Ordinary differential equations using infinite series

This lecture covers methods of solving ordinary differential equations using series. It will not cover computer assisted proofs, as that will be discussed separately.

Theory of ordinary differential equations

Initial Value Problem

$$\frac{dy}{dt} = G(t,y)$$
$$y(t_0) = y_0$$

Peano's Theorem (Existence) Let $F: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function.

Then for the initial value problem

$$\dot{y} = F(t,y) \qquad \text{IVP}$$
$$y(t_0) = y_0$$

There exists $\epsilon > 0$ such that there exists a solution to IVP on $(t_0-\epsilon, t_0+\epsilon)$.

Definition: Lipschitz functions

Given $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$

F is Lipschitz if there exists $K > 0$ so

$\|F(x) - F(y)\| \leq K \|x-y\|$ for all $x, y \in D$

Picard - Lindelöf Theorem

Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function.

Then for the initial value problem

$$\begin{cases} \dot{y} = F(t,y) \\ y(t_0) = y_0 \end{cases} \quad \text{IVP}$$

There exists $\epsilon > 0$ such that there exists a solution u to IVP on $(t_0 - \epsilon, t_0 + \epsilon)$.

Smoothness

If $F \in C^k$, $u \in C^{k+1}$

If $F \in C^\infty$, so is u

If $F \in C^\omega$ real analytic so is u.

Extends to PDEs as the Cauchy-Kovalevskaya Theorem

Given an ordinary differential equation with no prior knowledge of the solution, it is often a good strategy to find solutions in terms of a power series expansion

$$u(t) = \sum_{k=0}^{\infty} a_k (t-t_0)^k$$

This is often called the method of Frobenius

This analytical tool is also very use for computer assisted proofs.

Consider a power series centered at $t_0 = 0$.

$$u(t) = \sum_{j=0}^{\infty} a_j t^j$$

If u has a radius of convergence ρ then for all $t \in (t_0-\rho, t_0+\rho)$, this series converges absolutely.

If $u(t)$ is a C^∞ function on $I := (t_0 - \rho, t_0 + \rho)$ then we can write a formal power series

$$u(t) \sim \sum_{j=0}^{\infty} a_j t^j \qquad a_j = \frac{u^{(j)}(t_0)}{j!}$$

$u(t)$ is real analytic at t_0 if and only if this series converges and is equal to u with $\rho > 0$.

Theorem (Uniqueness)

$$\sum_{j=0}^{\infty} a_j t^j = 0 \iff a_j = 0 \text{ for all } j$$

Example Linear first order

Solve $\dot{u} = Ku \quad u(0) = u_0$

using the method of Frobenius, $t_0 = 0$.

Let $$u(t) = \sum_{j=0}^{\infty} a_j t^j$$

$$\dot{u}(t) = \sum_{j=1}^{\infty} j a_j t^{j-1}$$

Equation becomes

$$\sum_{\ell=1}^{\infty} \ell a_\ell t^{\ell-1} = K \sum_{j=0}^{\infty} a_j t^j$$

$j = \ell - 1$ $$\sum_{j=0}^{\infty} (j+1) a_{j+1} t^j = K \sum_{j=0}^{\infty} a_j t^j$$

$$\Leftrightarrow \quad \sum_{j=0}^{\infty} \left[(j+1) a_{j+1} - K a_j \right] t^j = 0$$

$$\Leftrightarrow \quad a_0 = u_0 \ \& \ a_{j+1} = \frac{K}{j+1} a_j \quad \text{for all } j > 0$$

Thus $$a_j = u_0 \frac{K^j}{j!}$$

$$u(t) = \sum_{j=0}^{\infty} u_0 \frac{(tK)^j}{j!} \overset{\text{recognise}}{=} u_0 e^{Kt}$$

Example Legendre

$$\begin{cases} \dfrac{d}{dt}\left((1-t^2)\dfrac{du}{dt}\right) + \beta u = 0 & -1<t<1 \\ u(1) = 1 \end{cases}$$

Find a bounded solution with bounded derivative

Let $u(t) = \sum_{k=0}^{\infty} a_k (t-1)^k$

$$\frac{d}{dt}\left((1-t^2)\frac{du}{dt}\right) = \frac{d}{dt}\left((1-t^2)\sum_{k=1}^{\infty} k a_k (t-1)^{k-1}\right)$$

$$= \sum_{k=1}^{\infty} k a_k \frac{d}{dt}\left((1-t^2)(t-1)^{k-1}\right)$$

$$= \sum_{k=1}^{\infty} k a_k \frac{d}{dt}\left(-(t-1)^k (t-1+2)\right)$$

$$= \sum_{k=1}^{\infty} k a_k \frac{d}{dt}\left(-(t-1)^{k+1} - 2(t-1)^k\right)$$

$$= \sum_{k=1}^{\infty} -k a_k \left[(k+1)(t-1)^k + 2k(t-1)^{k-1}\right]$$

Thus the differential equation becomes

$$0 = a_0\beta - \sum_{k=1}^{\infty}\left[(k^2+k-\beta)a_k(t-1)^k + 2k^2(t-1)^{k-1}\right]$$

change sum

$$0 = a_0\beta - 2a_1 - \sum_{k=1}^{\infty}\left[(k^2+k-\beta)a_k + 2(k+1)^2 a_{k+1}\right](t-1)^k$$

$$a_0 = 1, \quad a_0\beta - 2a_1 = 0 \Rightarrow a_1 = \frac{\beta}{2}$$

$$a_{k+1} = -\frac{k^2+k-\beta}{2(k+1)^2}\, a_k \qquad k > 1$$

If $\beta = n^2+n$ for some integer n, then the series terminates after a finite number of terms, and in fact we have a polynomial solution. For $n = k$, the polynomial is degree n

$$L_0(t) = 1$$

$$L_1(t) = 1 + (t-1) = t$$

$$L_2(t) = 1 + 3(t-1) + \frac{3}{2}(t-1)^2 = \frac{3t^2-1}{2}$$

These are the Legendre polynomials.
What about when $\beta \neq n^2+n$?
It turns out that in this case the only possible solution is

$$u(t) \equiv 0.$$

Why ?

- Sturm-Liouville Theory:

~~Distinct β values correspond to a linearly independent u.~~

Distinct β values correspond to <u>orthogonal</u> u ie $\beta \neq \hat{\beta}$

$$\boxed{\int_{-1}^{1} u_\beta \, u_{\hat{\beta}} \, dx = 0}$$

- Stone - Weierstraß Theorem:

a family of polynomials of all degrees is complete in the space of $L^2(-1,1)$
Therefore no other β can exist.

Tail Bounds

Generally we don't have the "recognize" step or the "terminate in polynomial" step. We instead can find a validated truncated series and a bound on a remainder term, the "tail"

$$u(t) = \underbrace{\boxed{\sum_{j=0}^{N} a_j (t-t_0)^j}}_{\text{FINITE SERIES}} + \underbrace{\boxed{\sum_{j=N+1}^{\infty} a_j (t-t_0)^j}}_{\text{TAIL TO BOUND}}$$

FINITE SERIES: S_N TAIL TO BOUND: S_∞

Norm: $\|u\|_\infty \le \|S_N\|_\infty + \|S_\infty\|_\infty$ BOUND THIS and/or SHOW CONVERGENCE

VARIOUS OPTIONS

- **Taylor's Theorem** as long as $u \in C^{N+1}$,

$$u(t) = \sum_{j=0}^{N} a_j (t-t_0)^j + \boxed{\frac{F^{(N+1)}(\xi)}{(N+1)!}(t-t_0)^{N+1}}$$

where ξ is between t & t_0.

- **Ratio test** $\left|\frac{a_{k+1}(t-t_0)}{a_k}\right| \to L < 1$ then the series converges.

- **Series bound** $\|S_\infty\|_\infty \le \sum_{k=N+1}^{\infty} |a_k| T^k \quad T = \max|t-t_0|$

- We will need other methods when a_k is unknown

Example Bessel

$$\frac{d}{dt}\left(t\,\frac{du}{dt}\right) - \frac{\ell^2}{t}\,u + \lambda t u = 0 \quad \text{on } (0,1)$$

$$u(1) = 0$$

Equivalently: $t^2 \ddot{u} + t\dot{u} + (\lambda t^2 - \ell^2)\,u = 0$

assume $u(t) = t^n \sum_{k=0}^{\infty} a_k t^k$, $a_0 \neq 0$.

Finding a series expression for LHS

$$t^n \sum c_k t^k = 0$$

gives $c_0 = (n^2 - \ell^2)\,a_0 = 0 \Rightarrow n = \ell$

$c_1 = (2\ell+1)\,a_0 = 0 \Rightarrow a_1 = 0$

$k \geq 0 \qquad c_k = (k^2 + 2k\ell)\,a_k + \lambda a_{k-2} = 0$

Since $a_1 = 0$, k is odd, $a_k = 0$.

If k is even $a_k = -\dfrac{\lambda}{k(2\ell+k)}\,a_{k-2}$

$$u(t) = \sum_{j=0}^{\infty} a_{2j}\, t^{2j+\ell}$$

Ratio test $\Rightarrow \left|\dfrac{a_k t}{a_{k-1}}\right| \leq \dfrac{|\lambda|\;1}{k(2\ell+k)} \to 0$

$\Rightarrow$ the series converges on $[0,1]$.

Working with power series

So far we avoided difficult power series manipulations.

Cauchy Products

$$\left(\sum_{j=0}^{\infty} \overset{f(t)}{a_j\, t^j}\right)\left(\sum_{j=0}^{\infty} \overset{g(t)}{b_j\, t^j}\right) = \sum_{j=0}^{\infty} \overset{h(t)}{c_j\, t^j}$$

where $c_n = (a * b)_n = \boxed{\sum_{k=0}^{n} a_{n-k}\, b_k}$

$(a*b)_0 = a_0 b_0$

$(a*b)_1 = a_1 b_0 + a_0 b_1$

$(a*b)_2 = a_2 b_0 + a_1 b_1 + a_0 b_2$

$(a*b)_3 = a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3$

etc.

<u>Note</u> $P_N(h(t))$ depends only on $P_N(f)$ & $P_N(g)$.

This is <u>not</u> true in general for $F(f,g)$.

We can iteratively find $\left(\sum a_j t^j\right)\left(\sum b_j t^j\right)\left(\sum c_j t^j\right)$ as a series.

Example $\dot{u} = u^2 \quad , \quad u(0) = u_0$

Let $u(t) = \sum_{j=0}^{\infty} a_j t^j \quad , \quad a_0 = u_0$

$$\dot{u}(t) = \sum_{\ell=1}^{\infty} \ell a_\ell t^{\ell-1} = \sum_{j=0}^{\infty} (j+1)\, a_{j+1}\, t^j$$

$$(u(t))^2 = \sum_{n=0}^{\infty} (a*a)_n\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n-k}\, a_k\, t^n$$

ie. $\sum_{j=0}^{\infty} (j+1)\, a_{j+1}\, t^j = \sum_{j=0}^{\infty} \sum_{k=0}^{j} a_{j-k} a_k\, t^j$

$$\Rightarrow \quad \boxed{(j+1)\, a_{j+1} - \sum_{k=0}^{j} a_{j-k}\, a_k = 0} \quad j = 0, 1, 2,$$

$1 \cdot a_1 - a_0^2 = 0 \qquad a_1 = u_0^2$

$2 \cdot a_2 - 2 a_0 a_1 = 0 \qquad a_2 = u_0^3$

$3 a_3 - (2 a_0 a_2 + a_1 a_1) = 0 \qquad a_3 = u_0^4$

$4 a_4 - 2(a_0 a_3 + a_1 a_2) = 0 \qquad a_4 = u_0^5$

Inductively assume $a_k = u_0^{k+1} \quad 0 \le k \le j-1$

$$j\, a_j = \sum_{k=0}^{j-1} u_0^{j-k+1}\, u_0^{k+1} = j\, u_0^j \Rightarrow a_j = u_0^{j+1} \quad \forall j$$

Note that in this case we could also solve the original ODE using the fact that it is separable. Thus we see that $\dot{u} = u^2$ can be written

$$\frac{du}{u^2} = dt$$

$$\Rightarrow \quad \frac{-1}{u} = t + c \quad \& \quad \frac{-1}{u_0} = t_0 + c \quad \Rightarrow \quad \boxed{-c = \frac{1}{u_0}}$$

Thus $$u(t) = \frac{1}{\frac{1}{u_0} - t} = \boxed{\frac{u_0}{1 - u_0 t}}$$ for $|u_0 t| < 1$

We can write this as a series using the geometric series

$$u(t) = u_0 \sum_{k=0}^{\infty} (u_0 t)^k$$

$$= \boxed{\sum_{k=0}^{\infty} u_0^{k+1} \, t^k}$$

As long as $|u_0 t| < 1$ this converges. consistent with the blowup point.