SLMath Summer School on OT: Day 6- Problem Set 2

1. (**Absolute Continuity of Push-Forward Measures**) Let $\mathcal{X}$ and $\mathcal{Y}$ be two measurable space, $f : \mathcal{X} \mapsto \mathcal{Y}$ be a measurable function. Let $\mu \in \mathcal{P}(\mathcal{X})$ be a probability measure and $v \in L^p(\mu; \mathcal{X})$. Then $f\#(v\mu)$ is a signed measure measure on $\mathcal{X}$ satisfying the following properties:

   - $f\#(v\mu) \ll f_{\#}(\mu)$.
   - Let w be the Radon-Nikodym derivative of $f\#(v\mu)$ w.r.t. $f\#(\mu)$. Then,
   $$\|w\|_{L^p(f\#\mu;\mathcal{X})} \leq \|v\|_{L^p(\mu;\mathcal{X})}$$

2. Suppose $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$ where $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$. Define $f_t : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^d$ for any $t \in [0,1]$ as $f_t(x,y) = (1-t)x + ty$. Let π^K be the Monge-Kantorovich coupling between μ and ν w.r.t. the squared-distance cost function. Define $\mu_t := f_t\#\pi^K$ and $\nu_t := f_t\#((y-x)\pi^K)$. Let v_t be the Radon-Nikodym derivative of ν_t w.r.t. μ_t. Then show the following:

   - $$\int \|v_t(x)\|^2 d\mu_t(x) = \int \|y-x\|^2 d\pi^K(x,y).$$
   - Show that (μ_t, v_t) satisfies the continuity equation, i.e.,
   $$\frac{d\mu_t}{dt} + \operatorname{div}(\mu_t v_t) = 0.$$