# Matrices, Norms, Projections, Gershgorin Circles

This lecture covers
- Interval matrices & eigensystems
- Matrix norms
- Gershgorin circle theorems
- Projections from function spaces to finite-dimensional spaces

## Interval matrices

$$A = (a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}} \qquad a_{ij} \in \mathbb{IR}$$

$A \in \mathbb{IR}^{m \times n}$

## Interval matrix arithmetic

$A+B$ has entries $(a_{ij}+b_{ij})$

If $A \in \mathbb{IR}^{m \times p}$ $B \in \mathbb{IR}^{p \times n}$

$$AB = \left\{ c_{ij} = \sum_{k=1}^{p} \alpha_{ik} \beta_{kj} \;:\; \alpha_{ik} \in a_{ik},\ \beta_{ik} \in b_{ik} \right\}.$$

- This definition results in overestimation & non-associativity!

**Example** **Overestimation** (of $AB$)

$A = ([1,2], [3,4])$   $[\text{infsup}(1,2), \text{infsup}(3,4)]$

$B = \begin{pmatrix} [5,6] & [7,8] \\ [9,10] & [11,12] \end{pmatrix}$

$AB = ([\overset{\boxed{32}}{31}, 52], [\overset{\boxed{40}}{39}, 64])$ according to INTLAB

However, there is no $\alpha \in A$, $\beta \in B$ such that $\alpha\beta = (32, 64)$

$\text{mig}(A)*\text{mig}(B) = \underline{A}\,\underline{B} = (1,3)\begin{pmatrix} 5 & 7 \\ 9 & 11 \end{pmatrix} = (32, 40)$

$\text{mag}(A)*\text{mag}(B) = \overline{A}\,\overline{B} = (2,4)\begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix} = (52, 64)$

How to get 31 & 39? Further overestimation because of the INTLAB method (chosen for efficiency)

midpoint-radius expansion   Calculate mid(A) mid(B) and add on radii.

**Example** **Non-associativity** (of $AB$)

$A = \begin{bmatrix} 1 & [0,1] \\ 1 & -1 \end{bmatrix} \Rightarrow A \times A = \begin{bmatrix} [1,2] & [-1,1] \\ 0 & [1,2] \end{bmatrix}$

$(A \times A) \times A = \begin{bmatrix} [0,3] & [-1,3] \\ [1,2] & [-2,-1] \end{bmatrix} \neq A \times (A \times A) = \begin{bmatrix} [1,2] & [-1,3] \\ [1,2] & [-3,0] \end{bmatrix}$

**Key fact**

**Theorem** $+, -, \times$ are inclusion monotonic for interval matrices. Thus for $\alpha \in A, \beta \in B$,

$$\alpha \circ \beta \subseteq A \circ B$$

- Can use mag, mig, abs, rad, diam

## Solving linear systems

Solve $Ax = b$ ie $\{x : \alpha \in A \,\&\, \beta \in B, \alpha x = \beta \ni x \in x\}$

A\b

verifylss (A,b)

Krawczyk & Hansen methods

- Cannot work if $0 \in \det A$.
- May not work in other cases due to overestimation issues.

## Operator norms

**p-norm** Let $1 \le p \le \infty$, $A \in \mathbb{R}^{m\times n}$ $A = (a_{ij})_{\substack{1\le i\le m \\ 1\le j\le n}}$

$$\|A\|_p = \sup_{\|x\|_p = 1} \|Ax\|_p$$ ← sup = max in nD

$p=1$ $$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|$$ max absolute column sum

$p=\infty$ $$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$ max absolute row sum

$p=2$ $$\|A\|_2 = \sqrt{\rho(A^T A)}$$ spectral norm

**Spectral radius** $\rho(B) = \max \{|\lambda| : \lambda \text{ eigenvalue of } B\}$

## Non-operator norm

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}}$$ Frobenius norm

**Theorem** $\|A\|_2 \le \|A\|_F$ (In fact more such statements hold)

## Interval matrix norms

$A \in \mathbb{IR}^{m\times n}$

$\|A\| =$ an interval containing $\{\|\alpha\| : \alpha \subseteq A\}$

# Eigensystems

$A \in \mathbb{R}^{n\times n}$ $\quad Av = \lambda v$

$\Leftrightarrow \quad \{\lambda, v\}$ $\quad$ eigenvalue eigenvector pair

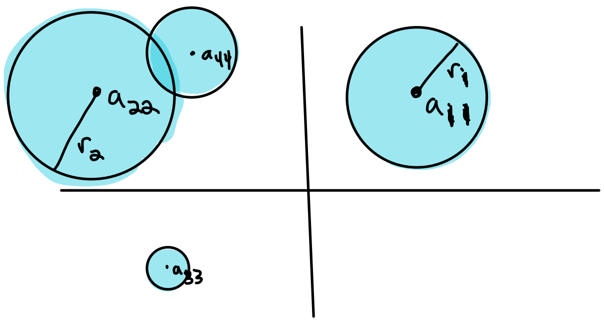
## Theorem (Gershgorin)

Let $A \in \mathbb{C}^{n\times n}$

Gershgorin disks

$$B_i(A) = B(a_{ii}, r_i) \qquad r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$$

$$B_G = \cup B_i$$




$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ a_{31} & \vdots & a_{33} & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

Then

- All eigenvalues lie in $B_G$.
- If $A$ is real & symmetric, each $B_i$ contains at least one eigenvalue.

# Finding eigenpairs in INTLAB

Finding approximate eigenpairs

$[V, D] = \text{eig}(\text{mid}(A))$ $\qquad A \in \mathbb{R}^{n \times n}$

## Verifying eigenpairs

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verifyeig (A, D(j,j), V(:,j))
```

The method for verifyeig is iterative, more sophisticated than Gershgorin, which would yield overly conservative estimates.

# Projections

Let $\ell_p = \{ x = (x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k|^p < \infty \}$

with norm $\|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$

$\ell_2$ is a Hilbert space.

## Projections

Define projection $P_N : \ell_p \to \ell_p^N$

by $P_N(x) = (x_1, \dots, x_N)$

$\ell_p^N$ is isometrically isomorphic to $\mathbb{R}^N$.

$Q_N = I - P_N$ is also a projection from $\ell_p$ into an infinite-dimensional subspace of $\ell_p$, which we call $\ell_p^\infty$

**Lemma** Given $x \in \ell_2$ and $N \in \mathbb{N}$, there is a unique $x_N \in \ell_p^N$, $x_\infty \in \ell_p^\infty$ such that

$$x = x_N + x_\infty$$

Furthermore $x_N = P_N x$, $x_\infty = (I - P_N) x$.

Keeping in mind the above lemma,

$$\|x\|_2 \le \|x_N\|_2 + \|x_\infty\|_2$$

Now consider a bounded linear operator

$$\boxed{T : l_2 \to l_2} \qquad \boxed{T \in \mathcal{L}(l_2, l_2) = \mathcal{L}(l_2)}$$

$y = Tx$ For emphasis $Q_N = P_\infty$

$$\begin{aligned}
y &= T(x_N + x_\infty) \\
&= Tx_N + Tx_\infty \\
&= P_N(Tx_N) + P_\infty(Tx_N) + P_N(Tx_\infty) + P_\infty(Tx_\infty) \\
&= \underbrace{T_N x_N}_{\in \mathbb{R}^N} \quad \underbrace{T_\infty x_N}_{l_2^\infty} \quad \underbrace{T_N x_\infty}_{\in \mathbb{R}^N} \quad \underbrace{T_\infty x_\infty}_{l_2^\infty}
\end{aligned}$$

$$\|y\| \le \|T_N x_N\| + \|T_\infty x_N\| + \|T_N x_\infty\| + \|T_\infty x_\infty\|$$

| $\|T_N x_N\|$ | $\|T_\infty x_N\|$ | $\|T_N x_\infty\|$ | $\|T_\infty x_\infty\|$ |
|---|---|---|---|
| N-dim'l norm | ∞-dim'l | N-dim'l norm | ∞-dim'l |
| computer | analysis | analysis | analysis |

Matrix form

$$T \sim A = \left(\begin{array}{ccc|cc}
a_{11} & a_{12} \cdots & a_{1N} & a_{1N+1} & \cdots \\
\vdots & & & & \\
a_{N1} & a_{N2} \cdots & a_{NN} & a_{NN+1} & \cdots \\
\hline
a_{N+1\,1} & \cdots & a_{N+1\,N} & a_{N+1\,N+1} & \cdots \\
\vdots & & & \vdots &
\end{array}\right)$$