Assignment for Day 2
SLMath Statistical Optimal Transport

Marcel Nutz, Columbia University

Exercise 1. Consider the entropic optimal transport problem EOT_ε with costs $c(x,y)$ and $\tilde{c}(x,y) = c(x,y) + a(x) + b(y)$ where $a \in L^1(\mu)$ and $b \in L^1(\nu)$. Show that the corresponding problems $\mathrm{EOT}_\varepsilon(c)$ and $\mathrm{EOT}_\varepsilon(\tilde{c})$ have the same optimal couplings. What is the relation between the optimal costs? And the dual solutions?

Exercise 2. Generalizing our problem

$$\mathrm{EOT}_\varepsilon := \inf_{\pi \in \Pi(\mu,\nu)} \int c\, d\pi + \varepsilon H(\pi|\mu \otimes \nu),$$

consider another pair (μ', ν') of marginals and

$$\mathrm{EOT}'_\varepsilon := \inf_{\pi \in \Pi(\mu,\nu)} \int c\, d\pi + \varepsilon H(\pi|\mu' \otimes \nu').$$

Show that

$$\mathrm{EOT}'_\varepsilon = \mathrm{EOT}_\varepsilon + \varepsilon H(\mu|\mu') + \varepsilon H(\nu|\nu')$$

and, if $\mathrm{EOT}'_\varepsilon$ is finite, that both problems have the same minimizer $\pi_\varepsilon \in \Pi(\mu,\nu)$.

Exercise 3. Explicitly solve the EOT_ε problem when the marginals (on $\mathbb{R}$) are $\mu = \mathcal{N}(m_0, \sigma^2)$ and $\mu = \mathcal{N}(m_1, \sigma^2)$ while $c(x,y) = |y-x|^2/2$.

Exercise 4. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported and let $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be L-Lipschitz. Detail the Arzelà–Ascoli argument (sketched in the lecture) to prove that the dual problem of EOT has a solution. Using that result, prove $\mathrm{DUAL}_\varepsilon \geq \mathrm{EOT}_\varepsilon$.

Complete the proof of strong duality by also showing the (easier) “weak duality” inequality $\mathrm{DUAL}_\varepsilon \leq \mathrm{EOT}_\varepsilon$.

Exercise 5. Let $c \in L^1(\mu \otimes \nu)$ be continuous and bounded from below. Let π_ε be the optimal coupling for EOT_ε. In the limit $\varepsilon \to 0$, show that there are cluster points (for weak convergence), and show that any cluster point π_0 is a solution of the unregularized optimal transport problem. If the latter has a unique solution π_0, conclude that $\pi_\varepsilon \to \pi_0$.

Exercise 6. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported and let $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be L-Lipschitz. Let $(\varphi_\varepsilon, \psi_\varepsilon)$ be EOT potentials for regularization parameter $\varepsilon > 0$, normalized such that $\int \varphi_\varepsilon \, d\mu = \int \psi_\varepsilon \, d\nu$. Show that in the limit $\varepsilon \to 0$, there are cluster points (for uniform convergence), and show that any cluster point (φ_0, ψ_0) is a dual solution of the unregularized optimal transport problem. If the latter has a unique solution (up to additive constant), conclude that $(\varphi_\varepsilon, \psi_\varepsilon)$ converges.