Introduction to interval arithmetic

Interval arithmetic is arithmetic for inequalities.

Example Given a table with side lengths

$$l_1 = [2.9, 3.1] = 3 \pm 0.1$$

and

$$l_2 = [4.5, 5.5] = 5 \pm 0.5$$

what is the area?

$$= (3 \pm 0.1) \times (5 \pm 0.5) \quad \text{midrad notation}$$

$$A = l_1 \times l_2 = [2.9, 3.1] \times [4.5, 5.5] \quad \text{infsup notation}$$

$$\approx [13.05, 17.05] \quad \text{computer uses floats}$$

among other things, a convenient shorthand.

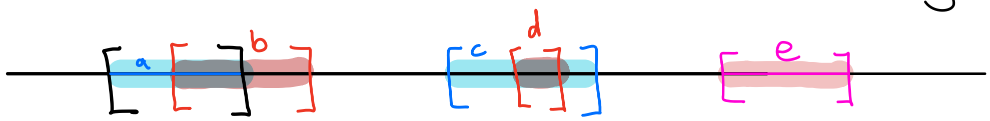
Let $\mathbb{IR} = \{ [\underline{a}, \bar{a}] : \underline{a} \leq \bar{a}, \underline{a}, \bar{a} \in \mathbb{R} \}$

Note that $[a, a]$ is an option (degenerate)

These intervals can be partially ordered

$a \leq b \iff \underline{a} \leq \underline{b}$ & $\bar{a} \leq \bar{b}$

$a < b \iff \bar{a} < \underline{b}$ Any element in a is less than any in b



$a \leq b$

a is not $<$ b

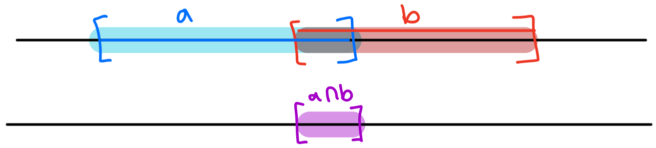
c is not $\leq$ d

$c < e$

$\Rightarrow$ "$<$" is transitive

We also can define unions and intersections

Intersection $a \cap b = [\max(\underline{a}, \underline{b}), \min(\bar{a}, \bar{b})]$

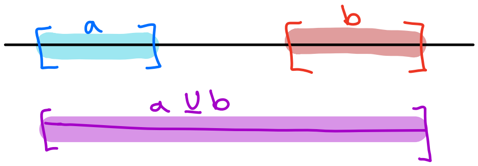


$a \cap b$ is always an interval (or $\emptyset$).

Goal of intersection If two people measure the same quantity and get $a = [10.1, 10.3]$ & $b = [10.2, 10.6]$ $\Rightarrow$ they can both be right because $a \cap b \neq \emptyset$.

Interval Hull $a \underline{\cup} b$ = hull of a & b — hull

$= [\min(\underline{a}, \underline{b}), \max(\bar{a}, \bar{b})]$



Because we want the union operation to preserve $\mathbb{IR}$.

Exercise $a = [-1, 2]$, $b = [3, 4]$, $c = [2, 3]$

Find $a \cap b$, $a \underline{\cup} b$, $a \cap c$.

Diameter $\text{diam}(a) = \bar{a} - \underline{a}$ — diam(a)

Radius $\text{rad}(a) = \dfrac{\text{diam}(a)}{2}$ — rad(a)

Absolute value $\{ |\alpha| : \alpha \in a \}$ — abs(a)

Eg. $\text{abs}([-10, 3]) = [0, 10]$

Magnitude $\text{mag}(a) = \max\{ |\underline{a}|, |\bar{a}| \} \in \mathbb{R}$ — mag(a)

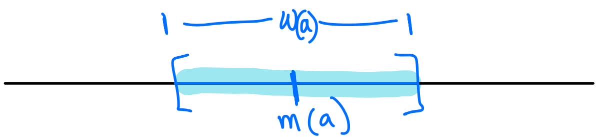
Bounds the absolute value of every point in a

Mignitude $\min(|\underline{a}|, |\bar{a}|)$ = smallest distance from 0. — mig(a)

Midpoint $m(a) = \dfrac{\underline{a} + \bar{a}}{2} \in \mathbb{R}$ — mid(a)

Positive $a > 0$

Negative $a < 0$



Exercise Find $m(a)$, $w(a)$, $|a|$ fr. above.

Arithmetic

$$a \odot b = \{ x \odot y : x \in a, y \in b \}$$

where $\odot$ is any operator $+, -, \times, \%$

Exercise $a = [5,6]$, $b = [-2,4]$

Find $a+b$, $a-b$, $a-a$ $(\neq 0)$

Exercise Let $f(x) = x^2$ & $g(x) = x \cdot x$

Show $f(a) \neq g(a)$ for $a = [-1,1]$.

This is due to interval dependency,

ie. $x \cdot x$ only means $x_1 \cdot x_2$, $x_1 \neq x_2 \in a$.

resulting in overestimation.

Careful how you set up your problem!

Exercise Let $a = [0,1]$.

$f(x) = x(1-x)$ $g(x) = x - x^2$ $h(x) = \frac{1}{4} - (x - \frac{1}{2})^2$

Show:

- For $x \in a$, $F(x) \in [0, \frac{1}{4}]$.
- For $x \in \mathbb{R}$, $f(x) = g(x) = h(x)$.
- Find $f(a)$, $g(a)$, $h(a)$. Use intlab!
- What can you conclude?

Extension of f If $f : \mathbb{R}^n \to \mathbb{R}$ then $F : \mathbb{R}^n \to \mathbb{R}$ is an extension if

$F([x,x]) = f(x)$.

Range Enclosure $R(f,a) = \{ f(x) : x \in a \}$

Inclusion isotonic extension means

$a_i \subseteq b_i \Rightarrow F(a_1, \ldots, a_n) \subseteq F(b_1, \ldots, b_n)$

all rational functions satisfy this.

Fundamental Theorem of Interval Analysis

If F is an inclusion isotonic extension of f then for all $a_i \in \mathbb{IR}$

$$\boxed{R(f, a_1, \dots, a_n) \subseteq F(a_1, \dots, a_n)}$$

That is, our interval function evaluation contain all the "real" values.
(but may contain more)

So far we have not discussed floating point errors. The important point is that all functions are calculated using interval extensions to guarantee that the answer interval contains the exact answer.

Intlab subtlety as a result of floating point

For non-representable numbers, one must use '·' to get an accurate rep.

Example intval(pi) − intval('pi') ≠ 0.

rad(intval('pi')) = 10^{-16}

which shows that it is really an enclosure.

Higher dimensions

We can also find 'interval vectors & matrices where each entry is an interval.

We can take norms, dot products, multiply, etc.

Integration via interval arithmetic

Our first application of interval arithmetic will be to compute definite integrals using Riemann sums.

Observe that for a partition of $[a,b]$

$\Delta x = \frac{b-a}{n}$, $x_i = a + i\,\Delta x$, $X_i = [x_i, x_{i+1}]$

$$\boxed{\sum_{i=0}^{n} \inf f(X_i)\,\Delta x \leq \int_a^b f(x)\,dx \leq \sum_{i=0}^{n} \sup f(X_i)\,\Delta x}$$

This is exactly in the interval form! Thus we can implement it without much extra trouble.