition, we can write

$$L = RDR',$$

where $R \in SO(m)$, $R' \in SO(k)$ and $D = (d_{ij})$ satisfies that $d_{ij} = 0$ if $i \neq j$. Observe that

$$\begin{aligned} \mathcal{H}^k(L(S)) &= \mathcal{H}^k(RDR'(S)) \\ &= \mathcal{H}^k(DR'(S) \\ &= |\det D'| \mathcal{L}^k(S), \end{aligned}$$

where

$$D = \begin{bmatrix} D' \\ 0 \end{bmatrix}$$

with D' diagonal. Note that

$$|\det D'| = \sqrt{\det(D^T D)} = \sqrt{L^T L}.$$

Using this idea, we can complete the proof. □

Exercise 2.3. Make this idea into a rigorous proof.

As a dual theorem, the coarea formula is introduced.

Theorem 2.4 (Coarea formula)**.** *Suppose that $l \geq m$, and let $f : \mathbb{R}^l \to \mathbb{R}^m$ be locally Lipschitz. Moreover, let $h : \mathbb{R}^l \to [0, \infty)$ be measurable. Then, for a Borel set $S \subset \mathbb{R}^l$, the set $f^{-1}(y)$ is k-rectifiable for almost all y, where $k = l - m$, and we have the formula*

$$\int_S h J_f dx = \int_{f(S)} \int_{f^{-1}(y)} h \, d\mathcal{H}^k dy.$$

Here, the Jacobian J_f is defined by

$$J_f = \sqrt{\det(df df^T)}.$$

The proof is hard, see [AFP00, pp. 100–108].

Exercise 2.5. Let $S \subset \mathbb{R}^k$ be a Borel set with $\mathcal{H}^k(S) = \mathcal{L}^k(S) \in (0, \infty)$. Given $0 < \sigma < \tau$, show that there exists $r \in (\sigma, \tau)$ such that

$$\mathcal{H}^{k-1}(S \cap \partial B_r(0)) \leq \frac{\mathcal{H}^k(S)}{\tau - \sigma}.$$

Tangent to a rectifiable set We now consider tangent objects to a rectifiable set. Recall that for $S \subset \mathbb{R}^n$ Borel, if it is k-rectifiable, then

$$S = E_0 \cup \bigsqcup_{j=1}^{\infty} E_j,$$

where $\mathcal{H}^k(E_0) = 0$ and $E_j \subset M_j$ for some C^1-manifold M_j. We define the following.

Definition 2.6. If $x \in E_j$, then we let

$$T_x S := T_x M_j,$$

called the *tangent space* at x.

Note that this notion is well-defined only in the following sense. If $S = E_0' \cup \bigsqcup_{j=1}^{\infty} E_j'$ is another expression for S, then the tangent spaces at x will agree for $\mathcal{H}^k$-almost every x.

Example 2.7. Let $S \subset \mathbb{R}^2$ given by $S = \{x\text{-axis}\} \cup \{y\text{-axis}\} = \{y = 0\} \cup \{x = 0\}$. If we decompose S as

$$S = \{y = 0\} \cup \{x = 0,\ y > 0\} \cup \{x = 0,\ y < 0\},$$

then $T_{(0,0)} S = \{y = 0\}$. However, if we decompose

$$S = \{y = 0,\ x \neq 0\} \cup \{x = 0\},$$

then $T_{(0,0)} S = \{x = 0\}$.

Remark 2.8. Suppose that $\mathcal{H}^k(S) < \infty$. Consider the measure $\mu = \mathcal{H}^k \llcorner S$. Recall that for μ-a.e. x, all blow-ups at x are multiples of the same k-plane, i.e. $c\mathcal{H}^k \llcorner P$. This plane P is actually $T_x S$ for $\mathcal{H}^k$-a.e. x,

Exercise 2.9. Show the fact in the remark.

Generalized area and coarea formulas We can generalize the formulas using the notion of tangent spaces.

Theorem 2.10 (Generalized area formula)**.** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, and let $S \subset \mathbb{R}^n$ be k-rectifiable. Then,*

$$\int_S hJ_f^S d\mathcal{H}^k = \int_{f(S)} \sum_{x \in f^{-1}(y)} h(x) d\mathcal{H}^k(y)$$

for a measurable function $f : S \to [0, \infty)$. Here,

$$J_f^S = \sqrt{\det(dg(x)^T dg(x)},$$

where $g = f|_{T_x S} : T_x S \to \mathbb{R}^n$ which is Lipschitz.

Exercise 2.11. State the generalized coarea formula.

Currents We first begin with the general definition.

Definition 2.12. Let M be a manifold. A *k-dimensional current* is an element of the dual space $(\Omega_c^k(M))^*$ of the space $\Omega_c^k(M)$ of k-forms with compact support.

Why is this definition natural? Consider an oriented compact k-dimensional submanifold Σ in M. Then, it defines a current $[\![\Sigma]\!]$ by the assignment

$$\omega \mapsto \langle [\![\Sigma]\!], \omega \rangle := \int_\Sigma \omega.$$

In fact, this defines the embedding

$$[\![\,\cdot\,]\!] : \{\text{compact oriented } k\text{-dimensional submanifolds}\} \hookrightarrow (\Omega_c^k(M))^*.$$

Moreover, if Σ has a nonempty boundary, then by Stokes' theorem,

$$\int_\Sigma d\eta = \int_{\partial\Sigma} \eta$$

for $\eta \in \Omega_c^{k-1}(M)$. In this regard, we define the boundary of a current in the following way.

Definition 2.13. Let T be a k-current on M. We define the $(k-1)$-current ∂_T, called the *boundary of* T, by

$$\langle \partial T, \eta \rangle := \langle T, d\eta \rangle.$$

By the Stokes' theorem, $\partial[\![\Sigma]\!] = [\![\partial\Sigma]\!]$ if Σ is a compact submanifold.

Remark 2.14. One can easily show that $\partial^2 T = 0$. Therefore, we can develop a homological theory for currents. That means, for $k = 0, \ldots, n$, where $n = \dim M$, if $G_k \subset \{k\text{-currents}\}$ such that $\partial G_k \subset G_{k-1}$, we define

$$H_k := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$$

Usually, (G_k) is taken as integral currents or normal currents, which will be defined below.

Fact 2.15. *For integral currents,*
$$H_k \cong H_k(M; \mathbb{Z}),$$
where the right-hand side is a singular homology group.

Sometimes, (M, g) has some curvature constraints. If $H_k(M, \mathbb{Z}) \neq 0$, we can minimize area among T such that $[T]$ is a generator and often reach a contradiction.

Masses and rectifiable currents

Definition 2.16. Let T be a k-current on a Riemannian manifold (M, g). Define the *mass* of T by
$$\mathbb{M}(T) \coloneqq \sup\{|\langle T, \omega\rangle| \mid \omega \in \Omega_c^k(M) \text{ with } |w| \leq 1 \text{ everywhere}\}.$$
Note that we need a Riemannian metric for defining this notion.

Exercise 2.17. Show that if Σ is a compact submanifold of (M, g), then $\mathbb{M}([\![\Sigma]\!])$ is the area of $\Sigma \subset M$.

Remark 2.18. Suppose that T has finite mass. Observe that
$$|\langle T, \omega\rangle| \leq \mathbb{M}(T)\|\omega\|_{C^0}.$$
Therefore, T is a bounded linear functional on $\Omega_c^k(M)$, then by the Riesz representation theorem in measure theory, there exists a unique measure μ and a unit section $\tau : M \to \Lambda^k TM$, we can write
$$\langle T, \cdot\,\rangle = \int_M \langle\,\cdot\,, \tau\rangle d\mu.$$
In particular, $\mu(M) = \mathbb{M}(T)$. We write this measure as $|T|$.

For defining rectifiable currents, we consider the following setting. Let S be a k-rectifiable set, and let $\nu : S \to [0, \infty)$ be measurable. Moreover, given $x \in S$, take an orientation $\tau(x)$ of $T_x S$, i.e. if $v_1, \ldots, v_k$ is an orthonormal basis of $T_x S$, then $\tau(x)$ is either $v_1 \wedge \cdots \wedge v_k$ or $-v_1 \wedge \cdots \wedge v_k$. Here, τ can be discontinuous even when S is clearly smooth.

Definition 2.19. A k-current T is called *k-rectifiable* if there are such ν and τ above such that
$$\langle T, \omega\rangle = \int_S \nu(x)\langle\omega(x), \tau(x)\rangle d\mathcal{H}^k(x)$$
for all ω. Here, we call μ the *multiplicity* of T.

Here, if $\int_S \nu < \infty$, then T has finite mass and the measure μ defined above is actually given by
$$d\mu = \nu\, d(\mathcal{H}^k \llcorner S).$$

Normal currents

Definition 2.20. A k-current is called *normal* if both T and ∂T have finite mass.

There is an interesting theorem by Smirnov.

Fact 2.21 (Smirnov)**.** *Every normal* 1*-current is a* superposition *of curves.*

More explicitly, this means the following. Let X be the space of all Lipschitz curves of finite length, where the domain of each curve is either an interval in $\mathbb{R}$ or S^1. Then, T is normal if and only if there exists a finite nonnegative measure λ on X such that

$$T = \int_X [\![\gamma]\!] d\lambda(\gamma).$$

For this fact, the essential part is the implication ($\Rightarrow$). For the other implication, we can always define a normal current in that way even for higher-dimensions.

Pushforward The class of currents which we are going to concern is the class of integral currents. For that, we introduce the notion of pushforward.

Definition 2.22. Let $\varphi : M \longrightarrow M'$ be a proper smooth map of Riemanniam manifolds. Given k-current T on M, define the *pushforward* $\varphi_* T$ by

$$\langle \varphi_* T, \omega \rangle := \langle T, \varphi^* \omega \rangle$$

for $\omega \in \Omega_c^k(M')$.

Remark 2.23. If T is a normal current, the pushforward even can be defined even when φ is only Lipschitz. In this case, we define

$$\langle \varphi_* T, \omega \rangle := \lim_{\epsilon \longrightarrow 0} \langle (\varphi_\epsilon)_* T, \omega \rangle,$$

where φ_ϵ is a regularization of φ.

Indeed, without loss of generality, we may assume that T is compactly supported and that M and M' are $\mathbb{R}^n$. The idea is to construct a homotopy between two different $(\varphi_\epsilon)_* T$ so that their difference is acutally a boundary. To do this, define

$$\Psi_{\delta,\epsilon}(t, x) := (1 - t)\varphi_\delta(x) + t\varphi_\epsilon(x)$$

for $\epsilon, \delta > 0$. For $I = [0, 1]$, define $S_{\delta,\epsilon} := \Psi_*([\![I]\!] \times T)$. Then,

$$\partial S_{\delta,\epsilon} = \Psi_*(\partial([\![I]\!] \times T)) = \Psi_*([\![\{1\}]\!] \times T - [\![\{0\}]\!] \times T) = (\varphi_\epsilon)_* T - (\varphi_\delta)_* T.$$

Then, given ω,

$$\begin{aligned} |\langle (\varphi_\epsilon)_* T - (\varphi_\delta)_* T, \omega \rangle| &\leq |\langle S_{\delta,\epsilon}, d\omega \rangle| \\ &\leq \mathbb{M}(S_{\delta,\epsilon}) \|d\omega\|_C^0. \end{aligned}$$

Moreover, for a $(k+1)$-form η on $\mathbb{R}_t \times \mathbb{R}^n_x$ with $|\eta| \leq 1$,

$$\begin{aligned}\langle S_{\delta,\epsilon}, \eta\rangle &= \langle [\![I]\!] \times T, \Psi^*\eta\rangle \\ &\leq \mathbb{M}(T) \cdot \|\partial_t \Psi\|_{C^0} \cdot \|\partial_x \Psi\|^k_{C^0} \cdot \|\eta\|_{C^0}.\end{aligned}$$

The inequality can be obtained by writing

$$\eta = dt \wedge \alpha + \beta,$$

where α, β are k-forms on $\mathbb{R}^n_x$. Here,

$$\|\partial_t \psi\|_{C^0} = \|\varphi_\epsilon - \varphi_\delta\|_{C^0},$$

and the other terms are finite. Therefore, $S_{\delta,\epsilon} \longrightarrow 0$ as $\delta, \epsilon \longrightarrow 0$, and consequently, $\mathbb{M}(S_{\delta,\epsilon}) \longrightarrow 0$. Therefore, the sequence $((\varphi_\epsilon)_* T)$ is weak-$*$ Cauchy, so the limit $\langle \varphi_* T, \omega\rangle$ exists.

Slicing Let see one more construction we need. Let T be a normal k-current such that $\partial T = 0$, and let $f : M \longrightarrow \mathbb{R}$ be Lipschitz. Consider the level set $\{f = t\}$. We want to make a correct definition for the slice $T \cap \{f = t\}$.

Definition 2.24. Given $t \in \mathbb{R}$, the *t-slice* of T along f is defined by

$$\partial(\mathbb{1}_{\{f<t\}} T).$$

If T has nonempty boundary, then the t-slice is defined by

$$\partial(\mathbb{1}_{\{f<t\}} T) - \mathbb{1}_{\{f<t\}} \partial T.$$

Remark 2.25. This makes sense for normal current (why?). Moreover, we can consider super-level sets instead of sub-level sets.

Theorem 2.26. *For almost all $t \in \mathbb{R}$, the t-slices are still normal currents.*

Remark 2.27. We can also define the slices for a Lipschitz function $f : M \longrightarrow \mathbb{R}^m$. The first way is to repeat above, or secondly, we can consider the following construction. Given $t_0 \in \mathbb{R}^m$, consider an approximation χ_ϵ of the $\delta_{\{f=t_0\}}$. Then, consider the current

$$\omega \mapsto \langle T, f^*(\chi_\epsilon dy_1 \wedge \cdots \wedge dy_m) \wedge \omega\rangle.$$

Letting $\epsilon \longrightarrow 0$, we get the t_0-slice.

Integral currents Now, we define the notion of integral currents.

Definition 2.28. A k-current T is an *integral current* if the following hold.

(a) T is k-rectifiable with integer multiplicity a.e.;

(b) ∂T also is rectifiable with integer multiplicity.

Theorem 2.29. *In fact, the second condition in the definition is redundant.*

We want to consider how pushforward behaves with integral currents. Let φ be smooth and proper, and let T be an integral current coming from a rectifiable set S, a multiplicity ν and an orientation τ as a rectifiable current. Then, for $y \in \varphi(S)$, define $\tau'(y)$ as $\varphi_*(\tau(x))$ where $x \in \varphi^{-1}(y)$, and let

$$\nu'(y) := \sum_{x \in \varphi^{-1}(y)} \varepsilon(\varphi, x)\nu(x),$$

where $\varepsilon(\varphi, x) = 1$ if $d\varphi$ maps $\tau(x)$ to $\tau'(y)$, and it is -1 otherwise. Then, $\varphi_* T$ is the rectifiable current defined by $\varphi(S)$, τ' and ν'. Moreover, in fact, for almost every y, all of $\varphi^{-1}(y)$ has the same tangent space. Thus, the previous one is well-defined. Moreover, one can allow φ to be just proper and Lipschitz.

For example, consider $\varphi(x) = |x|$ and $T = [\![\mathbb{R}]\!]$. Then, one can show that $\varphi_* T = 0$.

Next time, we see that integral currents can form a compact family.

3 Day 3: June 27th

Compactness of integral currents As we promised, we begin with the compactness of integral currents. For two integral k-currents T and T', we can write

$$T - T' = R + \partial S,$$

where R is an integral k-current and S is an inegral $(k+1)$-current. See Figure 2.

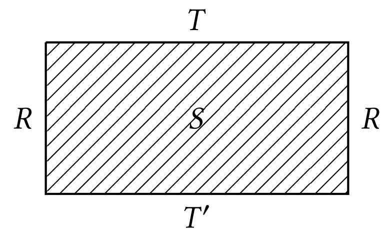


Figure 2: Difference of two integral currents

Then, we define the *flat metric*

$$d(T, T') \coloneqq \inf\{\mathbb{M}(R) + \mathbb{M}(S) \mid \text{for all such } R \text{ and } S\}.$$

Moreover, later, we need the following construction.

Definition 3.1. Let T be a k-current with $k \geq 1$. Let $\psi(t, x) \coloneqq tx$. Then, define the *cone* of T (with vertex at the origin) given by

$$\operatorname{cone}_0(T) \coloneqq \psi_*([\![I]\!] \times T),$$

where $I = [0, 1]$.

Exercise 3.2. When $k \geq 1$, show that $\partial \operatorname{cone}_0(T) = T$. Moreover, show that this is not valid when $k = 0$.

Why do we need this flat distance? One reason is that if $T_k \to T$ in flat metric, then $T_k \rightharpoonup T$, i.e. T_k converges to T weakly so that $\langle T_k, \omega \rangle \to \langle T, \omega \rangle$ for all ω. Therefore, by the following theorem, we obtain the compactness result.

Theorem 3.3. *Suppose that $K \subset M$ be compact and $C > 0$. Define the collection of integral k-currents*

$$\mathcal{S}_{K,C} \coloneqq \{T \mid \operatorname{supp}(T) \subset K,\ \mathbb{M}(T) + \mathbb{M}(\partial T) \leq C\}.$$

Then, this family $\mathcal{S}_{K,C}$ is compact in the flat topology.

Proof of Theorem 3.3. Recall that a metric space is compact if and only if it is complete and totally bounded. For completeness, we first need to argue that if (T_j) is Cauchy in flat metric, the weak limit T is actually a normal current. We want that this is an integral current, and actually, it is.

Theorem 3.4. *Let (T_j) be a Cauchy sequence in the flat metric. Then, the weak limit T is actually an integral current.*

Historically, the proof of this theorem relied on a very difficult fact. The fact is as follows.

> Let $\mathcal{H}^k(E) < \infty$, and assume that there is no rectifiable subset of positive $\mathcal{H}^k$-measure in E. Such a set E is said to be *purely rectifiable*. Then, for almost all k-planes P, the projection $\Pi_P(E)$ is negligible.

Nowadays, a modern approach is available using slicing. This fact can be stated as the following.

> A normal k-current is integral if and only if for each plane P, slices with respect to the projection Π_P provide a function of bounded variation with values in 0-currents.

For a reference, refer to [Jer02].

Proof of Theorem 3.3 (continued). Using the previous theorem, the family $\mathcal{S} = \mathcal{S}_{K,C}$ is complete. Now, we need tho show the total boundedness. The idea is to approximate the current T relative to the flat metric d by using a grid which covers the compact set K. For this, we need the following theorem.

Theorem 3.5 (Polyhedral deformation)**.** *Let T be an integral k-current wiht $k \geq 1$ and $\operatorname{supp} T \subset K$, where K is compact. For each $\rho > 0$, consider the cubical grid which covers K such that the side length of each cube is ρ. Then, we can write*

$$T = T' + R + \partial S,$$

where T' is supported on k-skeleton of the grid and has constant integer multiplicity on each face. Moreover, we have the following estimates:

$$\begin{aligned} \mathbb{M}(T') &\leq C(n)\mathbb{M}(T) \\ \mathbb{M}(\partial T') &\leq C(n)\mathbb{M}(T) \\ \mathbb{M}(S) &\leq C(n)\rho\mathbb{M}(T) \\ \mathbb{M}(R) &\leq C(n)\rho\mathbb{M}(\partial T). \end{aligned}$$

Here, if $\partial T = 0$, then $R = 0$.

Proof of Theorem 3.3 (continued). Note that if we have the theorem, then for sufficiently small ρ, by the polyhedral deformation, we can show the total boundedness. This completes the proof. □

Therefore, we discuss the proof of Theorem 3.5.

Proof of Theorem 3.5. We may assume that $\rho = 1$. In addition, assume that T is supported on l-skeleton of the grid for some $l > k$. By induction, if replace T with T' supported on $(l-1)$-skeleton, then we are done.

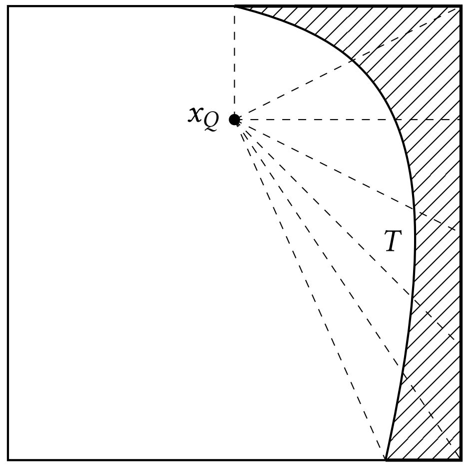


Figure 3: Radial projection in a cube

Given an l-cube Q, choose the *center* x_Q in a random way. By the radial projection, define

$$\Pi = \Pi^{x_Q} : Q \setminus \{x_Q\} \longrightarrow \partial Q.$$

See Figure 3. Note that approximately,

$$|d\Pi(x)| \cong \frac{1}{|x - x_Q|.}$$

Now, take $r > 0$ and consider $\mathbb{1}_{Q\setminus B_r(x_Q)}T$ and its pushforward by Π. Then,

$$\mathbb{M}(\Pi_*(\mathbb{1}_{Q\setminus B_r(x_Q)}T)) \lesssim \int_{Q\setminus B_r(x_Q)} \frac{1}{|x - x_Q|^k} d|T|(x).$$

Letting $r \longrightarrow 0$, and call the limit $\Pi_*^{x_Q}T$. By building a homotopy between Π^{x_Q} and the identity, we obtain

$$d|_Q(T, \Pi_*^{x_Q}T) \lesssim \lim_{r\to 0} \int_{Q\setminus B_r(x_Q)} \frac{1}{|x - x_Q|^k} d|T|(x).$$

We can glue all projections Π^{x_Q} together to get a map

$$\Pi : (l\text{-skeleton}) \setminus \{x_Q \,|\, Q \text{ a } l\text{-cube}\} \longrightarrow ((l-1)\text{-skeleton}).$$

Then, $T' = \Pi_* T$, and define S to be the pushforward of $[\![I]\!] \times T$ under a homotopy between Π and the identity. Finally, $R = T - T' - \partial S$. See Figure 3 again to understand these constructions.

Now, it remains to show the estimates. Observe that

$$\begin{aligned}\int_{Q'}\int_Q \frac{1}{|x - x_Q|^k} d|T|(x) dx_Q &= \int_Q \int_{Q'} \frac{1}{|x - x_Q|^k} dx_Q d|T|(X) \\ &\lesssim C_l \int_Q \int_0^{\sqrt{n}} r^{l-k-1} dr d|T|(x).\end{aligned}$$

Now, since $l \geq k$, we have

$$(\cdots) \leq C(l,n) \int_Q d|T|(x)$$
$$= |T|(Q) < \infty.$$

By this estimate, we can find x_Q so that the estimates hold. This completes the proof. □

As a corollary, we have the following.

Corollary 3.6 (Nonsharp isoperimetric inequality)**.** *Suppose that T is an integral k-current with $k \geq 1$. Suppose also that $\partial T = 0$. Then, there exists an integral $(k+1)$-current S such that $\partial S = T$ and*

$$\mathbb{M}(S) \leq C(n,k)\mathbb{M}(T)^{\frac{k+1}{k}}.$$

Idea of Proof. By the theorem, we can write

$$T = T' + \partial S.$$

Then,

$$\mathbb{M}(S) \leq C\rho\mathbb{M}(T)$$
$$\mathbb{M}(T') \leq C\mathbb{M}(T).$$

Note that we also have $\mathbb{M}(T') \gtrsim \rho^k$, and we can choose ρ so that $\rho \cong \mathbb{M}(T)^{1/k}$. Therefore, by the second inequality, we have $T' = 0$, and moreover,

$$\mathbb{M}(S) \lesssim \mathbb{M}(T)^{1+1/k}.$$

This completes the proof. □

Exercise 3.7. For a compact manifold M, by an embedding $M \hookrightarrow \mathbb{R}^N$ with $N \gg 1$, show that the previous statement also holds on M if $\mathbb{M}(T)$ is small enough.

Exercise 3.8. Suppose that T is an integral k-current supported on an connected embedded k-dimensional manifold in a Euclidean space. Show that T has constant multiplicity.

Plateau problem We consider the following variational problem. Let Γ be a k-dimensional embedded smooth submanifold without boundary in a Euclidean space.[6] We assume that $k \geq 2$. Now, consider the family of integral k-currents

$$\mathcal{S} \coloneqq \{T \mid \partial T = \Gamma\}.$$

We want to *minimize mass in $\mathcal{S}$*. That is, we want to find T that attains

$$\Lambda \coloneqq \inf_{T \in \mathcal{S}} \mathbb{M}(T).$$

[6]This is assigning the boundary of solutions. In a compact manifold M, instead, we can also assign a homology class in $H_k(M)$.

By taking the cone of Γ, the collection $\mathcal{S}$ is nonempty. We now can consider a minimizing sequence (T_j) such that $\mathbb{M}(T_j) \to \Lambda$. The question is whether we can find the limit of (T_j) possibly by passing to a subsequence.

Note that by projecting onto a large ball, we may assume that $\operatorname{supp}(T_j) \subset K$ for a compacf set K. Then, by Theorem 3.3, we can assume that $T_j \rightharpoonup T$. We need the regularity of T, but this is actually a hard task. We are going to talk about this more after discussing the notion of varifolds.

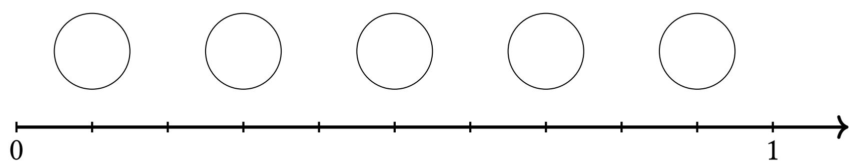
Remark 3.9. Note that mass is *lower semi-continuous* with respect to weak limit of currents. Suppose that $T_j \rightharpoonup T$ and $\mathbb{M}(T_j) \to \Lambda$. Then, for ω with $|\omega| \leq 1$ everywhere,

$$\begin{aligned} |\langle T, \omega \rangle| &= \lim |\langle T_j, \omega \rangle| \\ &\leq \lim \mathbb{M}(T_j) \\ &= \Lambda. \end{aligned}$$

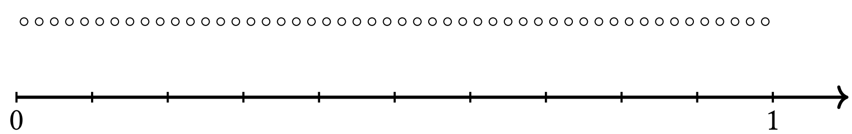
Therefore,

$$\mathbb{M}(T) \leq \Lambda.$$

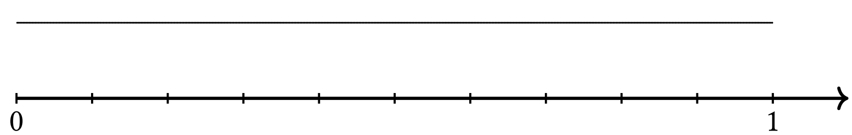
In fact, the equality can be failed as in the following example.



(a) $k = 1$



(b) $k = 2$



(c) $k = 3$

Figure 4: Train of circles

Example 3.10. For each $k = 1, 2, \ldots$, consider a train of circles T_k as in Figure 4. Here, the centers of circles are given by

$$\left(\frac{2i+1}{10^k}, 1 \right), \quad i = 0, \ldots, \frac{10^k}{2} - 1,$$

and the radius of each circle is

$$r_k = \frac{1}{2 \cdot 10^k}.$$

Then, as 1-currents, the total mass is

$$\mathbb{M}(T_k) = \frac{10^k}{2} \cdot 2\frac{1}{2 \cdot 10^k} \cdot \pi = \frac{\pi}{2}.$$

Therefore, the mass is constantly $\pi/2$. However, $T_k \rightharpoonup 0$ (why?).

Varifolds Suppose that a sequence of k-rectifiable sets (Σ_j), $\Sigma_j \subset M$, satisfies

$$0 < c < \mathcal{H}^k(\Sigma_j) < C < \infty$$

for all j. This (Σ_j) is *almost critical*. We want to take the limit $j \to \infty$ and obtain a critical point of area.

A naive idea is the following. Consider the measures $\mu_j := \mathcal{H}^k \llcorner \Sigma_j$. If $\mu_j \rightharpoonup \mu$, then $\mu(M) \geq C$. However, typically, we think criticality in the following way.

Given family of diffeomorphisms $(\Phi_t)_{t \in \mathbb{R}}$ on M with $\Phi_0 = \mathrm{id}_M$,

$$\mathcal{H}^k(\Phi_t(\Sigma)) = \mathcal{H}^k(\Sigma) + o(t).$$

This is always true for measures.

For resolving this issue, we try to *remember* the *direction* (without orientation) of a tangent space like Young measures.[7]

Definition 3.11. Let $M = \mathbb{R}^n$. A *k-varifold* V is a positive radon measure on $\mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n)$. Moreover, we can define the measure $|V|$ on $\mathbb{R}^n$ which is the underlying measure of V on $\mathbb{R}^n$.

Remark 3.12. Here, $\mathrm{Gr}_k(\mathbb{R}^n)$ is the Grassmannian, i.e. the set of all k-planes passing through the origin. It has a distance defined as follows:

$$d(P, P') := \|\Pi_P - \Pi_{P'}\|,$$

where the norm can be taken any norms for n-by-n symmetric matrices. For example, one can use the operator norm or the Hilbert-Schmidt norm. These norms are all equivalent. In this regard, we can consider Borel measures on it.

We mainly focus on recifiable varifolds.

Definition 3.13. Let $E \subset \mathbb{R}^n$ be a k-rectifiable set, and let $f : E \to (0, \infty)$ be a multiplicity. Define a *rectifiable k-varifold* by

$$V(x, P) := f\,\mathcal{H}^k \llcorner E(x) \otimes \delta_{T_x E}(P).$$

[7]European Mathematical Society, *Young measure*, Encyclopedia of Mathematics, Springer, 2020, URL: http://encyclopediaofmath.org/index.php?title=Young_measure&oldid=50866

Therefore, this is a rectifiable measure with *fiber information*. In fact, we have the following fact.

Fact 3.14. *There is a one-to-one correspondence between k-rectifiable measures and k-rectifiable varifolds.*

Example 3.15. Consider Example 3.10 again. Now, we consider the sequence as a sequence (V_k) of 1-varifolds. Then, in fact, $V_k \to V$, where

$$V = c \cdot (\mathcal{H}^1 \llcorner I) \otimes \mu,$$

where $I = [0, 1]$, and μ is the uniform measure on $\mathrm{Gr}_1(\mathbb{R})$. One can visually understand this in Figure 4.[8] Note that the limit V is *not* rectifiable.

We now define pushforward for varifolds. Let ϕ be a diffeomorphism on $\mathbb{R}^n$. Then, it induces a diffeomorphism $\widehat{\Phi}$ on $\mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n)$, where

$$\widehat{\Phi}(x, P) \coloneqq (x, d\Phi(x)(P)).$$

We define the pushforward by

$$\Phi_* V \coloneqq \widehat{\Phi}_* [J_\Phi^P(x) V(x, P)].$$

Exercise 3.16. Suppose that a varifold V comes from a submanifold Σ with multiplicity 1. For a diffeomorphism Φ, consider the pushforward $\Phi_* V$. Show that $\Phi_* V$ is also a varifold associated to $\Phi(\Sigma)$.

Stationary varifolds To think of varifolds as critical points for area, we now consider the notion of stationary varifolds. First, we define the variation of varifolds. Let $X \in C_c(M, TM)$ be a compactly supported vector field, and let (Φ_t^X) be the flow of X. Let $\operatorname{supp} X \subset K$, where K is compact. Then, define

$$\langle \delta V, X \rangle \coloneqq \left. \frac{d}{dt} \right|_{t=0} \left| (\Phi_t^X)_* V \right| (K).$$

Definition 3.17. We say that a varifold V is *stationary* if $\langle \delta V, X \rangle = 0$ for all X.

Exercise 3.18. Show that the derivative

$$\frac{d}{dt} \left| (\Phi_t^X)_* V \right| (K)$$

always exists, and in fact,

$$\frac{d}{dt} \left| (\Phi_t^X)_* V \right| (K) = \int_{\mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n)} \operatorname{div}^P X(x) \, dV(x, P).$$

[8] Author's comment: Try zooming the subfigure of $k = 3$. That is not a line!

Theorem 3.19 (Allard, Rectifiability Criterion)**.** *Let V be a stationary k-varifold, and suppose that $\Theta_k^*(|V|, x) > 0$ for $|V|$-a.e. Then, V is rectifiable.*

Remark 3.20. In fact, for a k-varifold V, the density

$$\Theta_k(|V|, x) = \lim_{r \to 0} \frac{|V|(B_r(x))}{\omega_k r^k}$$

exists for $|V|$-a.e.

Theorem 3.21 (Allard, Compactness)**.** *Suppose that (V_j) be a sequence of stationary varifolds which are rectifiable with integer multiplicity. Then, passing to a subsequence, $V_j \rightharpoonup V$, and V has the same properties.*

In fact, the assumption on stationary varifolds can be weaken by the following notion.

Definition 3.22. We say that a varifold V has *(locally) bounded first variation* if for every compact set K and for every vector field X with $\operatorname{supp} X \subset K$,

$$|\langle \delta V, X \rangle| \le C(K) \|X\|_{C^0}.$$

Remark 3.23. Note that in this case, by the Riesz representation theorem, we can write

$$\langle \delta V, X \rangle = -\int_M X \cdot dH,$$

where H is a vector measure on M.

Exercise 3.24. Suppose that a varifold V arises from a smooth submanifold Σ. Show that

$$dH = (\text{mean curvature of } \Sigma) \cdot d(\mathcal{H}^k \llcorner \Sigma).$$

Remark 3.25. The assumption of Allard's rectifiability criterion can be weakened as V to have bounded first variation. See [Sim18, Theorem 5.5, p. 246] (or [Sim84]).

Moreover, for Allard's compactness theorem, instead of stationary varifolds, a sequence of *uniformly* bounded first variations will also provides the same result. See [Sim18, Theorem 5.8, p. 248] (or [Sim84]).

4 Day 4: June 28th

Some results on varifolds Recall that a k-varifold on a manifold M is a Radon measure on the Grassmannian bundle $\pi : G_k(M) \to M$. Moreover, the underlying measure $|V|$ on M is defined by $\pi_* V$. If $M = \mathbb{R}^n$, the Grassmannian bundle is trivial, so a k-varifold on $\mathbb{R}^n$ is a Radon measrue on $\mathbb{R}^n \times \mathrm{Gr}_k(\mathbb{R}^n)$. We also write $\mathrm{Gr}_k(\mathbb{R}^n)$ as $\mathrm{Gr}_{n,k}$. Note that if

$$\mu = f\,(\mathcal{H}^k \llcorner S)$$

is a rectifiable measure, then we can consider the varifold

$$V(x, P) = f(x)(\mathcal{H}^k \llcorner S)(x) \otimes \delta_{T_x S}(P),$$

which is called a rectifiable k-varifold. A motivation of the notion of varifolds is to take account for the Jacobian factor arising from the tangent space part cosidering the area formula.

Theorem 4.1 (Allard, rectifiability criterion)**.** *Let V be a k-varifold with locally bounded first variation, and suppose that*

$$\Theta_k^*(|V|, x) > 0$$

for $|V|$-a.e. x. Then, V is rectifiable.

Example 4.2. Consider a varifold on $\mathbb{R}^n$

$$V = (\mathcal{H}^k \llcorner (\text{segment})) \otimes (\text{uniform measure on } \mathrm{Gr}_{n,k}).$$

Then, the first variation is not locally bounded. Therefore, this is not rectifiable.

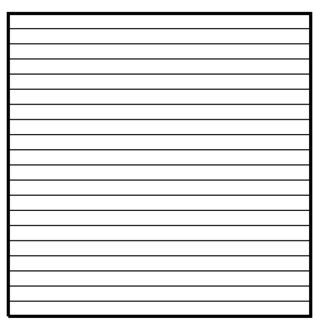


Figure 5: Bounded first variation without density assumption

Example 4.3. Consider a varifold on $\mathbb{R}^2$ defined by

$$V = (\mathcal{H}^2 \llcorner (\text{square})) \otimes \delta_{(x\text{-axis})}.$$

See Figure 5. Then, this is of bounded first variation, but it does not satisfy the positivity of the density. This is not rectifiable. Here, the idea is that even though the set is 2-dimensional, each tangent space is 1-dimensional.

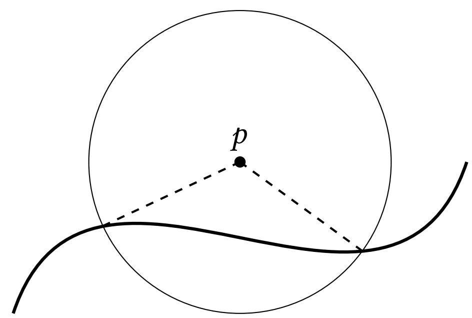


Figure 6: For monotonicity formula

For the proof of the theorem, we only consider stationary varifolds. The key idea is the *monotonicity formula*. To get some idea, consider a stationary k-rectifiable set Σ. Now, define

$$f(r) := \mathcal{H}^k(\Sigma \cap B_r(p)).$$

Note that Σ is locally area minimizing, and one can show that

$$\mathcal{H}^{k-1}(\Sigma \cap \partial B_r(p)) \leq f'(r).$$

This inequality is equivalent to the fact that $\Sigma \perp \partial B_r(p)$. Here, we consider the cone joining p and the intersection $\Sigma \cap \partial B_r(p)$ as in Figure 6. Then, the area of the cone is $\mathcal{H}^{k-1}(\Sigma \cap \partial B_r(p)) \cdot r/k$, and one can estimate

$$f(r) \leq \mathcal{H}^{k-1}(\Sigma \cap \partial B_r(p)) \cdot \frac{r}{k} \leq f'(r)\frac{r}{k}.$$

Therefore, we have an differential inequality

$$f'(r)\frac{r}{k} \geq f(r).$$

Using some ODE theory, one can show that $f(r)/r^k$ is increasing.

We want to consider now the varifold case.

Proposition 4.4 (Monotonicity formula)**.** *Let V be a k-varifold. Then, for $a < b$, we have*

$$\frac{|V|(B_b(p))}{b^k} - \frac{|V|(B_a(p))}{a^k} = \int_{(B_b(p)\setminus B_a(p))\times \mathrm{Gr}_{n,k}} \frac{|\nabla^{P^\perp} r(x)|^2}{r(x)^k} dV(x,P),$$

where r is the distance from p, and $\nabla^{P^\perp}$ is the projection of ∇ onto $P^\perp$.

Proof. We may assume that $p = 0 \in \mathbb{R}^n$. We want a vector field like

$$X(x) := \mathbb{1}_{B_r(0)} x.$$

For a smooth one, first, consider a smooth function $0 \leq \varphi \leq 1$ with $\varphi(t) = 0$ for $t \geq 1$, and define

$$X(x) := \varphi\left(\frac{|x|}{r}\right) x.$$

Then, by some computations for $\langle \delta V, X \rangle$, we obtain the formula. ◻

By this formula, we also obtain the following.

Corollary 4.5. *The density*

$$\Theta_k(|V|, P) = \lim_{r \to 0} \frac{|V|(B_r(p))}{\omega_k r^k}$$

exists $|V|$*-a.e.*

Proof of theorem. We want to use the first rectifiability criterion for measures. Fix $x_0 \in \mathbb{R}^n$, and define

$$\delta_{x_0,s}(y) := \frac{y - x_0}{s}.$$

Now, let (s_j) be a sequence with $s_j \longrightarrow 0$. Then, define

$$V_j := (\delta_{x_0,s_j})_* V,$$

and we want to consider the limit of V_j. In the definition of varifold pushforward, the Jacobian factor is s_j^{-k}, so

$$|V_j| = \frac{(\delta_{x_0,s_j})_*|V|}{s_j^{-k}}.$$

Note that this sequence is bounded above by the monotonicity formula with some scaling argument. Therefore, passing to a subsequence, let V_∞ be the limit (in weak-$*$ topology) of that subsequence. We want to show that V_∞ is associated to a single k-plane with multiplicity.

Note that since

$$\int_{\mathbb{R}^n \times \mathrm{Gr}_{n,k}} \mathrm{div}^P(X)(x) dV(x, P) = 0$$

for every $X \in C_c^1$, V_∞ is still stationary. Now, we use the monotonicity formula. Define

$$h(r) := \frac{|V|(B_r(x_0))}{r^k}.$$

We know that

$$\lim_{r \to 0} h(r) = \omega_k \Theta_k(|V|, x_0).$$

Meanwhile, for a fixed $r > 0$,

$$h_j(r) := h(s_j r) \overset{j \to \infty}{\longrightarrow} \omega_k \Theta_k(|V|, x_0),$$

but also at the same time,

$$\lim_{j \to \infty} h_j(r) = h_\infty(r) := \frac{|V_\infty|(B_r(x_0))}{r^k}.$$

Therefore, $h_\infty(r)$ is actually independent of r. Now, if we use the monotonicity formula for V_∞, then

$$\left| \nabla^{P^\perp} d(x) \right|^2 = 0$$

for $|V_\infty|$-a.e. x. Thus, $x - x_0 \in P$ for $|V_\infty|$-a.e. x.

We now want to choose x_0 in a clever way so that our conclusion holds. In fact, *generic* x_0 will satisfy. In this regard, we remark the following.

- Like Lebesgue points, we can take x_0 as an approximate continuity point of $\Theta_k(|V|, \cdot)$ with some additional uniformity for the limit of density around x_0. Therefore, $|V_\infty|$ has constant density, and
$$\Theta_k(|V_\infty|, x) = \Theta_k(|V|, x) > 0$$
for $|V_\infty|$-a.e. x.

- Therefore, $|V_\infty|$ is rectifiable. That means, we can write
$$|V_\infty| = \Theta_k(|V|, x_0) \cdot (\mathcal{H}^k \llcorner S),$$
where S is a rectifiable set.

- In general, we can write
$$V_\infty(x, P) = |V_\infty|(x) \otimes \lambda_x(P),$$
where λ_x is a probability measure on $\mathrm{Gr}_{n,k}$. Thus, take x_0 also to be an approximate continuity point for λ_x. Then,
$$V_\infty = \Theta_k(|V|, x_0) \cdot (\mathcal{H}^k \llcorner S) \otimes \lambda,$$
and λ is independent of x, i.e. we can freeze the $\mathrm{Gr}_{n,k}$ part.

Now, take *distributional* supports of $\mathcal{H}^k \llcorner S$ and λ, namely, $\widetilde{S}$ and T. Then, since V_∞ is not trivial, $\widetilde{S}$ is also not trivial.

Without loss of generality, assume $x_0 = 0$. Then, $x \in P$ for V_∞-.a.e (x, P). Therefore, for all $x \in \widetilde{S}$ and $P \in T$, $x \in P$. Now, fix any $\widetilde{P}$ in T, then $\widetilde{S} \subset \widetilde{P}$. In fact, $S \subset \widetilde{S} \subset \widetilde{P}$. It follows that T is a singleton set. Indeed, otherwise, for $\widetilde{P}$ and $\widehat{P}$ in T, $S \subset \widetilde{P} \cap \widehat{P}$, but then $\mathcal{H}^k(S) = 0$. This is a contradiction. Hence, $\lambda = \delta_{\widetilde{P}}$.

Moreover, the area ratio is constant, i.e.
$$\frac{|V_\infty|(B_r(0))}{r^k}$$
is constant. This means that $S = \widetilde{P}$. This proves the assertion. □

Compactness of stationary integral varifolds Now, we see Allard's compactness result.

Theorem 4.6. *Let (V_j) be a sequence of stationary k-rectifiable varifolds with integer multiplicity. Suppose that $V_j \rightharpoonup V$. Then, V has the same properties.*

Note that a rectifiable varifold with integer multiplicity is called an *integral varifold.*

Exercise 4.7. Let V be a stationary integral k-varifold. Show that
$$\Theta_k(|V|, x) \geq 1$$
for x in the distributional support of $|V|$.

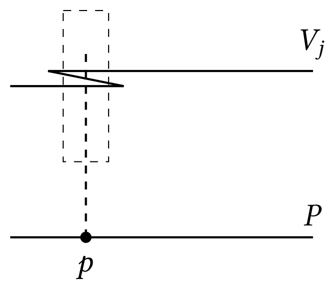


Figure 7: Convergence of (V_j)

Remark 4.8. One can show that for a stationary k-varifold, the density $\Theta_k(|V|, x)$ is upper semi-continuous. Indeed, consider two lines intersecting at a point p. Then, at p, $\Theta_k(|V|)$ is 2, but at other points on each line, $\Theta_k(|V|)$ is 1.

Idea of Proof. In this proof, we use the notation supp for distributional supports. Note that by the upper semicontinuity of the density, the limit V satisfies $\Theta_k(|V|, x) \geq 1$ for $x \in \text{supp}(|V|)$. Therefore, by a rectifiability criterion, V is rectifiable. Moreover, at $|V|$-a.e. point, the blow-up is $c\dot{P}$, where P is a k-lane. In fact, by a diagonal arguments, without loss of generality, one can show that $V = c \cdot P$. Here, the integral multiplicity is obtained by the monotonicity formula, and moreover, the tangent plane to each j is very close to the same plane P. Now, consider a small cylinder of radius r over a point p as in Figure 7. Here, because V_j is *almost parallel* to P, we have that

$$|V_j|(\text{cylinder}) \cong \omega_k r^k \cdot (\text{multiplicity near } p).$$

In this way, we can show that V is a stationary integral k-varifold. □

Author's comment For the remaining part of the lecture, Dr. Pigati discussed the area mininizing problem with Allen-Cahn energy. Possibly, I will update this in these notes after his official lecture notes become available through SLMath webpage. Moreover, we have not discussed the regulrity results much, and for those, one can refer to [Sim18] or [Sim84].

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