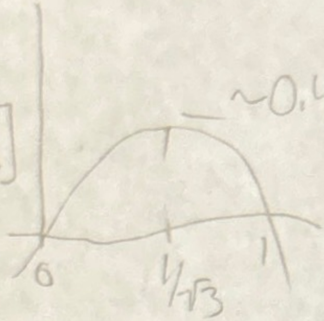
Branch & Bound Methods & Integration

Fix $f(x) = x - x^3$

$R(f;[0,1]) = [0, \frac{2}{3\sqrt{3}}]$



~~Calc 1: • Eval end points~~

~~• Find all $f'(x^*) = 0$~~

~~... NP Hard~~

Interval Ext $F: \mathbb{IR} \to \mathbb{IR}$

$F([0,1]) = [0,1] - [0,1]^3 = [-1, 1]$

$F([0,.1]) = [0,.1] - [0, 10^{-3}] = [-10^{-3}, .1]$

Spz $f: D \to \mathbb{R}$ is Lipschitz

$\exists K > 0$ s.t. $\forall x,y \in D$ then

$|f(x) - f(y)| \leq K|x-y|$

• $\forall X \subseteq D$, $X \subseteq \mathbb{IR}$

$\text{rad}(R(f,D)) \leq K \,\text{rad}(X)$

Th'm | Spz f is composed of Elementary Functions, Lip. on D, & $F($

e.g. $\frac{1}{x} + e^{\sin^2(\sqrt{1x1})}$ on $[2,3]$,

$\exists K$ s.t. if $X = \bigcup_{i=1}^{n} X_i \subseteq D$, & $F(X)$ is well def.

then $\text{rad}\left(\bigcup_{i=1}^{n} F(X_i)\right) \leq \text{rad}(R(f,X)) + K \max_{i=1,n} \text{rad}(X_i)$

Cover on $X = [0,1]$ by $\{X_i\}_{i=1}^{n} \subseteq \mathbb{IR}$

$\{X = \bigcup_{i=1}^{n} X_i$,

Then

$R(f,X) = \bigcup_{i=1}^{n} R(f, X_i)$ ← Range Enclosure

$\subseteq \bigcup_{i=1}^{n} F(X_i)$ ← Isotonic Inclusion

$\subseteq F(X)$

~~See computer for demo~~

~~• How Much is this Over Estimating~~

~~• Do We need to subdivide Everything?~~

~~Q1 Find Min of:~~

$\cancel{f(x) = \frac{\sin(e^x)}{1+x^2} \text{ on } [-100, 100]}$

Optimization Problem:

Fix $f: \mathbb{R}^d \to \mathbb{R}$ & I.E. $F: \mathbb{IR}^d \to \mathbb{IR}$ & compact domain X.

$$\boxed{\text{Prob. Find } \min_{x \in X} f(x)}$$

Moore-Skelboe Alg. | One Variant

Data: $\mathcal{L} = \{X_0, \ldots, X_n\}$ partition of X.

$\mathcal{R} = \emptyset$ — Result list

$\overline{m} \geq \min_{x \in X} f(x) \leq \overline{m}$

$\varepsilon > 0$ preset search tol

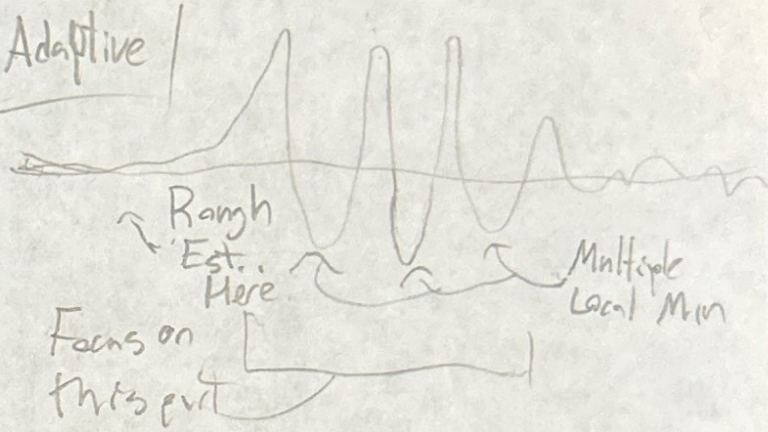
while $\mathcal{L} \neq \emptyset$
  Remove X_{loc} from $\mathcal{L}$
  $\overline{m} := \min(\overline{m}, \sup(f(\text{mid}(X_{loc}))))$
  if $f(X_{loc}) > \overline{m}$
    continue
  else if $w(X_{loc}) < \varepsilon$
    Add X_{loc} to $\mathcal{R}$
  else subdivide X_{loc} & add to $\mathcal{L}$
  end
end

Exhaustive Cover $X = \{X_i\}_{i=1}^n$

$$\min_{x \in X} f(x) = \min_{i=1..n} \left\{ \min_{x \in X_i} f(x) \right\} \in \min_{i=1,..n} \{F(X_i)\} = [\underline{m}, \overline{m}]$$

$$\min\{[a,b],[c,d]\} = [\min(a,c), \min(b,d)]$$

Adaptive |



Subdivide X_{loc} ~ Split in Half in each dim (2^d pieces)

- Other heuristics for $d \geq 4$

Thm · Algorithm finishes

- $\overline{m}$ monotonically decreases
- If $f(x^*) = \min_{x \in X} f(x)$ w/ $x^* \in X$, then $x^* \in X_i$ for some $X_i \in \mathcal{R}$.

Adding/Removing Boxes

$[0,1], [1,2], [2,3]$

$[0,\tfrac{1}{2}], [\tfrac{1}{2},1]$

Last In First Out (Depth First)

First In First Out (Breadth First)

Riemann Integration

Spz $\{X_i\}_{i=1}^N$ is a cover of X & $w(X_i \cap X_j) = 0$ for $i \neq j$

$$I = \int_a^b f(x)\,dx \quad , [a,b] = X$$

$$= \sum_{i=1}^N \int_{x \in X_i} f(x)\,dx$$

$$\in \sum_{i=1}^N F(X_i)\, w(X_i)$$

Convergence of $w(\mathbb{I})$: $O(1/N)$.

Trapezoidal Rule: $O(1/N^2)$

$$I \approx \int_a^b f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\,dx$$

Note F_c is Inclusion Isotonic iff $c = \text{mid}(X)$.

Taylor Approx

Fix $f \in C^1(\mathbb{R})$ & I.E. F, F' of f, f'.

Fix $X \in \mathbb{IR}$ & $c \in X$. Then $\forall x \in X$

$$f(x) = f(c) + \int_c^x f'(y)\,dy$$

Lemma Spz $g, h : X \to \mathbb{R}$ & $h \geq 0$, $g(x) \in G(X)$ $\forall x \in X$. Then

$$\int_{x \in X} g(x)h(x)\,dx \in G(X) \int_{x \in X} h(x)\,dx$$

Hence $f(x) \in f(c) + F'([X])(x-c)$

$$R(f, X) \subseteq \underbrace{f(c) + F'(X)(X-c)}_{\text{Centered Form } F_c : \mathbb{IR} \to \mathbb{IR}}$$

Integration

$X = [c-h, c+h]$, $\text{rad}(X) = h$

$$\int_c^{c+h} f(x)\,dx = \int_c^{c+h} f(c) + \int_c^x f'(y)\,dy\;dx$$

$$\in f(c)h + \int_c^{c+h} F'(X)\underbrace{(x-c)}_{\geq 0}\,dx$$

$$\in f(c)h + F'(X)\tfrac{1}{2}(x-c)^2\Big|_c^{c+h}$$

$$= f(c)h + F'(X)\frac{h^2}{2}$$

$$\int_c^{c+h} f(x)\,dx \in f(c)h - F'(X)\frac{h^2}{2}$$

$$\int_{c-h}^{c+h} f(x)\,dx \in 2f(c)h + \frac{h^2}{2}\underbrace{\left(F'(X) - F'(X)\right)}_{\neq 0!}$$

If f' is Lip., then $\exists K$ s.t.

$$\text{rad}(F'(X) - F'(X)) \leq K\,\text{rad}(X) = K \cdot h$$

Fix disjoint cover $\{X_i\}_{i=1}^N$ of $X = [a,b]$. & $c_i = \text{mid}(X_i)$. Then

$$I = \int_a^b f(x)\,dx$$

$$= \sum_{i=1}^N \int_{x \in X_i} f(x)\,dx$$

$$\in \sum_{i=1}^N \left[2\,\text{rad}(X_i) f(c_i) + \frac{\text{rad}(X_i)^2}{2}(F'(X) - F'(X))\right]$$

Convergence of $w(\mathbb{I})$: $O(1/N^2)$

Uniform Cover: $w(X_i) = w(X_j)\ \forall i,j$

Adaptive Cover: Subdivide X_i with largest $\mathrm{rad}\left(\frac{\mathrm{rad}(X_i)^2}{2}[F''(X_i) - F''(X_i)]\right)$

Higher Order Methods | Sppz $f \in C^{k+1}$, $c \in X \subset \mathbb{R}$

$$f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(c)}{i!}(x-c)^i + R_k(x)$$

$$R_k(x) = \int_c^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\, dt$$

$$\in F^{(k+1)}(X)\frac{(x-c)^{k+1}}{(k+1)!}$$

$$\subseteq F^{(k+1)}(X)\,(\mathrm{rad}\,c)^{k+1}$$

Integrale | Sppz $X = [c-h, c+h]$ & $k+1$ is even

$$\int_{c-h}^{c+h} f(x)\,dx$$

$$\in \int_{c-h}^{c+h} \sum_{i=0}^{k} \frac{f^{(i)}(c)}{i!}(x-c)^i + F^{(k+1)}(X)\underbrace{\frac{(x-c)^{k+1}}{(k+1)!}}_{\geq 0} dx$$

$$= \sum_{i=0}^{k} \frac{f^{(i)}(c)}{(i+1)!}\left(h^{i+1} - (-h)^{i+1}\right) + F^{(k+1)}(X)\int_{c-h}^{c+h} \underbrace{\frac{(x-c)^{k+1}}{(k+1)!}}\,dx$$

$$= 2\sum_{i=0}^{\lfloor k/2 \rfloor} \frac{f^{(2i)}(c)\cdot 2h^{2i+1}}{(2i+1)!} + 2F^{(k+1)}(X)\frac{h^{k+2}}{(k+2)!}$$