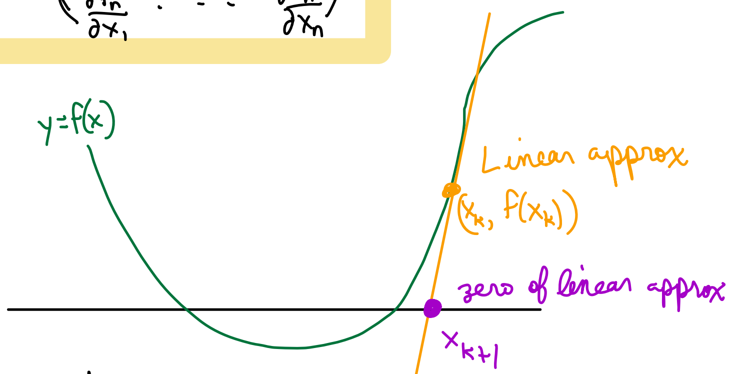
Interval Newton's Method

This lecture covers the interval (validated) version Newton's method: Interval Newton method (1D) & Krawczyk method (n D), both theoretical & practical considerations. The function space version will appear as the Newton-Kantorovich Theorem in a separate lecture.

Jacobian $f : \mathbb{R}^n \to \mathbb{R}^n \quad Df(x_k) \in \mathbb{R}^{n \times n}$

$$\boxed{Df(x_k) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{pmatrix}}$$

Traditional Newton's Method



an iterative method for x_{k+1} based on x_k starting from an initial point x_0.

Idea : Find the zero of the linearized map at x_k

1D: $x_{k+1} = x_k - f(x_k) / f'(x_k)$

nD: $x_{k+1} = x_k - (Df(x_k))^{-1} f(x_k)$ ← DO NOT DO THIS!

Numerically: Never invert a matrix!

Mathematically Equivalently, numerically better:

① Solve $DF(x_k) v_k = -f(x_k)$ for v_k

② Let $x_{k+1} = x_k + v_k$

Theorem (Traditional Newton's Method)

Let $f \in C^2$, $f(x_*) = 0$, $DF(x_*)$ nonsingular

Then there exists $\epsilon > 0$ such that

for all $x_0 \in B_\epsilon(x_*)$,

① $\lim_{k \to \infty} x_k = x_*$

② Let $e_k = \|x_* - x_k\|$ — Error at k^{th} step

Then $\lim_{k \to \infty} \frac{e_{k+1}}{e_k^2} = C < \infty$

This is called quadratic convergence

③ In 1D, $C = \left| \frac{f''(x_*)}{2f'(x_*)} \right|$

Problems : - What is ϵ ? ie Will it even converge?

- What is the error ?

Being that error depends on x_*.

Solution :

Interval Newton's Method in 1D

Assume

$a = [\underline{a}, \bar{a}] \in \mathbb{IR}$

$f : a \to \mathbb{R}$ be $C'(a)$

$f(x_*) = 0$ for some $x_* \in a$

F', an inclusion isotonic extension of f', exists.

$0 \notin F'(a)$ ie $f'(x) \neq 0$ for all $x \in a$.

Then by the mean value theorem for any $x \in a$, $\exists\ \xi \in a$ such that

$$f(x) = f(x_*) + f'(\xi)(x - x_*)$$

Therefore

$$\boxed{x_* = x - \frac{f(x)}{f'(\xi)} \in x - \frac{f(x)}{F'(a)}}$$

This formula is a key important fact that allows us to define the method as follows.

① Let $\mathbb{X}_0$ = a the original interval.

② Iteratively, `Xnew = newtonstep(f, fprime, X)`

$$\boxed{N(\mathbb{X}_k) = m(\mathbb{X}_k) - \frac{f(m(\mathbb{X}_k))}{F'(\mathbb{X}_k)}}$$

```
midX = infsup(mid(X), mid(X))
NX   = midX - f(midX)/fprime(X)
```

indicates type

may not be exact representable

$$\mathbb{X}_{k+1} = N(\mathbb{X}_k) \cap \mathbb{X}_k$$

```
Xnew = intersect(NX, X)
```

Clearly $\mathbb{X}_{k+1} \subset \mathbb{X}_k$

also by the MVT plus interval result on N,

$x_* \in \mathbb{X}_k$ for all k

Theorem (Interval Newton's Method)

Under the assumptions given on $a, f, F', \ldots$ $\{\mathbb{X}_k\}_{k=1}^{\infty}$ is a nested sequence of intervals converging to x_*.

Proof ① If $m(X_k) = x_*$ for some k, then

$$N(X_k) = m(X_k)$$

② assume $m(X_k) \neq x_*$

Claim $m(X_k) \notin N(X_k)$

Proof assume $\xi \in N(X_k)$ such that

$$\boxed{m(X_k) = m(X_k) - \frac{f(m(X_k))}{F'(\xi)}}$$

Then $f(m(X_k)) = 0 \iff m(X_k) = x_*$.

since $F' \neq 0$ only one zero!

Thus $m(X_k) \notin N(X_k)$ which implies

$$\boxed{\mathrm{width}(X_{k+1}) < \frac{1}{2}\,\mathrm{width}(X_k)}$$

Thus we have a nested nonempty sequence of compact sets $\Rightarrow$ convergence

▲

Theorem **Quadratic convergence of interval Newton's method**

Same theorem as the traditional case.

Example Find $\sqrt{5}$ to 8 digits.

Exercise (a) Find $\sqrt{2}$ to 8 digits starting with $X_0 = [1,2]$.

(b) For traditional Newton, any guess will do. What happens if you start with interval $X_0 = [4,6]$ in the above?

(c) What about $X_0 = [-2,2]$?

How does this violate hypotheses?

What goes wrong in practice?

How to fix this: Extended interval arithmetic

Eg. $$\frac{1}{[-9,9]} = \left(-\infty, -\tfrac{1}{3}\right] \cup \left[\tfrac{1}{3}, \infty\right)$$

Automatic Differentiation

Rather than finding the derivative by hand, Intlab contains the rules of differentiation and can create an inclusion isotonic extension of a derivative itself using the $\boxed{\text{gradientinit}}$ command

```
XI = gradientinit(X)
FX = f(XI)
```

FX.x is $F(X)$
FX.dx is $F'(X)$

Exercise (a) Rewrite your newtonstep code

```
Xnew = newtonstep2(f, X)
```

to find the derivative automatically.

(b) Use this to compute zeros of

$$f(x) = x^3 - 5x - 1$$

Note Multivariate Newton's method is analogous.

pronounced "CROF-CHICK"

Krawczyk's method for root finding in 1D

$\forall x \in \underline{X}$, MVT $\Rightarrow \exists \xi \in \underline{X}$ s.t.

$$f(x) = f(x_*) + f'(\xi)(x - x_*)$$

$$f(x) = 0 + f'(\xi)(x - x_*)$$

$$M f(x) = M f'(\xi)(x - x_*)$$

$$x_* - x + M f(x) = x_* - x + M f'(\xi)(x - x_*)$$

$$x_* = x - M f(x) - (1 - M f'(\xi))(x - x_*)$$

$$\Rightarrow \quad x_* \in x - M f(x) - (1 - M F'(\underline{X}))(x - \underline{X}) := K(\underline{X})$$

Thus we no longer divide by $F'(\underline{X})$!

Krawczyk method starting with $\underline{X}_0$. Iteratively

1. Let $M \approx (F'(m(\underline{X}_k)))^{-1}$
2. Define $m = m(\underline{X}_k)$
3. $K(\underline{X}_k) = m - M f(m) + (1 - M F'(\underline{X}_k))(\underline{X}_k - m)$
4. $\underline{X}_{k+1} = \underline{X}_k \cap K(\underline{X}_k)$

Theorem Krawczyk's method converges

(a) If X_k contains a zero, so does X_{k+1}

(b) If $X_0 \cap K(X_0) = \emptyset$ then f has no zeros in X_0.

(c) If $K(X_k) \subset X_k$ then X_k contains exactly one zero.

Convergence is quadratic if $F'(X)$ is Lipschitz

Krawczyk's method for root finding in nD

Search for solutions to

$f(x) = 0$ where $f : \mathbb{R}^n \to \mathbb{R}^n$

Let F & DF be the extensions of f, Df on interval X.

Newton's method is possible but Krawczyk's method is more practical in higher dimensions.

Theorem Krawczyk's method converges (nD)

Let $f: \mathbb{R}^n \to \mathbb{R}^n$ $f \in C^1(\Omega \subset \mathbb{R}^n)$, F, DF extensions, & $\mathbf{X} \subset \mathbb{IR}^n$.

assume $M \approx (DF(m(\mathbf{X})))^{-1}$ nonsingular & $x \in \mathbf{X}$.

Define

$$\boxed{K(\mathbf{X}) = x - M f(x) + [I - M\, DF(\mathbf{X})](\mathbf{X} - x)}$$

Then a, b, c hold as in 1D case.

For example $K(\mathbf{X}) \subset \mathbf{X} \Rightarrow$ unique zero.

Implementation of Krawczyk's method

① Let $m = m(\mathbf{X}_k)$

② Choose $M_k \cong (DF(m))^{-1}$ Preconditioning matrix

③ $K(\mathbf{X}_k) = m - M_k f(m) + (I - M_k DF(\mathbf{X}_k))(\mathbf{X} - m)$

④ $\mathbf{X}_{k+1} = \mathbf{X}_k \cap K(\mathbf{X}_k)$

Implemented in INTLAB as the command

verifynlss (F, interval) (with "ε-inflation")

Example Find (x_1, x_2) such that

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 - 1 \\ x_1 - x_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

On interval $[0.5, 0.8] \times [0.6, 0.9]$.