Def| A Banach space $(V, \|\cdot\|)$ is a complete normed vector space

Def| Fix Banach spaces V, W & $F: V \to W$.

Def| F is Fréchet differentiable @ $x_0 \in V$ if $\exists\, A \in B(V,W)$ s.t. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall \|h\| < \delta$

$$\|F(x_0+h) - F(x_0) - Ah\|_W \le \varepsilon \|h\|_V$$

Write $DF(x_0) = A$.

Thm| Mean Value Inequality

Spz $U \overset{\circ}{\subseteq} V$ & $f \in C^1(U,W)$. Fix $x_0 \in U$, $\delta > 0$ s.t. $\overline{B_\delta(x_0)} \subseteq U$.

$$\|f(x) - f(y)\|_W \le \left(\sup_{z \in \overline{B_\delta(x_0)}} \|Df(z)\|_{B(V,W)} \right) \cdot \|x - y\|_V.$$

Cor| $\mathrm{Lip}_B(f) \le \sup_{z \in U} \|Df(z)\|_{B(V,W)}$

Say $F \in C^1(V,W)$ if $DF: V \to B(V,W)$ is continuous & bdd

Th'm| Spz $F \in C^1(V,V)$ & $A \in B(V,V)$ & $G(x) = AF(x)$. Then

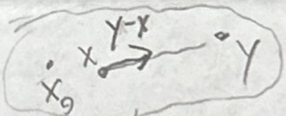
$$DG(x_0) = A\, DF(x_0)$$

E.g.| Define $F: C^0([a,b], \mathbb{R}) \circlearrowleft$ by

$$[F(u)](t) = \int_a^t u^2(x)\, dx$$

Claim: $\forall u_0, h \in C^0([a,b], \mathbb{R})$

$$[DF(u_0) \cdot h](t) = \int_a^t 2u_0(x)\, h(x)\, dx$$

Proof| 

$$\|f(y) - f(x)\| = \left\| \int_0^1 Df((1-t)x + ty) \cdot (y-x)\, dt \right\|_W$$

$$\le \int_0^1 \|Df((1-t)x + ty)\|_{B(V,W)} \|y - x\|_V\, dt$$

$$\le \left(\sup_{z \in \overline{B_\delta(x_0)}} \|Df(z)\|_{B(V,W)} \right) \int_0^1 \|y - x\|_V\, dt$$

∎

Fixed Pt Thm's

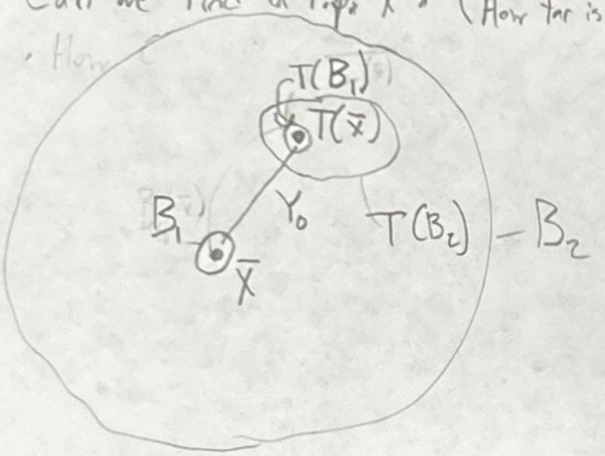
Contraction: B, complete metric space

$T: B \to B$, $\text{Lip}(T) = k$.

If $k<1$, $\exists!\ \tilde{x} \in B$ s.t. $T(x) = x$

Spz $\|T(\bar{x}) - \bar{x}\| \ll 1$. ~~Can We find a FP?~~

Can we find a F.P. $\tilde{x}$? (How close is it? How far is it unique?)



"Radii Polynomial Thm" / Parametrized Newton-Kantorovich Thm

Spz X, B space, $T \in C^1(X,X)$, $\bar{x} \in X$ &

$$\|T(\bar{x}) - \bar{x}\|_X \le Y_0$$

$$\sup_{x \in B_r(\bar{x})} \|DT(x)\|_{B(X)} \le Z(r) \quad \forall r>0$$

Def $p(r) = Z(r)r - r + Y_0$. ← Radii Polynomial

If $\exists\, r_0 > 0$ s.t. $p(r_0) < 0$, then

then $\exists!\ \tilde{x} \in \overline{B_{r_0}(\bar{x})}$ s.t. $T(\tilde{x}) = \tilde{x}$.

Proof | WTS $T: \overline{B_{r_0}(\bar{x})} \circlearrowleft$ is a contraction

- Self Map
- $\text{Lip}(T) < 1$

By MVT, $\text{Lip}_{B_r(\bar{x})}(T) \le \sup_{x \in B_r(\bar{x})} \|DT(x)\|$

$\le Z(r)$.

Note $Z(r_0)r_0 - r_0 + Y_0 < 0$, $r_0 > 0$

$(Z(r_0) - 1)r_0 < 0$

$Z(r_0) < 1$ ∴ T is contraction

Self Map | Show $\|y - \bar{x}\| < r \Rightarrow \|T(y) - \bar{x}\| < r$

$$\|T(y) - \bar{x}\| \le \|T(y) - T(\bar{x})\| + \|T(\bar{x}) - \bar{x}\|$$
$$\le Z(r)\|y - \bar{x}\| + Y_0$$
$$\le Z(r_0) r_0 + Y_0 < r_0 \quad \blacksquare$$

Alt Version | Fix $F \in C^1(X,Y)$, $\bar{x} \in X$

$A^\dagger \in B(X,Y)$, $A \in B(Y,X)$, 1-1

Def Quasi Newton Map: $T(x) = x - AF(x)$

Let

$$Y_0 \ge \|AF(\bar{x})\|_X$$
$$Z_0 \ge \|I - AA^\dagger\|_{B(X)}$$
$$Z_1 \ge \|A[DF(\bar{x}) - A^\dagger]\|_{B(X)}$$

& $\forall c \in \overline{B_r(\bar{x})}$, $r > 0$

$$Z_2(r) \ge \|A[DF(c) - DF(\bar{x})]\|_{B(X)}$$

Def Radii poly

$$p(r) = (Z_0 + Z_1 + Z_2(r))r - r + Y_0$$

...

Eg. Solve $F\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1^2+x_2^2-1\\x_1-x_2^2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

Approx Sol: $\bar{x} = \begin{pmatrix}\bar{x}_1\\\bar{x}_2\end{pmatrix} = \begin{pmatrix}0.6\\0.8\end{pmatrix}$

$F(\bar{x}) = \begin{pmatrix}0.0\\-0.04\end{pmatrix}$

Quasi-Newton: $T(x) = x - AF(x)$

$A^{\dagger} = DF(\bar{x}) = \begin{pmatrix}2\bar{x}_1 & 2\bar{x}_2\\1 & -2\bar{x}_2\end{pmatrix} = \begin{pmatrix}1.2 & 1.6\\1 & -1.6\end{pmatrix}$

Approx Inverse: A

$A := \begin{pmatrix}0.5 & 0.5\\0.3 & -0.3\end{pmatrix}$

Fix Norm $\|\cdot\|_1$.

Radii-Poly: Calc Y_0, Z_0, Z_2.

$Y_0 \geq \|AF(\bar{x})\|_1 = \left\|\begin{pmatrix}-.02\\.012\end{pmatrix}\right\|_1$

$= \left\|\begin{pmatrix}-.02\\.012\end{pmatrix}\right\| = 0.032$

$\boxed{Y_0 = 0.04}$

$\|I - A\cdot DF(\bar{x})\|_1 = \left\|\begin{pmatrix}-.1 & 0\\-.06 & .04\end{pmatrix}\right\|_1$

$= \max\{.16, .04\}$

$= 0.16$

$\boxed{Z_0 = 0.2}$

$\sup_{x\in\overline{B_r(\bar{x})}} \|A[DF(x) - A^{\dagger}]\|_1$

$\leq \left\|\begin{pmatrix}.5 & .5\\.3 & -.3\end{pmatrix}\right\|_1 \sup_{h\in\overline{B_r(\bar{x})}} \|$

$\leq \left\|\begin{pmatrix}.5 & .5\\.3 & -.3\end{pmatrix}\right\|_1 \sup_{x\in\overline{B_r(\bar{x})}} \left\|\begin{pmatrix}2x_1 & 2x_2\\1 & -2x_2\end{pmatrix} - \begin{pmatrix}1.2 & 1.6\\1 & -1.6\end{pmatrix}\right\|_1$

$\doteq 0.8 \sup_{h\in\overline{B_r(0)}} \left\|\begin{pmatrix}2h_1 & 2h_2\\0 & -2h_2\end{pmatrix}\right\|_1$

$= 0.8 \sup_{\|h\|_1 = r} \max_h \{|2h_1|, |4h_2|\}$

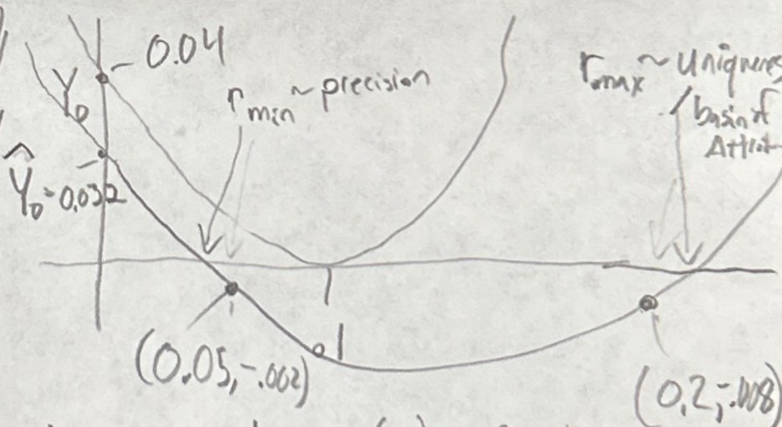
$= 0.8 \cdot 4r$

$\boxed{Z_2(r) = 4r}$

$\rho(r) = Z_2(r)r + (1-Z_0)r + Y_0$

$= 4r^2 - (1-.2)r + 0.04$

$= 4(r - 1/10)^2$



No r_0 s.t. $\rho(r_0) < 0$!

Improve Estimates

$\hat{Y}_0 = 0.032$, $\hat{Z}_0 = 0.16$, $\hat{Z}_2(r) = 3.2r$

- $\forall r \in [0.05, 0.2]$, $T: \overline{B_r(\bar{x})} \to \overline{B_r(\bar{x})}$ is a contraction; $\exists!\ \tilde{x} \in \overline{B_r(\bar{x})}$ s.t. $F(\tilde{x}) = \vec{0}$

Compare $\tilde{x}_{\pm} = \left(\frac{\sqrt{5}-1}{2}, \pm\sqrt{\frac{\sqrt{5}-1}{2}}\right)$

$\|\tilde{x}_+ - \bar{x}\|_1 \approx 0.032$, $\|\tilde{x}_- - \bar{x}\|_1 \approx 1.6$