Functional Framework of Taylor methods

General Approach

Color

$$\begin{tikzcd}
\text{Function Space} \arrow[r, shift left, "\varphi\text{; Taylor transform}"] \arrow[r, phantom, "\text{Equivalent}"] & \text{Sequence Space} \arrow[l, shift left, "\varphi^{-1}"] \arrow[d, shift right, "\text{project}"'] \\
& \text{Finite Dim.} \arrow[u, shift right, "\text{include}"']
\end{tikzcd}$$

Want: Validated Approx

Consider $\dot{x} = f(x)$, f analytic

Let $r>0$, $z_0\in\mathbb{C}$,

$$\mathbb{D}_r(z_0) := \{ z\in\mathbb{C} \mid |z-z_0|<r \}$$

Def| Inverse Taylor transform: For z_0,

$$\varphi^{-1}\left(\{a_n\}_{n=0}^{\infty}\right) = f, \qquad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Domain/Range? Need Sequence Space

Def| For $\mathbb{I} = \{\mathbb{N}, \mathbb{Z}\}$ (index set)

$$S_{\mathbb{I}}(\mathbb{C}) = \{ a = \{a_n\} \mid a_n\in\mathbb{C},\ n\in\mathbb{I} \}$$

$$\mathbb{I} = \begin{cases} \mathbb{N} & \text{1-sided seq, (Taylor)} \\ \mathbb{Z} & \text{2-sided seq, (Fourier)} \end{cases}$$

Norms| For a sequence of weights

$$\{\omega_n\}_{n\in\mathbb{I}} \subseteq \mathbb{R}_{>0}$$

Def| $C^{\omega}(\mathbb{D}_r(z_0), \mathbb{C}) \equiv C^{\omega}(z_0, r)$

$$:= \{ f : \mathbb{D}_r(z_0) \to \mathbb{C} \mid f \text{ is bdd \& analytic} \}$$

w/ norm $\|f\|_\infty$.

This is a Banach Space.

Def| Taylor transform: for $f\in C^{\omega}(\mathbb{D}_r(z_0), \mathbb{C})$

$$\varphi(f) = \{a_n\}_{n=0}^{\infty}, \qquad a_n = \frac{1}{n!}\frac{d^n}{dz^n} f(z_0)$$

Thm| (Cauchy Est). If $\rho<r$, then

$$|a_n|\rho^n \le \sup_{z\in\mathbb{D}_\rho(z_0)} |f(z)|$$

define norms

$$\|a\|_{p,\omega,\mathbb{I}} := \left(\sum_{n\in\mathbb{I}} |a_n|^p \omega_n \right)^{1/p}$$

$$\|a\|_{\infty,\omega,\mathbb{I}} := \sup_{n\in\mathbb{I}} |a_n| \omega_n$$

Th'm| For $1\le p\le\infty$, the spaces

$$\ell^p_{\omega,\mathbb{I}} := \{ a\in S_{\mathbb{I}}(\mathbb{C}) \mid \|a\|_{p,\omega,\mathbb{I}} < \infty \}$$

are Banach spaces.

Notation| For $\nu>0$, def geometric weights $\omega = \{\omega_n = \nu^{|n|}\}_{n\in\mathbb{I}}$

$$\ell^p_{\nu,\mathbb{I}} \equiv \ell^p_{\omega,\mathbb{I}}$$

We'll mostly use $\varphi = \mathcal{I}, \mathbb{N}$, | $\ell^1_v \equiv \ell^1_{v,\mathbb{N}}$

$\mathrm{Dom}(\varphi)$ & $\mathrm{Rng}(\varphi)$?

Th'm | (Cauchy Est) Let $f \in C^\omega(\mathbb{D}_r(z_0), \mathbb{C})$

If $\rho < r$, then

$$|a_n|\rho^n \le \sup_{z \in \mathbb{D}_\rho(z_0)} |f(z)|$$

Th'm | • If $0 < v < r$, then w/ $\sum |a_n| < \infty$

$$\varphi_{z_0,r,v} : C^\omega(\mathbb{D}_r(z_0), \mathbb{C}) \to \ell^1_{v,\mathbb{N}}$$

is a 1-1, bdd, linear op.

• If $0 < r \le v$, then

$$\varphi^{-1}_{z_0,r,v} : \ell^1_v \to C^\omega(\mathbb{D}_r(z_0), \mathbb{C})$$

is 1-1, bdd, Lin. op w/ $\|\varphi^{-1}_{z_0,r,v}\| = 1$

Fix $r = v > v' = r' > 0$

$$\begin{tikzcd}
C^\omega(\mathbb{D}_r(z_0),\mathbb{C}) \arrow[r, hook] \arrow[dr, "\varphi_{z_0,r,v'}"] & C^\omega(\mathbb{D}_{r'}(z_0),\mathbb{C}) \\
\ell^1_v \arrow[u, "\varphi^{-1}_{z_0,r,v}"] \arrow[r, hook] & \ell^1_{v'} \arrow[u, "\varphi^{-1}_{z_0,r',v'}"']
\end{tikzcd}$$

This commutes!

$\varphi \circ \varphi^{-1} = id$ is a densely defined operator

Goal: Translate $\dot{x} = f(x)$ to eq on seq.

Derivatives | If $r' < r$ then

$$\frac{d}{dz} : C^\omega(\mathbb{D}_r(z_0), \mathbb{C}) \to C^\omega(\mathbb{D}_r(z_0), \mathbb{C})$$

is bdd.

$$\frac{d}{dz} f(z) = \frac{d}{dz} \sum_{k=0}^{\infty} k a_k (z - z_0)^{k-1}$$

$$= \sum_{k=0}^{\infty} (k+1) a_{k+1} (z - z_0)^k$$

Def | Pseudo-differential Operator $\mathcal{D} : \ell^1_v \to \ell^1_{v'}$

$$(\mathcal{D} a)_k \overset{\text{def}}{=} (k+1) a_{k+1}$$

Claim | $\mathcal{D} a = \varphi\left(\frac{d}{dz} \varphi^{-1}(a)\right)$

where defined.

Nonlinearity Spz $f:\mathbb{C}\to\mathbb{C}$ is a ~~poly~~ M-degree poly

$$f(z)=\sum_{n=0}^{M} f_n z^n \quad , \; f_n\in\mathbb{C}$$

Induces Map on functions...

$$f: C^\omega(\mathbb{D}_r(z_0),\mathbb{C}) \circlearrowleft$$

$$g(z)\mapsto f(g(z))$$

... and on seq:

$$f: \ell^1_r \to \ell^1_r \quad , \quad a\mapsto f(a)$$

Recall Cauchy Product: For $a,b\in\ell^1_r$,

$$(a*b)_n=\sum_{k=0}^{n} a_{n-k}\, b_k$$

$$f(a)\overset{\text{df}}{=}\sum_{n=0}^{M} f_n \underbrace{a*\cdots *a}_{n\text{-times}}$$

Note; for $f,g\in C^\omega(\mathbb{D}_r(z_0),\mathbb{C})$

$$\|f\cdot g\|_\infty \le \|f\|_\infty \|g\|_\infty$$

Merten's Th'm For $a,b\in\ell^1$, then $c=(a*b)\in\ell^1$

$$A\cdot B=\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)=\sum_{n=0}^{\infty} c_n = C \quad c_n=(a*b)_n$$

Proof Partial Sums $A_N=\sum_{n=0}^{N} a_n$, $B_N=\cdots$, $C_N=\cdots$

Note $C_N=\sum_{k=0}^{N} a_{N-k} B_k$

$$\& \; C_N = A_N B + \underbrace{\sum_{k=0}^{N} a_{N-k}(B_k-B)}$$

take $N\gg 1$ so either a_{N-k} or (B_k-B) is small

Def A commutative Banach Algebra $(X,*)$ is a B-space X w/ mult $*: X\times X\to X$ s.t.,

- $x*(y*z)=(x*y)*z$
- $x*y=y*x$
- $(x+y)*z=x*z+y*z$
- $\alpha(x*y)=\alpha x*\alpha y$
- $\|x*y\|_X\le\|x\|_X\|y\|_X$

~~Th'm~~ $(C^\omega(\mathbb{D}_r(z_0),\mathbb{C}),\cdot)$ &

Th'm $(\ell^1_r,\cdot)$ are B-algebras

Th'm Let $r'<r\le v$. Spz $x\in C^\omega(\mathbb{D}_r(z_0),\mathbb{C})$ satisfies,

$$\frac{d}{dz}x=f(x) \qquad (1)$$

Then $a=\varphi_{z_0,r,r'}(x)\in\ell^1_{r'}$ satisfies

$$\mathcal{D}a=f(a) \qquad (2)$$

If $a\in\ell^1_r$ satisfies (2) then $x=\varphi^{-1}_{z_0,r,r}(a)$ satisfies (1).

Cor Result generalizes for

$$f:\mathbb{C}^d\to\mathbb{C}$$

$$x\in\left[C^\omega(\mathbb{D}_r(z_0),\mathbb{C})\right]^N=C^\omega(\mathbb{D}_r(z_0),\mathbb{C}^N)$$

$$a=(a^1,\dots,a^N)\in\left[\ell^1_r\right]^N=\underbrace{\ell^1_r\times\cdots\times\ell^1_r}$$

Eg.1 $\dot{x} = x(x-1) \qquad x(t_0) = x_0$ (I)

for $x \in C^\omega(\mathbb{D}_r(t_0), \mathbb{C})$ & some r

Transform $a = \varphi_{t_0,r,\nu}(x) \in \ell^1_\nu$:

(II) $$\mathcal{D} a = a * (a - 1)$$

$1 = (1,0,0,\ldots)$

&

$$a_0 = x_0$$

$$(\mathcal{D} a)_n = (a * (a-1))_n$$

$$(n+1)\, a_{n+1} = (a*a)_n - a_n$$

Def $F : \ell^1_\nu \to \ell^1_\nu$

$$F(a)_n = \begin{cases} a_0 - x_0 & \text{if } n = 0 \\ (\mathcal{D}a - a*(a-1))_{n-1} & \text{if } n \geq 1 \end{cases}$$

Spz $F(\tilde{a}) = 0$, $\tilde{a}^N = (\tilde{a}_0, \ldots, \tilde{a}_N, 0, \ldots)$
$\tilde{a} = \tilde{a}^N + \tilde{a}^\infty$

Spz $(\tilde{a})_n \in A_n \in \mathbb{IR}$ for $n = 0, \ldots, N$

& $\sum_{n=N+1}^{\infty} |\tilde{a}_n| \nu^n < \varepsilon^\infty$

Then $\sum_{n=0}^{N} \tilde{a}_n t^n \in \sum_{n=0}^{N} A_n t^n \quad \forall t \in (-\nu, \nu)$

Def $\bar{a}_n = \text{mid}(A_n)$.

$$\|\varphi^{-1}(\tilde{a}) - \varphi^{-1}(\bar{a})\|_\infty = \sup_{t \in (-\nu,\nu)} \left| \sum_{n=0}^{N} (\tilde{a} - \bar{a})_n t^n + \sum_{n=N+1}^{\infty} \tilde{a}_n t^n \right|$$

$$\leq \sum_{n=0}^{N} |\tilde{a}_n - \bar{a}_n| \nu^n + \sum_{n=N+1}^{\infty} |\tilde{a}_n| \nu^n$$

$$\leq \left(\sum_{n=0}^{N} \text{rad}(A_n) \nu^n \right) + \varepsilon^\infty$$

Thm) If $a \in \ell^1_\nu$ satisfies $F(a) = 0$ then $x : (-\nu, \nu) \to \mathbb{R}$, given as $\varphi^{-1}_{t_0,\nu,\nu}(a)$ satisfies (I) $\forall t \in (-\nu, \nu)$.

Numerics | Define a_n recursively

$$a_0 = x_0$$

$$a_{n+1} = \frac{1}{n+1} \left[-a_n + (a*a)_n \right]$$

Use interval arithmetic to enclose soln

$a_n \in A_n \in \mathbb{IR}$, for $n = 0, \ldots, N$

Note! For $\nu' < \nu$

$$\sum_{n=N+1}^{\infty} |\tilde{a}_n| (\nu')^n = \sum_{n=N+1}^{\infty} |\tilde{a}_n| \nu^n \left(\frac{\nu'}{\nu}\right)^n$$

$$\leq \underbrace{\varepsilon^\infty \left(\frac{\nu'}{\nu}\right)^{N+1}}_{\to 0 \text{ quickly}}$$