Continuation with respect to a variable

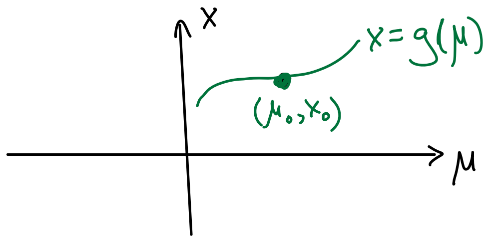
We have discussed methods for solving $f(x) = 0$, $f: \mathbb{R}^n \to \mathbb{R}^n$.

Now we consider one-parameter families $f(\mu, x)$, $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.

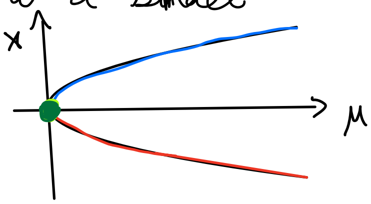
We wish to solve the parameter dependent equation $\{(\mu,x): f(\mu,x) = 0\}$.

Our goal is to discuss

continuation Given (μ_0, x_0) such that $f(\mu_0,x_0) \approx 0$ Finding nearby points (μ,x) with $f(\mu,x)$.



bifurcation a qualitative change in dynamics occuring due to a small change in parameter



Application: $\dot{x} = f(\mu, x)$, $x(0) = x_0$

If $f(\mu_0, x_0) = 0$ then $x(t) \equiv x_0$ is an equilibrium solution for $\mu = \mu_0$.

Definition **Hyperbolic matrix**

a matrix $A \in \mathbb{R}^{n \times n}$ is hyperbolic if it has no eigenvalues with real part equal to zero.

Hyperbolic $\Rightarrow$ non singular

Definition **Topological equivalence**

① $\dot{x} = f(x)$ $f: E_1 \to \mathbb{R}^n$ & ② $\dot{y} = g(y)$ $g: E_2 \to \mathbb{R}^n$

are called **topologically equivalent** if there is a function $H: E_1 \to E_2$ that preserves trajectories and their orientations.

That is, phase portraits of ① & ② are equivalent in terms of topological properties.

Definition **Bifurcation point for ODE**

Consider $\dot{x} = f(\mu, x)$

and equilibrium at (μ_0, x_0)

ie. $f(\mu_0, x_0) = 0.$

(μ_0, x_0) is a bifurcation point

there exist $\mu_k \to \mu_0$ such that if

$\dot{x} = f(\mu_k, x)$ have phase portraits

which are not topologically equivalent

to $\dot{x} = f(\mu_0, x)$ near x_0.

Theorem Sufficient conditions for continuation

OR

Necessary conditions for a bifurcation

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be C^2 and assume

$\dot{x} = f(\mu, x)$ has an equilibrium at (μ_0, x_0).

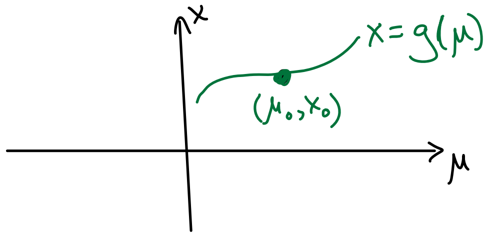
ie $f(\mu_0, x_0) = 0$

assume $D_x f(\mu_0, x_0)$ is hyperbolic

Then (μ_0, x_0) is not a bifurcation point.

Proof Since $D_x f(\mu_0, x_0)$ is nonsingular, the implicit function theorem implies that there exists $\delta > 0$ such that there is a function $g: [\mu_0 - \delta, \mu_0 + \delta] \to B_\delta(x_0)$ such that for all $\mu \in [\mu_0 - \delta, \mu_0 + \delta]$,

1. $f(\mu, g(\mu)) = 0$
2. this is the unique point in $B_\delta(x_0)$ with this property.



If we only care about existence of a unique path, we can stop here.

Top Conj: Since $D_x f(\mu_0, x_0)$ is hyperbolic, we can choose $\delta > 0$ so that

$D_x f(\mu, g(\mu))$ is hyperbolic for all $\mu \in B_\delta(\mu_0)$

Note that hyperbolic linear systems with the E^u of the same dimension are topologically conjugate. Thus by the Hartman-Grobman Theorem, there is no bifurcation in $B_\delta(\mu_0, x_0)$. ▲

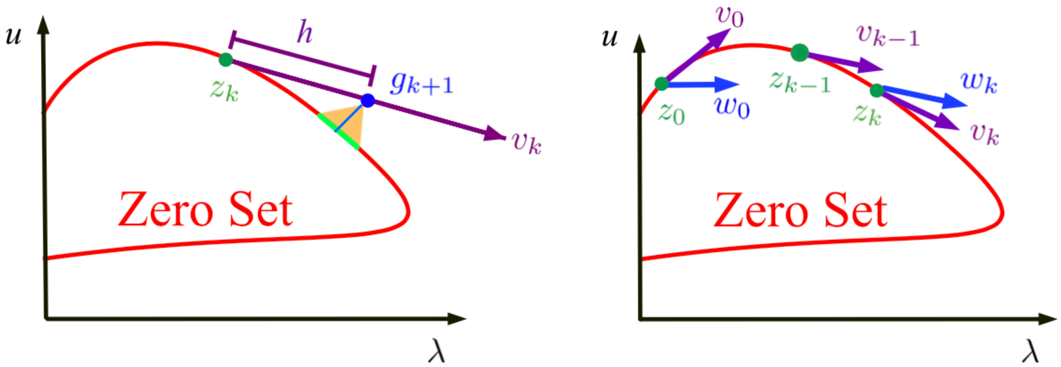
assume ① $f(\mu_0, x_0) = 0$

② $D_x f(\mu_0, x_0)$ is non singular

How can we find a local part of $\{ z = (\mu, x) : f(\mu, x) = 0 \}$ given by $x = g(\mu)$

Predictor - corrector method

Numerical pseudo arclength continuation - iterative process



Predictor : Based on the previously known point on the zero set, find a prediction for the next point.

Corrector : Restrict to a line and use root finding to correct the original guess.

Pseudo arclength continuation

Prediction

Observe that $Z = \{z : F(z) = 0\}$ is orthogonal to $DF(z)$

$\Rightarrow$ If $DF(z)\,v = 0$ then v is tangent to Z

Predictor g_{k+1}: Let z_k be on the zero curve

& Let h be small.

Define v_k to be a normalized tangent vector:

$$\boxed{\begin{pmatrix} DF(z_k) \\ v_{k-1}^t \end{pmatrix} \tilde{v}_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}} \quad \& \quad \boxed{v_k = \frac{\tilde{v}_k}{\|\tilde{v}_k\|}}$$

Predict using the tangent line

$$\boxed{g_{k+1} = z_k + h\, v_k}$$

Corrector We make the assumption that z_{k+1} lies along the hypersurface orthogonal to the tangent line, solving the following extended system

$$G(z) = \begin{pmatrix} F(z) \\ v_k \cdot (z - g_{k+1}) \end{pmatrix} = 0 \qquad G : \mathbb{R}^{(n+1)} \circlearrowleft$$

For Newton's method we also require DG

$$DG(z) = \begin{pmatrix} DF(z) \\ v_k^t \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

Code & simple examples as follows.

So far, this is the non-validated version. The validated version involves finding constructive conditions on the region for which a unique solution exists.

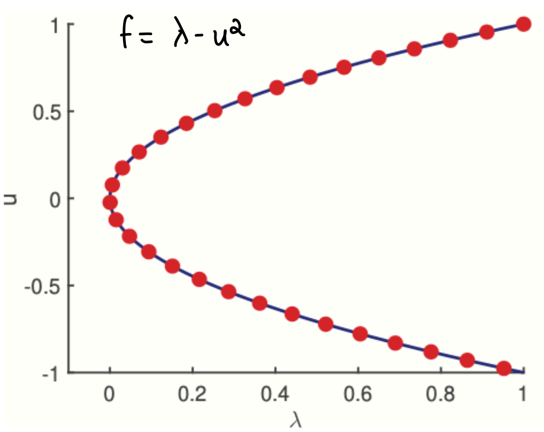
Matlab Code 5.5. *continuation.m*

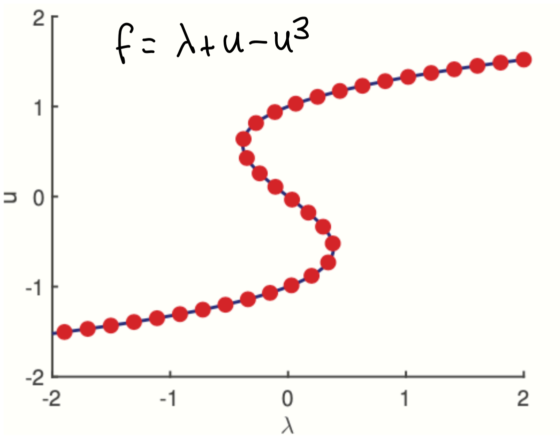
```
% continuation(@filename,lambda,g,h)
% lambda is the first value of lambda
% g is the initial guess
% h is the stepsize
% The sign of h denotes the direction of the initial search

function z = continuation(fval,lambda,g,h)

tol = 0.0001;
MaxSteps = 150;

N = length(g)+1;
rvec = [zeros(N-1,1); 1];

% Find the first point by fixing lambda and using root finding
u = newton(fval,lambda,g,tol);
z(:,1) = [lambda;u];
v = sign(h)*[1;zeros(N-1,1)];

% Now the continuation loop using the extended system
for i = 2:MaxSteps
    [y,yp,ylambda] = fval(z(:,i-1));
    v = [ylambda yp; v'] \ rvec;
    v = v/norm(v);
    g = z(:,i-1) + abs(h)*v;  % Continue in the direction of v, so use |h|
    z(:,i) = newtonextended(fval,g,tol,g,v);
end

function u  = newton(fval,lambda,guess,tolerance)
    u = guess;
    err = 1;
    maxcount = 200;
    count = 1;
    while (err > tolerance & count < maxcount)
        [y,yp] = fval([lambda;u]);
        res = yp \ y;
        err = max(abs(res));
        u = u - res;
        count = count + 1;
    end

function z  = newtonextended(fval,zguess,tolerance,g,v)
    z = zguess;
    err = 1;
    maxcount = 200;
    count = 1;
    while (err > tolerance & count < maxcount)
        [y,yp,ylambda] = fval(z);
        yext = [y; dot(z-g,v)];
        ypext = [ylambda, yp; v'];
        res = ypext \ yext;
        err = max(abs(res));
        z = z - res;
        count = count + 1;
    end
```

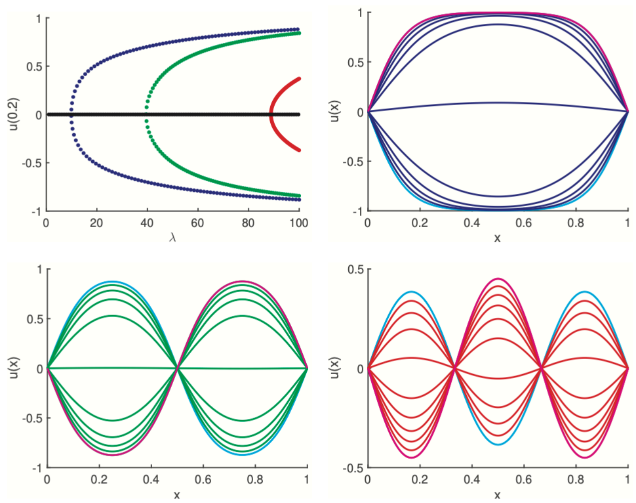
```
function [y,yp,ylambda] = quadratic(z)
lambda = z(1);
u = z(2);
y = lambda - u^2;
yp = -2*u;    % derivative with respect to u
ylambda = 1; % derivative with respect to lambda
```

This implementation provides the function value $f(\lambda,u) = \lambda - u^2$, as well as the two partial derivatives $f_\lambda(\lambda,u) = 1$ with respect to λ and $f_u(\lambda,u) = -2u$ with respect to u. Once this file is accessible, the command

```
bif = continuation(@quadratic,1,1,-0.1);
```







$F(\lambda,u) = \Delta u + \lambda(u - u^3)$

using finite differences

Validated continuation

The proof of continuation relies on the Implicit Function Theorem.

We now state a constructive version of this theorem that allows for validated continuation.

Theorem Constructive Implicit Function Theorem

assumptions

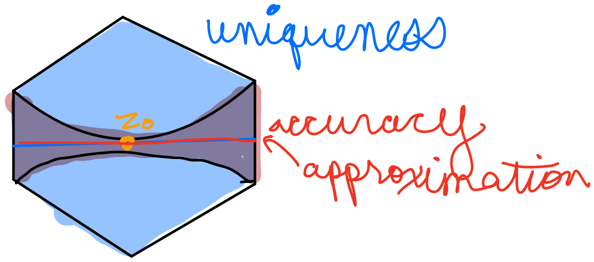
① $\|f(z_0)\| \le \rho$ small residual $z=(\lambda,u)$

② Assume $\|(DG(0,0))^{-1}\| \le K$ (replacing assuming nonsingular)

③ Assume Df is Lipschitz with constants

$$\|D_u f(\lambda,u) - D_u f(\lambda_0,u_0)\| \le L_1\|u-u_0\| + L_2\|\lambda-\lambda_0\|$$
$$\|D_\lambda f(\lambda,u)\| \le L_3 + L_4\|\lambda-\lambda_0\|$$

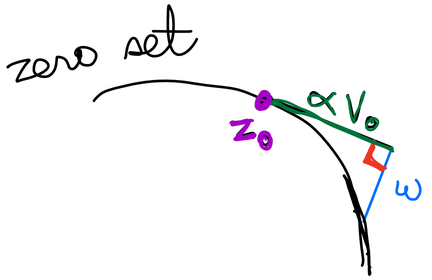
Then as long as $4K^2\rho < 1$, we have a list of specific conditions on existence and uniqueness of the zero set.

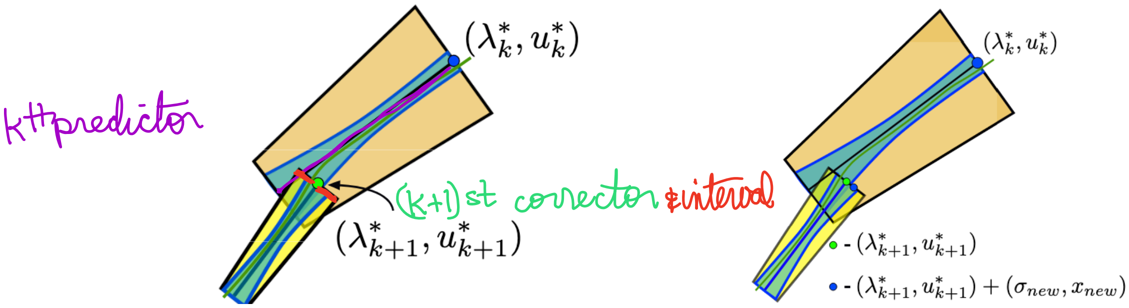


What about pseudo arclength continuation?

Use the theorem on the extended system

$$G(\alpha, \omega) = \begin{pmatrix} f(z_0 + \alpha v_0 + \omega) \\ v_0 \cdot \omega \end{pmatrix} = 0 \qquad G: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$





Linking condition

Since we want to bound an entire path, we need to use many boxes. Thus we require $\alpha=0$ accuracy region of the $(k+1)^{st}$ corrector point to be inside the k^{th} uniquess box.

another approach: series methods.
Write $u(\lambda)$ as a series and use root finding techniques.

Example $f(\lambda,u) = \lambda - u^2 = 0$ near $(1,1)$

We seek to solve for $u(\lambda)$.

More conveniently we rescale and consider

$$1+\lambda - (u(\lambda))^2 = 0 \quad \text{near} \quad (0,1)$$

Let $u(\lambda) = \sum_{j=0}^{\infty} a_j \lambda^j$ a Taylor series (where we see that $a_0 = 1$)

$$1+\lambda - \sum_{j=0}^{\infty} (a*a)_j \lambda^j = 0$$

$$F(a) = 1 - a_0 + (1-2a_0a_1)\lambda + \sum_{j=2}^{\infty} (a*a)_j \lambda^j = 0$$

$$F(a)_j = \begin{cases} 1 - a_0 & j=0 \\ 1-2a_0a_1 & j=1 \\ -(a*a)_j = -\sum_{k=0}^{j} a_{j-k}a_k & j \geq 2 \end{cases}$$

Note that $(a*a)_j$ depends on $a_0,\ldots,a_j$.
Therefore F_j depends only on $a_0,\ldots,a_j$.

Consider the projection

$$F^N(a_0, \ldots, a_N) = (F_0, \ldots, F_N) \qquad F^N : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$$

If we find $a_0, \ldots, a_N$ such that $F^N(a_0, \ldots, a_N) = 0$

How accurate is $u_N = \sum_{j=0}^{N} a_j \lambda^j$

Let $-\delta < \lambda < \delta$. Then $\|u - u_N\| \leq \sum_{j=N}^{\infty} |a_j| \delta^j$

We need to use a Newton-Kantorovich theorem on the operator

$$T(a) = a - A F(a)$$

where A is an approximate inverse of DF.

$$DF(a) = -\begin{pmatrix} 1 & & & 0 \\ 2a_1 & 2a_0 & & \\ 2a_2 & 2a_1 & 2a_0 & \\ 2a_3 & 2a_2 & 2a_1 & 2a_0 \\ \vdots & & & \ddots \end{pmatrix} = \begin{pmatrix} B_N & 0 \\ B_{N\infty} & B_{\infty\infty} \end{pmatrix}$$

Lower triangular

Define $A_N = B_N^{-1}$. Recall $a_0 = 1$.

Let A be given by $\begin{pmatrix} A_N & \\ & \frac{1}{2} \ddots \end{pmatrix}$

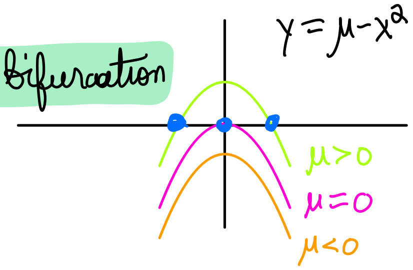
Use techniques discussed to find the error estimates.

These notes will only occur if time allows.

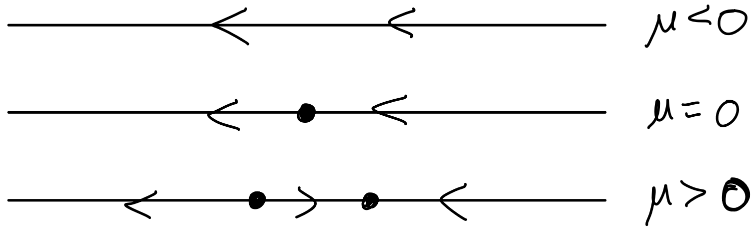
Bifurcations of equilibria in 1D

Example Saddle-node bifurcation

$\dot{x} = \mu - x^2$

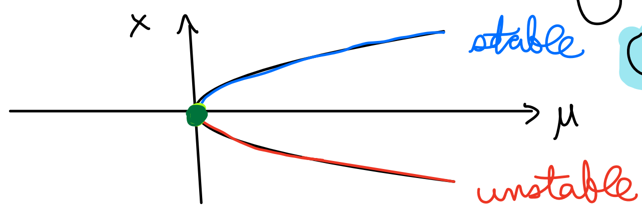


Phase portraits



$f(\mu,x) = \mu - x^2$

Bifurcation diagram showing equilibria and their stability

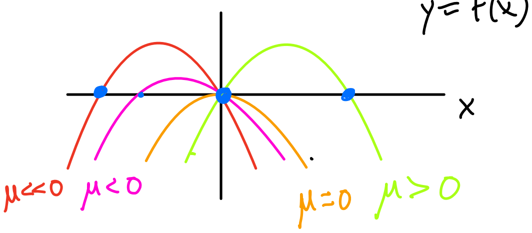


(0,0) is a bifurcation point, since the number of equilibria changes.

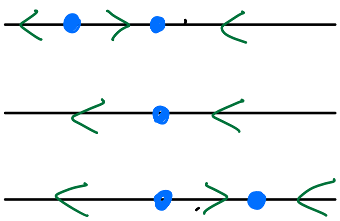
Example Transcritical bifurcation

$$\dot{x} = \mu x - x^2$$

$$f(\mu, x) = \mu x - x^2$$



Phase Portraits



$$f(0,0) = 0$$

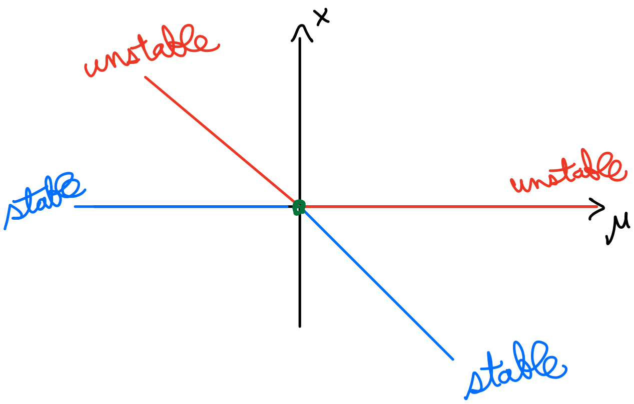
$$D_x f(0,0) = 0$$

$$D_\mu f(0,0) = 0$$

$$D_{x\mu} f(0,0) = 1 \neq 0$$

$$D_{xx} f(0,0) = -2 \neq 0$$

Bifurcation diagram

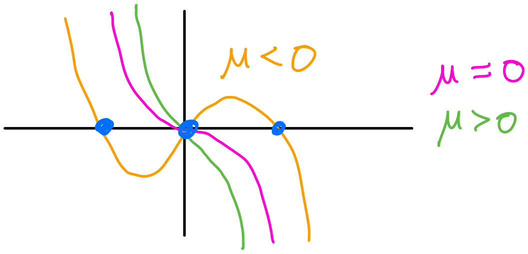


Exchange of stability at $(0,0)$.

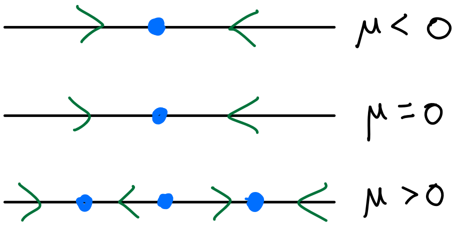
Example   Pitchfork Bifurcation

$$\dot{x} = \mu x - x^3$$

$$f(\mu, x) = \mu x - x^3$$



Phase Portraits



$$f(0,0) = 0$$

$$D_x f(0,0) = 0$$

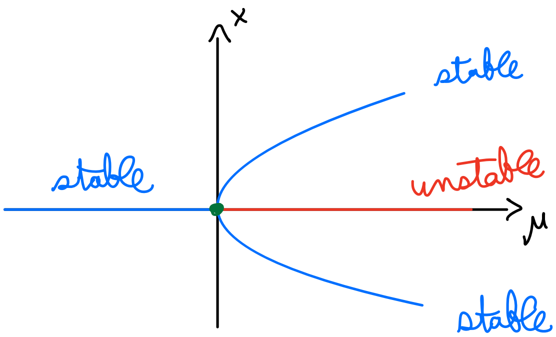
$$D_\mu f(0,0) = 0$$

$$D_{x\mu} f(0,0) = 1 \neq 0$$

$$D_{xx} f(0,0) = -6x\big|_{x=0} = 0$$

$$D_{xxx} f(0,0) = -6 < 0$$

Bifurcation diagram



Change of number of equilibria at $(0,0)$.

Calculus in higher dimensions

Define $f : \mathbb{R}^n \to \mathbb{R}^n$

$D_x f$ is an $n \times n$ matrix

$D_x f(x_0)\, v \in \mathbb{R}^n$ directional derivative

How about higher order derivatives?

$$D_{xx} f(x_0)(u,v) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} u_i v_j \in \mathbb{R}^n$$

$$D_{xxx} f(x_0)(u,v,w) = \sum_{i,j,k} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} u_i v_j w_k \in \mathbb{R}^n$$

Example

$$f(x_1,x_2) = \begin{pmatrix} x_1 x_2 + 7x_1^2 \\ x_2 \sin x_1 \end{pmatrix}, \quad D_x f = \begin{pmatrix} x_2 + 14x_1 & x_1 \\ x_2 \cos x_1 & \sin x_1 \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \begin{pmatrix} 14 \\ -x_2 \sin x_1 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial x_2^2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = \begin{pmatrix} 1 \\ \cos x_1 \end{pmatrix}$$

$$D_{xx} f(0,1)\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix}\right) = \begin{pmatrix} 14 \\ 0 \end{pmatrix} 1(-2) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}(3 \cdot 4) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 \cdot 4 + 3(-2))$$

$$= \begin{pmatrix} -28 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} -30 \\ -2 \end{pmatrix}$$

Theorem Sufficient conditions for a bifurcation

Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be C^2, and consider $\dot{x} = f(\mu, x)$. Assume

Extended System (μ_0, x_0, v)

① $f(\mu_0, x_0) = 0$

② Singularity $A := D_x f\,(\mu_0, x_0)$ has a simple eigenvalue of 0

$D_x f(\mu_0, x_0)\, v = 0$

Notation: $\boxed{\begin{array}{l} \text{right eigenvector } v \text{ ie } Av = 0 \cdot v \\ \text{left eigenvector } w \text{ ie } A^t w = w^t A = 0 w^t \end{array}}$

Then the following holds

ⓐ If ① $Q_1 := w^t D_\mu f(\mu_0, x_0) \neq 0$ Transversality

② $Q_2 := w^t D_{xx} f(\mu_0, x_0)(v, v) \neq 0$ Nondegeneracy

Then there is a saddle-node bifurcation at (μ_0, x_0). Birth of 2 equilibria from none.

The sign of $\frac{Q_2}{Q_1}$ determines the direction it opens

$\boxed{\frac{Q_2}{Q_1} > 0 \Rightarrow \supset} \quad \boxed{\frac{Q_2}{Q_1} < 0 \Rightarrow \subset}$

(b) If [(1) $w^t D_\mu f(\mu_0, x_0) = 0$ extra condition!

(2) $w^t D_{x\mu} f(\mu_0, x_0) v \neq 0$ Transversality

(3) $w^t D_{xx} f(\mu_0, x_0)(v, v) \neq 0$ Nondegeneracy

then there is a transcritical bifurcation.

at (μ_0, x_0). i.e. Crossing of equil. branches & exchange of stability

(c) If (1) [$w^t D_\mu f(\mu_0, x_0) = 0$

(2) $w^t D_{\mu x} f(\mu_0, x_0) v \neq 0$ Transversality

(3) $w^t D_{xx} f(\mu_0, x_0)(v, v) = 0$

(4) $w^t D_{xxx} f(\mu_0, x_0)(v, v, v) \neq 0$ Nondegeneracy

then there is a pitchfork bifurcation

at (μ_0, x_0). Birth of 2 new equilibria $1 \to 3$ with change of stability of one.

Note: There are situations not covered by this theorem. These are all the typical cases with a simple zero. another typical case is for A to have purely imaginary eigenvalues.

Example

$$\boxed{\dot{x} = \mu + x - \ln(1+x)}$$

$$f(0,0) = 0$$

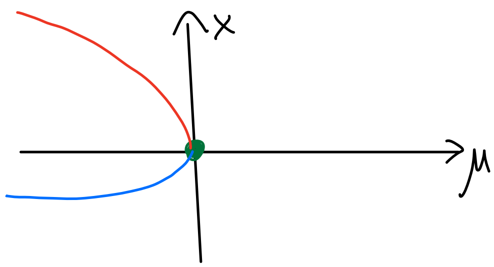
$$D_x f(0,0) = \left(1 + \frac{1}{1+x}\right)\Big|_{x=0} = 0$$

$$D_\mu f(0,0) = 1 \neq 0$$

$$D_{xx} f(0,0) = +\frac{1}{(1+x)^2} = +1 \neq 0$$

$$\frac{D_{xx} f}{D_\mu f} = \frac{1}{1} > 0$$

$\Rightarrow$ Saddle-node bifurcation at $(0,0)$ opening to the left.



Example $\dot{x} = \mu x - \ln(1+x)$

$$f(1,0) = 0$$

$$D_x f(1,0) = \mu - \frac{1}{1+x}\Big|_{(1,0)} = 0$$

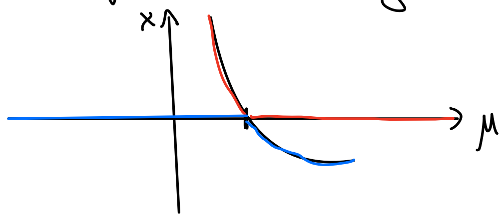
$$D_\mu f(1,0) = x \Big|_{(1,0)} = 0$$

$$D_{x\mu} f(1,0) = 1 \neq 0$$

$$D_{xx} f(1,0) = \frac{1}{(1+x)^2}\Big|_{(1,0)} = 1 \neq 0$$

Transcritical bifurcation

Bifurcation diagram



Example $\dot{x} = \mu x + x^3 - x^5$

$$f(0,0) = 0$$

$$D_x f(0,0) = \mu + 3x^2 - 5x^4\Big|_{(0,0)} = 0$$

$$D_\mu f(0,0) = x \Big|_{(0,0)} = 0$$

$$D_{\mu x} f(0,0) = 1 \neq 0$$

$$D_{xx} f(0,0) = 6x - 20x^3\Big|_{(0,0)} = 0$$

$$D_{xxx} f(0,0) = 6 \neq 0$$

Pitchfork

$$\frac{f(\mu,x)}{x} = \mu + x^2 - x^4$$

∴ the pitchfork opens rightward