Solving Periodic Orbits w/ Fourier Series

General Approach

$$\begin{tikzcd}
\text{Function Space} \arrow[r, shift left, "\mathcal{F}:\ \text{Fourier transf.}"] & \text{Sequence Space} \arrow[l, shift left, "\mathcal{F}^{-1}"] \arrow[d, shift right, "\text{proj}"'] \\
 & \text{Finite Dim} \arrow[u, shift right, "\text{include}"']
\end{tikzcd}$$

Spz $\dot{x} = f(x)$ $\quad f: \mathbb{R}^n \to \mathbb{R}^n$

has a τ-periodic solution

i.e. $x(t+\tau) = x(t)$

Then $\dot{x} = \frac{\tau}{2\pi} f(x)$ has a

2π-periodic solution; wlog $\tau = 2\pi$.

Def) For $g \in L^2([0,2\pi], \mathbb{C})$ def Fourier coeff

$$\hat{g}(k) := \frac{1}{2\pi} \int_0^{2\pi} g(t)\, e^{-ikt}\, dt, \quad k \in \mathbb{Z}$$

Def) Fourier transform

$$\mathcal{F}(g) = \hat{g} = \{\hat{g}(k)\}_{k \in \mathbb{Z}} \in S_{\mathbb{Z}}(\mathbb{C})$$

Note $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

Def) Inverse Fourier transform for $a \in \ell^1_{\mathbb{Z}}$

$$\mathcal{F}^{-1}(a) = \sum_{k \in \mathbb{Z}} a_k e^{ikx}$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{ikx} + a_{-k} e^{-ikx}\right)$$

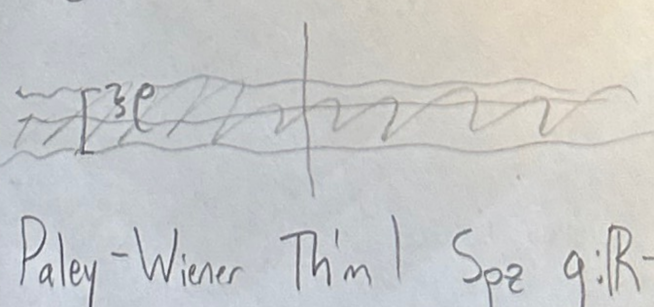
$$= a_0 + \sum_{k=1}^{\infty} (a_k + a_{-k}) \cos kx + i(a_k - a_{-k}) \sin kx$$

Trig $\equiv$ Fourier

Function Spaces: $\|a\|_{\ell^1_{\nu,\mathbb{Z}}} = \sum_{k \in \mathbb{Z}} |a_k| \nu^{|k|}$, $\nu \geq 1$

Def) For $\rho > 0$ def complex strip

$$A_\rho = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) < \rho \}$$



Paley-Wiener Thm) Spz $g: \mathbb{R} \to \mathbb{R}$ is

- real analytic, 2π-periodic, $g = \sum a_k e^{ikt}$
- $\exists\ \rho > 0$ s.t. $g: A_\rho \to \mathbb{C}$ is analytic
- $\exists\ \nu > 1$ s.t. $a \in \ell^1_{\nu,\mathbb{Z}}$

$C^\omega(A_\ell) \approx \{g: A_\ell \to \mathbb{C} \mid g \text{ is bdd, } 2\pi \text{ periodic, analytic}\}$

Let $\ell' < \ell$, $r' < r$, & $r = e^\ell$

$$\begin{tikzcd}
C^\omega(A_\ell, \mathbb{C}) \arrow[r, hook] \arrow[dr, "\mathcal{F}"] & C^\omega(A_{\ell'}, \mathbb{C}) \\
\ell^1_{r,\mathbb{Z}} \arrow[r, hook] \arrow[ur, "\mathcal{F}^{-1}"] & \ell^1_{r',\mathbb{Z}} \arrow[u]
\end{tikzcd}$$

$\mathcal{F} \circ \mathcal{F}^{-1} = id|_{\ell^1_r}$ & $\mathcal{F}^{-1} \circ \mathcal{F} = id|_{C^\omega}$

where defined

Why Fourier, & not Trig?

- equiv, but Fourier simplifies Notation.

Derivatives)

Let $a \in \ell^1_{r,\mathbb{Z}}$, $r > 1$, & $g = \mathcal{F}^{-1}(a)$.

Def) $(\mathcal{K} a)_k = ik a_k$

Thm) $\mathcal{F}\left(\frac{d}{dt} g\right) = \mathcal{K} a = \mathcal{K} \mathcal{F}(g)$

$$\frac{d}{dt} \sum_{k \in \mathbb{Z}} a_k e^{ikt} = \sum_{k \in \mathbb{Z}} ik a_k e^{ikt}$$

Non Linearity)

Def) Let $a, b \in \ell^1_r$, $r \geq 1$. Def Convolution product $* : \ell^1_r \times \ell^1_r \to \ell^1_r$

$$a * b = \mathcal{F}\left(\mathcal{F}^{-1}(a) \cdot \mathcal{F}^{-1}(b)\right),$$

Fix e_m, e_n w/

$$(e_n)_k = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{else} \end{cases}$$

$$\alpha e_m * \beta e_n = \mathcal{F}\left(\alpha e^{imt} \cdot \beta e^{int}\right)$$
$$= \alpha\beta \mathcal{F}\left(e^{i(m+n)t}\right)$$
$$= \alpha\beta e_{m+n}.$$

Note $e_{n-k} * e_k = e_n \quad \forall k \in \mathbb{Z}$

Hence

$$(a * b)_n = \sum_{k \in \mathbb{Z}} a_{n-k} b_k$$

Thm! $(\ell^1_{\nu,\mathbb{Z}}, *)$ is a Banach Algebra $\nu \geq 1$

$$\|a*b\|_{\ell^1_\nu} \leq \|a\|_{\ell^1_\nu} \|b\|_{\ell^1_\nu}$$

Spz $f(x) = \sum_{n=0}^{\infty} f_n x^n$ $f_n \in \mathbb{R}$ is analytic

then $f: \ell^1_\nu \to \ell^1_\nu$ w/

$$f(a) = \sum_{n=0}^{\infty} f_n \underbrace{a * \dots * a}_{n \text{ terms}}$$

is continuous & analytic.

Note!
We can compute $a*b$ of finite sequences Explicitly & w/ fft.

Define (densely defined) function $F: \ell^1_\nu \to \ell^1_\nu$ as

$$F(a) = (K^2 + \varepsilon K + \alpha) a + \beta a*a*a + b$$

$$(F(a))_k = (-k^2 + i\varepsilon k + \alpha) a_k + b_k + \beta \sum_{k_1+k_2+k_3=k} a_{k_1} a_{k_2} a_{k_3}$$

If $F(a) = 0$ for $a \in \ell^1_\nu$, then $x = \mathcal{F}^{-1}(a)$ solves (I) for $t \in \mathbb{R}$

Converting Diff Eq / Duffing

(I) $x'' + \varepsilon x' + \alpha x + \beta x^3 + \gamma \cos t = 0$.

- $\mathcal{F}(\gamma \cos t) \equiv b \in \ell^1_\nu$

$$b_n = \begin{cases} \gamma/2 & \text{if } n = \pm 1 \\ 0 & \text{else} \end{cases}$$

- $\mathcal{F}(\alpha x) = \alpha a \in \ell^1_\nu$
- $\mathcal{F}(x'' + \varepsilon x') = K^2 a + \varepsilon K a$
- $\mathcal{F}(\beta x^3) = \beta a*a*a$

Note! $F(a)_k$ depends on all coeff of a!

Numerics! Use Galerkin truncation

$$\pi_N : \ell^1_{\nu,\mathbb{Z}} \to (\ell^1_{\nu,\mathbb{Z}})^N \cong \mathbb{R}^{2N+1}$$

$$a \mapsto (0\,0 \dots a_{-N} \dots a_N\, 0\,0 \dots)$$

Def! $F_N = \pi_N \circ F \circ \pi_N$

Numerically Solve $F_N(a^N) = 0 \in \mathbb{R}^{2N+1}$

Validate w/ Quasi Newton

$$T(a) = a - AF(a), \quad A \approx DF(a)^{-1}$$

What is $DF(a)$?

$$F(a) = (K^2 + \varepsilon K + \alpha)a + \beta a * a * a + b$$

Thm: For comm. B-alg $(X, *)$, if

$$S(u) = \underbrace{u * \dots * u}_{n} \text{ then}$$

$$DS(u)\cdot h = n \underbrace{u * \dots * u}_{n-1} * h$$

$$DF(a)\cdot h = (K^2 + \varepsilon K + \alpha)a + 3\beta a * a * h$$

How to Represent op $h \mapsto c * h$, $c, h \in \ell^1$? $c = 3\beta a * a$

$$\left[\sigma^{-n}(c)\right]_k = c_{k-n}$$

Then

$$c * h = \sum_{n \in \mathbb{Z}} \sigma^{-n}(c)\, h_n$$

We Obtain Matrix Rep for $c * h$

$$\begin{array}{c c c}
 & \text{ind } 0 & \\
 & \begin{pmatrix} \ddots & & & & & \\ & c_{-1} & c_{-2} & & & \\ & c_0 & c_{-1} & c_{-2} & & \\ \cdots & c_1 & c_0 & c_{-1} & \cdots & \\ & & c_1 & c_0 & c_{-1} & \\ & & c_2 & c_1 & c_0 & \\ & & & c_2 & c_1 & \ddots \end{pmatrix} & \begin{pmatrix} \vdots \\ h_{-2} \\ h_{-1} \\ h_0 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix}
\end{array}$$

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Fix basis element $\hat{e}_k$, e.g.

$$c = \sum_{k \in \mathbb{Z}} c_k \hat{e}_k.$$

Then

$$c * h = \sum_{k \in \mathbb{Z}} (c * h)_k \hat{e}_k$$

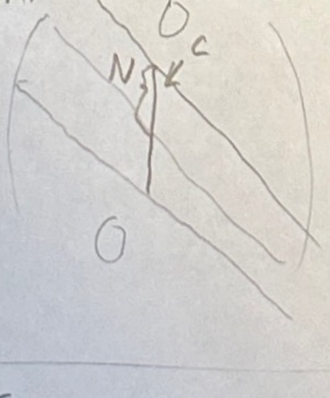
$$= \sum_{k \in \mathbb{Z}} \left(\sum_{n \in \mathbb{Z}} c_{k-n} h_n \right) \hat{e}_k$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\left(\sum_{k \in \mathbb{Z}} c_{k-n} \hat{e}_k \right)}_{} h_n$$

$$= \sum_{n \in \mathbb{Z}} \underbrace{\left(\sum_{k' \in \mathbb{Z}} c_{k'} \hat{e}_{k'+n} \right)}_{\overset{\text{def}}{=} \sigma^{-n}(c)} h_n$$

This is a Toeplitz Matrix

In ∞-dim, if $c = \pi_N c$ then mat for $c *$ is



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