SLMath Summer School on OT: Day 6 - Problem Set 1

1. Let γ_ϵ denote the entropic optimal transport plan between μ and ν, with corresponding potentials f_ϵ, g_ϵ. Define $\varphi_\epsilon := \frac{1}{2}\|\cdot\|^2 - f_\epsilon$. Compute the derivatives of φ_ϵ and conclude that

$$\nabla\varphi_\epsilon(x) = \mathbb{E}_{\gamma_\epsilon}[Y \mid X = x]\,,$$
$$\nabla^2\varphi_\epsilon(x) = \epsilon^{-1}\mathrm{Cov}_{\gamma_\epsilon}(Y \mid X = x)\,.$$

2. Recall the definition of the rate function of the large deviation of EOT from the class (with $\Gamma := \mathrm{supp}(\pi^*)$-support of Monge-Kantorovich solution):

$$I(x,y) := \sup_{k\in\mathbb{N}_+} \sup_{(x_i,y_i)_{i=2}^k\subset\Gamma} \sup_{\sigma\in\mathrm{Perm}([1,k])} \Big[\sum_{i=1}^k \big(c(x_i,y_i) - c(x_i,y_{\sigma(i)})\big)\Big]$$

where $(x_1,y_1) := (x,y)$. Suppose that dual Kantorovich problem between μ and ν w.r.t. the cost function c has unique solution upto some additive constant. Then show the following

- $$I(x,y) = c(x,y) - \phi^*(x) - \psi^*(y)$$

  where ψ^* and ψ^* are solution to the Kantorovich dual problem.
  Hint: Use the variational description of $\phi^*(x)$ and $\psi^*(y)$, i.e.,

  $$\begin{aligned}\phi^*(x) := \sup_{y\in\mathcal{Y}} \sup_{m\in\mathbb{N}_+} \sup_{(x_i,y_i)_{i=1}^{m-1},(x_m,y)\subset\Gamma} &\Big[c(x,y) - c(x_m,y) + c(x_m,y_{m-1}) - c(x_{m-1},y_{m-1}) + \dots \\ &+ c(x_1,y_0) - c(x_0,y_0)\Big]\end{aligned}$$

- Furthermore, if $(x,y),(x',y')$ are such that $(x',y),(x,y')\in\Gamma$, then,

  $$I(x,y) + I(x',y') = c(x,y) + c(x',y') - c(x',y) - c(x,y').$$

3. Suppose $\pi\in\Pi(\mu,\nu)$, but π is not absolutely continuous w.r.t. $\mu\otimes\nu$. Then show that

$$\mu\otimes\mu\Big(\Big\{(x_1,x_1) : \exists U_1,U_2\in\mathcal{B}(\mathcal{Y}), \mathbb{P}_{x_i}(U_i) > 0, \mathbb{P}_{x_i}(U_{i+1}) = 0\Big\}\Big) > 0$$

where $\mathbb{P}_x$ denotes conditional probability under π and $U_3 = U_1$.

4. (**Identity behind the equivalence between primal and dual EOT**) Consider the probability spaces $(\mathcal{X},\mathcal{B}(\mathbb{R}^d),\mu)$ and $(\mathcal{Y},\mathcal{B}(\mathbb{R}^d),\nu)$. Suppose Ω be the space of all $\gamma:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}_+$ such that $\int\gamma(x,y)d\mu(x)d\nu(y) < \infty$. Then for any measurable $h:\mathcal{X}\times\mathcal{Y}$, show that

$$\begin{aligned}&\sup_{\gamma\in\Omega}\Big\{\int h(x,y)\gamma(x,y)d\mu(x)d\nu(y) - \int\gamma(x,y)(\log\gamma(x,y)-1)d\mu(x)d\nu(y) + 1\Big\} \\ &\quad = \int(e^{h(x,y)}-1)\gamma(x,y)d\mu(x)d\nu(y).\end{aligned}$$