Assignment for Day 1
SLMath Statistical Optimal Transport

Marcel Nutz, Columbia University

Exercise 1. Consider the optimal transport problem with costs $c(x, y)$ and $\tilde{c}(x, y) = c(x, y) + a(x) + b(y)$ where $a \in L^1(\mu)$ and $b \in L^1(\nu)$. Show that the corresponding optimal transport problems have the same optimal couplings. What is the relation between the optimal costs? And the dual solutions?

Specialize this to $c(x, y) = \|x - y\|^2/2$ and $\tilde{c}(x, y) = -\langle x, y\rangle$, assuming that μ, ν have finite second moments.

Exercise 2. Detail the connection between the assignment problem (in its Kantorovich formulation) and the Birkhoff polytope of doubly stochastic $n \times n$ matrices. Use this to explain geometrically the uniqueness or non-uniqueness of the optimizers, both in the Monge sense and in the Kantorovich sense. Argue that uniqueness in the Monge sense already implies uniqueness in the Kantorovich sense.

Exercise 3. In the special case of the assignment problem, show (by elementary arguments) the equivalence of optimality of a coupling and c-cyclical monotonicity of its support.

Exercise 4. Detail the meaning of c-cyclical monotonicity for $c(x, y) = |y - x|^2$ on $\mathbb{R} \times \mathbb{R}$. Show that the condition for two pairs of points already implies the general version.

Show that the c-cyclical monotonicity property does not depend on c as long as $c(x, y) = h(y - x)$ with h strictly convex or $\partial^2_{xy} c(x, y) < 0$. (Those are two popular classes of costs; you may generalize further.) Note: This is specific to the one-dimensional setting.

Exercise 5. Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ where μ is atomless. Show that for cost $c(x, y) = h(y - x)$ with h strictly convex, there is a unique optimal coupling, given by the map $T = F_\nu^{-1} \circ F_\mu$ where F_μ is the cdf of μ and F_ν^{-1} is the pseudo-inverse

$$F_\nu^{-1}(x) = \inf\{t \in \mathbb{R} : F_\nu(t) \geq x\}.$$

Suppose that μ, ν admit densities f_μ, f_ν w.r.t. Lebesgue measure, where $\{y : f_\nu(y) > 0\}$ is convex. Determine the regularity of T in terms of the regularity of the densities. What happens if the convexity condition does not hold? Compare with Caffarelli's theorem.

Exercise 6. Let $\mu = \mathcal{N}(m_0, \sigma_0^2)$ and $\nu = \mathcal{N}(m_1, \sigma_1^2)$ be Gaussian probability measures on $\mathbb{R}$ and consider the quadratic cost $c(x,y) = \frac{1}{2}|x-y|^2$. Explicitly compute the optimal transport map T. (You may use the formula $T = F_\nu^{-1} \circ F_\mu$ of Exercise 5.) Compute the optimal transport cost.

Exercise 7. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported and let $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be L-Lipschitz. Use the Arzelà–Ascoli theorem to prove that the dual optimal transport problem has a solution. Moreover, prove that any solution is L-Lipschitz and bounded. Use those results to make rigorous the proof of strong duality via the minimax theorem.