Bifurcations

Return to e-vals & e-vectors.

Fix $M \in M_N(\mathbb{C})$. Solve

$$Mv = \lambda v \quad , \quad \lambda \in \mathbb{C},\ v \in \mathbb{C}^N$$

Newton method?

$$g(\lambda, v) = (M - \lambda I)v = 0$$

Dimen Problem $g: \mathbb{C}^{N+1} \to \mathbb{C}^N$

Symmetry Prob! $g(\tilde{\lambda}, \tilde{v}) = 0 \Rightarrow g(\tilde{\lambda}, \mu\tilde{v}) = 0,\ \forall \mu \in \mathbb{C}$

No Uniqueness? No Contraction.

Newton $T(\lambda, v) = \binom{\lambda}{v} - AF(\lambda, v)$

$DT(\lambda,v) = I - ADF(\bar{\lambda},\bar{v}) - A(DF(\bar{\lambda},\bar{v}) + A(DF(\bar{\lambda},\bar{v}) - DF(\lambda,v))$

Rad. Poly:

$$Y_0 \geq \left\| \binom{\bar{\lambda}}{\bar{v}} - AF(\bar{\lambda}, \bar{v}) \right\|_1$$

$$Z_0 \geq \| I - A\, DF(\bar{\lambda}, \bar{v}) \|_1$$

$$Z_2(r) \geq \sup_{(\mu,h) \in B_r(0)} \| A(DF(\bar{\lambda}+\mu, v+h) - D(\bar{\lambda},\bar{v})) \|$$

$$\geq \|A\| \sup_{(\mu,h) \in B_r(0)} \left\| \left(\begin{array}{c|c} 0 & 0 \\ \hline -h & -\mu Id \end{array} \right) \right\|$$

$$= \|A\|_1 \sup_{(\mu,h) \in B_r(0)} \max\{\|h\|_1, |\mu|\}$$

$$= r\|A\|_1$$

Need a phase condition.

- $\|\tilde{v}\|^2 = 1$, only if $\tilde{v} \in \mathbb{R}^N$
- $\tilde{v}_k = 1$ $\quad 1 \leq k \leq N$
- More gen. $\ell(\tilde{v}) = \xi \cdot \tilde{v} = 1$ for any $\xi \in \mathbb{C}^N$ (choose $\xi \approx \tilde{v}$)

Define $F(\lambda, v) = \begin{pmatrix} \ell(v) - 1 \\ Mv - \lambda v \end{pmatrix}$

$$DF(\lambda, v) = \left(\begin{array}{c|c} 0 & \xi^T \\ \hline -v & M - \lambda Id \end{array} \right)$$

Fix approx sol $(\bar{\lambda}, \bar{v})$ & $A \approx (DF(\bar{\lambda}, \bar{v}))^{-1}$

Def rad poly

$$p(r) = \|A\|_1 r^2 - (1 - Z_0)r + Y_0$$

IF $\exists\, r_0 > 0$ s.t. $p(r_0) < 0$

$\Rightarrow \exists!\, (\tilde{\lambda}, \tilde{v})$ w/ $F(\tilde{\lambda}, \tilde{v}) = 0 \ldots$

Note: This is robust!

Spz $M = M(\mu) \in C^1(\mathbb{R}^d, M_N(\mathbb{C}))$

Then $F_\mu(\lambda, v)$ & $DF_\mu(\lambda, v)$ vary smoothly w/ α

Def $A \approx DF_{\mu_0}(\bar{\lambda}, \bar{v})^{-1}$. The

Bounds Y_0, Z_0, Z_2 vary cont w/ μ

IF $p_{\mu_0}(r_0) < 0 \Rightarrow p_\mu(r_0) < 0\ \forall |\mu - \mu_0| < \epsilon$

(Similarly for M)

(Simple) eval & e-vect of M vary cont. w/r μ

Evaluate) $p_{I_\mu}(r_0) < 0$ w/ $I_\mu \in \mathbb{IR}$

$\implies$ eval bounds on $M(\mu)$ $\forall \mu \in I_\mu$

Application: Continuation

Fix $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$, ODE $\dot{x} = f_\mu(x)$

Spz $f(\tilde{x}, \tilde{\mu}) = 0$, Hyperbolic Eq

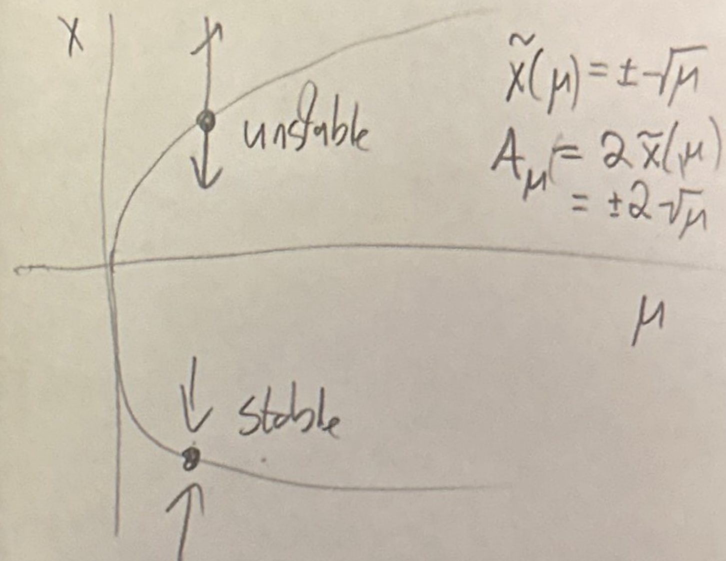
ie $\sigma(D_x f(\tilde{x}, \tilde{\mu})) \cap i\mathbb{R} = \emptyset$

Implicit Fnct Th'm

$\implies \exists$ branch $f(\tilde{x}(\mu), \mu) = 0$

Define linearization: $A_\mu := D_x f(\tilde{x}(\mu), \mu)$

Eq) $\dot{x} = x^2 - \mu = f(x, \mu)$

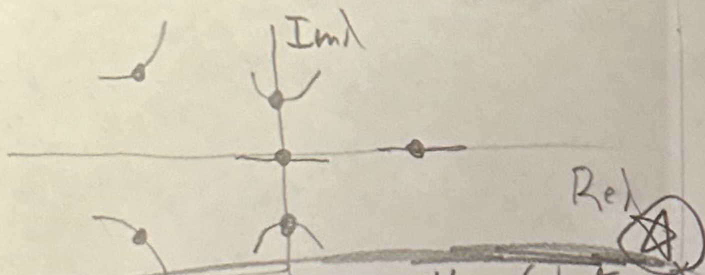


$\tilde{x}(\mu) = \pm\sqrt{\mu}$

$A_\mu = 2\tilde{x}(\mu) = \pm 2\sqrt{\mu}$

Simple Eval are robust!

$\exists \delta > 0$ s.t. $\sigma(A_\mu) \cap i\mathbb{R} = \emptyset$ $\forall |\mu - \tilde{\mu}| < \delta$



Def) Morse index $(\tilde{x})$

$Mor(\tilde{x})$ = # e-vals w/ positive Real part (w/ Multiplicity)

Spz $Mor(\tilde{x}_\mu) = Mor(\tilde{x}_{\tilde{\mu}})$ $\forall |\mu - \tilde{\mu}| < \varepsilon$

Stability Does Not Change

Hartman Grobman) Local Dynamics are the same; No Bifurcations

Bifurcations generally occur when
- Stability Changes
- Qual. dyn. Changes
- Linear Dynamics are not enough

↓

Reduce to (nonlinear) normal form

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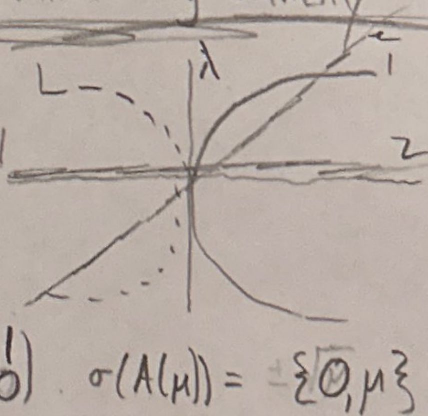
Classification ~ Alg. Geometry

Repeated Eval are Hard



$A(\mu) = \begin{pmatrix} \mu & 1 \\ 0 & 0 \end{pmatrix}$ $\sigma(A(\mu)) = \{0, \mu\}$

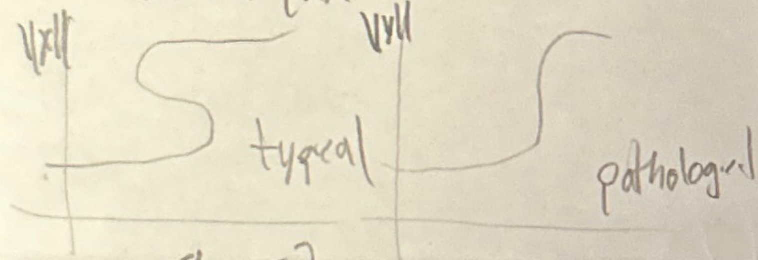
$B(\mu) = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix}$ $\sigma(B(\mu)) = \pm\sqrt{\mu}$

Simplest Case of Bifurcation: Saddle Node

Def] A saddle node for f is a pt $(\tilde{x},\tilde{\mu}) \in \mathbb{R}^N \times \mathbb{R}$ s.t.

i) $f(\tilde{x},\tilde{\mu}) = 0$ &

ii) $0 \in \sigma(D_x f(\tilde{x},\tilde{\mu}))$ w/ alg. mult 1 & other eval have non-zero rational part



Thm] [see Chicone] Suppose $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is C^1, $(\tilde{x},\tilde{\mu})$ is a s.n. & $\tilde{v}$: $\ker(D_x f(\tilde{x},\tilde{x})) = \text{span}\{\tilde{v}\} \neq \{0\}$,

If

- $u_1 = D_\mu f(\tilde{x},\tilde{\mu}) \neq \vec{0}$
- $u_2 = D_x^2 f(\tilde{x},\tilde{\mu}) \cdot [\tilde{v},\tilde{v}] \neq \vec{0}$
- $u_1, u_2 \notin \text{im}(D_x f(\tilde{x},\tilde{\mu}))$

Then $(\tilde{x},\tilde{\mu})$ is a S.N.B.

~~Easy These are all robust conditions we can check, given $(\tilde{x},\tilde{\mu})$ is a S.N.~~

~~More Involved~~

Def] A saddle node bifurcation happens at s.n. $(\tilde{x},\tilde{\mu})$ if

(SNB1) (curve of eq.) $\exists$ smooth $g: (-\delta,\delta) \to \mathbb{R}^N \times \mathbb{R}$, $s \mapsto (g_1(s), g_2(s))$ s.t. $g(0) = (\tilde{x},\tilde{x})$ & $f(g_1(s), g_2(s)) = 0$

(SNB2) (Non degeneracy) Curve g has quadratic tangency @ $(\tilde{x},\tilde{\mu})$ i.e.

$g_2(0) = \tilde{\mu}$, $g_2'(0) = 0$, $g_2''(0) \neq 0$

(SNB3) (transv) If $s \neq 0$, then $D_x f(g_1(s), g_2(s))$ is hyperbolic & If $\lambda(s) \in \sigma(D_x f(g_1(s), g_2(s)))$ w/ $\lambda(0) = 0 \Rightarrow \mu'(0) \neq 0$

Thm] Let $X = (x,\mu,v) \in \mathbb{R}^{2N+1}$ & def $F: \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$ by

$$F(X) = \begin{pmatrix} f(x) \\ \ell(v) - 1 \\ D_x f(x,\lambda) v \end{pmatrix} \leftarrow \text{phase}$$

w/ $\ell(v) = \xi \cdot v$ for fixed $\xi \in \mathbb{R}^N$.

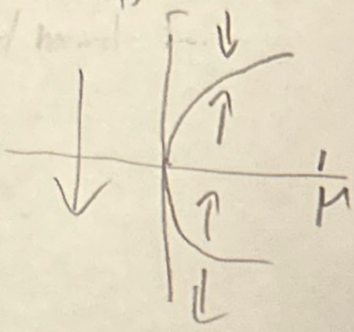
If $\tilde{X} \in \mathbb{R}^{2N+1}$ is a non-degen zero of F, & $(N-1)$ e-val of $D_x f(\tilde{x},\tilde{\mu})$ have non-zero real part, then a S.N.B. occurs at the S.N. $(\tilde{x},\tilde{\lambda})$

Note | $0 = F(X)$ being non-degenerate is Robust, provable w/ CAP.

Various normal form Bifurcations to Know

Saddle Node

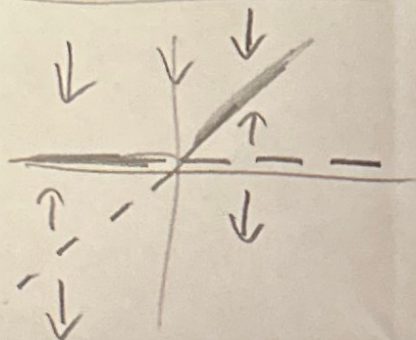
$$\dot{x} = \mu - x^2 + O(x^3)$$



Transcritical

$$\dot{x} = \mu x - x^2 + O(|x|^3)$$

(less common)



Linearization

$$\begin{aligned}\dot{x} &= \\ \dot{y} &= \end{aligned} \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ spiral sink/source

Normal Form (in Polar coord)

$$\dot{r} = (a\mu + br^2) r$$

$$\dot{\theta} = (\omega + c\mu + br^2)$$

$\dot{r} = 0 \Rightarrow r = 0$ equilibria

$\mu = -\frac{br^2}{a}$ ~ periodic orbits

For e-val $\lambda(\mu), \lambda(\mu)^*$, then

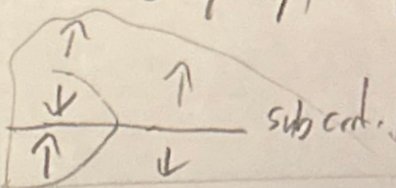
$$\frac{d}{d\mu}\left(Re(\lambda(\mu))\right)\Big|_{\mu=\mu_0} = a \overset{\text{want}}{\neq} 0$$

Speed of e-val crossing $i\mathbb{R}$

Pitch Fork

$$\dot{x} = \mu - x^3 + O(x^4)$$

(Needs Symmetry)



Hopf Bifurcation

Spz $\pm i\omega \in \sigma(D_x f_{\mu_0}(0))$ only complex e-val

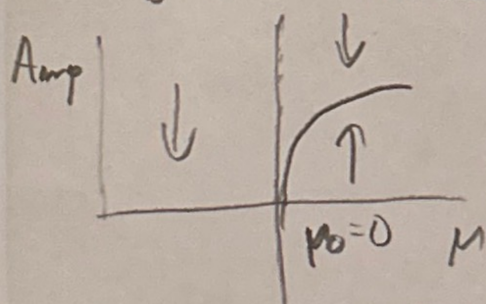
Then $D_x f_{\mu_0}(0)$ is invertible,

No New equilibria

If $b \neq 0$, there is a family of periodic orbits

$b < 0$ ~ stable

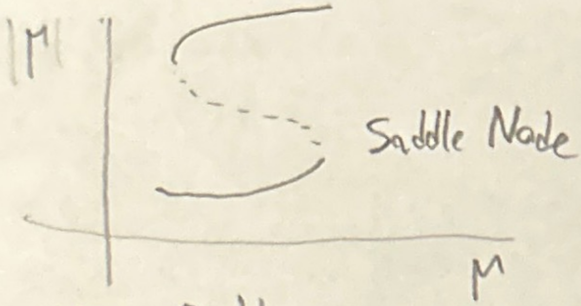
$b > 0$ ~ unstable



Amplitude ~ $\sqrt{\frac{\mu a}{b}}$ singular @ $\mu = 0$!

Solution: Fix Amplitude as param, Solve for μ.

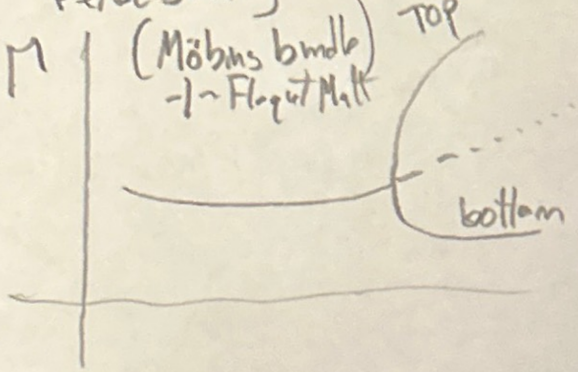
Periodic Orbits Can bifurcate too!



Period Doubling

(Möbius bundle)

$-1 \sim$ Floquet Mult



Secondary Map?

Periodic Orbit $\longrightarrow$ Invariant torus $S^1 \times S^1$