CAP in ∞-dim | Solving an IVP

Recall Henon-Heiles

(☆) $z'' + z + i\bar{z}^2 = 0, \quad z(t_0) = z_0, \; z'(t_0) = z_1$

Taylor transform

$\mathcal{T}(z) = \{a_n\}_{n=0}^{\infty}, \qquad \mathcal{T}^{-1}(z) = \sum_{n=0}^{\infty} a_n (t-t_0)$

Pseudodiff Op $\mathfrak{D} : \ell^1_\nu \to \ell^1_{\nu'} \qquad \nu > \nu' > 0$

$\mathfrak{D}(a)_n = (n+1)\, a_{n+1}$

Cauchy Prod.: $* : \ell^1_\nu \times \ell^1_\nu$

$(a*b)_n = \sum_{k\in\mathbb{Z}} a_{n-k} b_k$

~~IVP in Sequence Space~~

$\mathfrak{D}^2 a + a + i\,\bar{a}*\bar{a} = 0, \quad a_0 = z_0, \; a_1 = z_1$

Fixed point Operator: $T : \ell^1_\nu \to \ell^1_\nu$

$$(T(a))_n = \begin{cases} z_0 & \text{if } n=0 \\ z_1 & \text{if } n=1 \\ \frac{-1}{n(n-1)}\left(a_{n-2} + i(\bar{a}*\bar{a})_{n-2}\right) & \text{if } n\geq 2 \end{cases}$$

Same

Truncation | Fix $N \in \mathbb{N}$

Def $X_N \times X_\infty = \ell^1_\nu$

$X_N = \{a \in \ell^1_\nu \mid a_n = 0 \;\forall n \geq N+1\} \cong \mathbb{R}^{N+1}$

$X_\infty = \{a \in \ell^1_\nu \mid a_n = 0 \;\forall n \leq N\}$

Proj $\pi_N : \ell^1_\nu \to X_N$

$\pi_\infty : \ell^1_\nu \to X_\infty$

IVP in Sequence Space

$a_0 = z_0$

$a_1 = z_1$

$0 = \mathfrak{D}^2 a + a + i\bar{a}*\bar{a}$

$0_n = (n+2)(n+1)a_{n+2} + a_n + i(\bar{a}*\bar{a})_n \quad n\geq 0$

Reindex $n+2 \to n$; def $F : \ell^1_\nu \to \ell_\nu$

$$F(a)_n = \begin{cases} a_0 - z_0 & n=0 \\ a_1 - z_1 & n=1 \\ n(n-1)a_n + a_{n-2} + i(\bar{a}*\bar{a})_{n-2} & n\geq 2 \end{cases}$$

$a \in \ell^1_\nu \;\&\; F(a) = 0 \implies \mathcal{T}^{-1}(a)$ solves ☆

Compute w/ Interval Arithmetic

$\hat{A} = \{A_n\}_{n=0}^{N} \in \mathbb{IR} \qquad \hat{a} \equiv \hat{a}_{(N)} = (\hat{a}_0, \ldots, \hat{a}_N)$

Contains true solution $\hat{a}^{(N)}_n \in A_n$; ✓

$\pi_N\left(T(\pi_N \hat{a}_{(N)})\right) = \pi_N \hat{a}_{(N)}$

Spz $a_{(N)} \in X_N \;\&\; a_\infty \in X_\infty$

$\pi_N T(a_{(N)} + a_\infty) = \pi_N T(a_{(N)})$

WANT $a_\infty \in X_\infty$ s.t.

$\pi_\infty T(\hat{a}_{(N)} + a_\infty) = \hat{a}_{(N)} + a_\infty$

$\pi_\infty T(\hat{a}_{(N)} + a_\infty) = a_\infty$

Define Operator: $T_\infty : X_\infty \to X_\infty$
for $a_\infty \in X_\infty$ as

$$T_\infty(a_\infty) := \pi_\infty T(\hat{a}_{(N)} + a_\infty)$$
$$\in \pi_\infty T(\hat{A} + a_\infty)$$

Want to show T_∞ has a fixed Point

Newton-Kantorovich Th'm

Spz X, B-space $T \in C^1(X, X)$, $\bar{x} \in X$

$$\| T(\bar{x}) - \bar{x} \|_X \leq Y_0$$

$$\sup_{x \in B_r(\bar{x})} \| DT(x) \|_{B(X)} \leq Z(r) \ \forall r > 0$$

$$= \frac{|\hat{a}_{N-1}|}{(N+1)N} + \frac{|\hat{a}_N|}{(N+2)(N+1)} + \sum_{n=N+1}^{2(N+1)} \frac{|(\hat{a} * \hat{a})_{n-2}| \nu^{-n}}{n(n-1)}.$$
$$=: Y_0$$

Finite Sum! Perform bounds w/ Interval Arithmetic $\hat{A}_n \ni \hat{a}_n$

$\boxed{Z(r) \text{ bound}}$ Need Frechet Deriv

Calc $DT_\infty(a)$. Recall

$$(T(a))_n = \frac{-1}{n(n-1)} \left[a_{n-2} + i (\bar{a} * \bar{a})_{n-2} \right] \quad n \geq 2$$

Define shift up map

$$\sigma(a)_n = \begin{cases} 0 & \text{if } n = 0 \\ a_{n-1} & \text{if } n \geq 1 \end{cases}$$

Def $p(r) = Z(r) r - r + Y_0$

If $\exists r_0 > 0$ s.t. $p(r_0) < 0$,
then $\exists ! \tilde{x} \in B_{r_0}(\bar{x})$ s.t. $T(\tilde{x}) = \tilde{x}$

Here $X = X_\infty$, $\bar{x} = 0$

Calculate Bounds

$$\| T(\bar{x}) - \bar{x} \|_X$$
$$= \| T_\infty(0) - 0 \|_{\ell^1_\nu}$$
$$\leq \left\| \left\{ \frac{-1}{n(n-1)} \left(\hat{a}_{n-2} + i \left(\overline{\hat{a} * \hat{a}} \right)_{n-2} \right) \right\}_{n=N+1}^{\infty} \right\|$$
$$\leq \sum_{n=N+1}^{\infty} \frac{|\hat{a}_{n-2}| + |(\hat{a} * \hat{a})_{n-2}|}{n(n-1)} \nu^{-n}$$

Define op $L : \ell^1_\nu \to \ell^1_\nu$

$$(La)_n = \begin{cases} 0 & \text{if } n = 0, 1 \\ \frac{1}{n(n-1)} & \text{if } n \geq 2 \end{cases}$$

Then we may write for $n \geq 2$,

$$T(a) = -L \sigma^2 (a + i \bar{a} * \bar{a})$$

Hence $\forall h \in \ell^1_\nu$,

$$DT(a) \cdot h = -L \sigma^2 (h + 2i \bar{a} * \bar{h})$$

Recall $T_\infty(a_\infty) = \pi_\infty T(\hat{a} + a_\infty)$

Hence for $h_\infty \in X_\infty$

$$DT_\infty(a_\infty) \cdot h_\infty = \pi_\infty \left(-L \sigma^2 \left(h_\infty + 2i \left(\overline{\hat{a} + a_\infty} \right) * \bar{h}_\infty \right) \right)$$

$DT_\infty(a_\infty)\cdot h = \pi_\infty\left[-L\sigma^2\left(h + 2i(\overline{\hat{a} + a_\infty}) * h\right)\right]$

Want $Z_2(r)$ to bound

$\cancel{Z_2(r)}$ $\sup_{a_\infty \in \overline{B_r(0)}} \|DT_\infty(a_\infty)\|_{B(\ell^1_r, \ell^1_r)}$

$$= \sup_{a_\infty \in \overline{B_r(0)}} \sup_{\|h\|_{\ell^1_r} = 1} \left\| -\pi_\infty L\sigma^2\left(h + 2i(\overline{\hat{a}+a_\infty}) * h\right)\right\|_{\ell^1_r}$$

$$\leq \sup_{a_\infty \in \overline{B_r(0)}} \sup_{\|h\|_{\ell^1_r}=1} \|\pi_\infty L\sigma^2\|_{B(\ell^1_r)} \cdot \|h + 2i(\overline{\hat{a}+a_\infty}) * h\|_{\ell^1_r}$$

$$\leq \|\pi_\infty L\sigma^2\|_{B(\ell^1_r)} \sup_{a_\infty \in \overline{B_r(0)}} \sup_{\|h\|_{\ell^1_r}=1} \left(\|h\|_{\ell^1_r} + \|2i(\overline{\hat{a}+a_\infty}) * h\|_{\ell^1_r}\right)$$

$$\leq \|\pi_\infty L\sigma^2\|_{B(\ell^1_r)} \sup_{a_\infty \in \overline{B_r(0)}} \sup_{\|h\|_{\ell^1_r}=1} \left(1 + 2\|\hat{a}+a_\infty\|_{\ell^1_r} \cdot \|h\|_{\ell^1_r}\right)$$

$$\leq \|\pi_\infty L\sigma^2\|_{B(\ell^1_r)} \sup_{a_\infty \in \overline{B_r(0)}} \left(1 + 2\left(\|\hat{a}\|_{\ell^1_r} + \|a_\infty\|_{\ell^1_r}\right)\right)$$

$$\boxed{= \|\pi_\infty L\|_{B(\ell^1_r)} \|\sigma^2\|_{B(\ell^1_r)} \left(1 + 2\|\hat{a}\|_{\ell^1_r} + 2r\right)}$$

$$\|\sigma^2\|_{B(\ell^1_r)} = \sup_{\|h\|_{\ell^1_r}=1} \|\sigma^2 h\|_{\ell^1_r}$$

$$= \sup_{\|h\|_{\ell^1_r}=1} \sum_{n\in\mathbb{N}} |(\sigma^2 h)_n| r^n$$

$$= \sup_{\|h\|=1} \sum_{n=2}^{\infty} |h_{n-2}| r^n$$

$$= \sup_{\|h\|=1} r^2 \sum_{n=2}^{\infty} |h_{n-2}| r^{n-2}$$

$$= \sup_{\|h\|=1} r^2 \|h\|_{\ell^1_r} = \boxed{r^2}$$

$$\|\pi_\infty L\|_{B(\ell^1_r)} = \sup_{\|h\|=1} \|\pi_\infty L h\|_{\ell^1_r}$$

$$= \sup_{\|h\|=1} \sum_{n=N+1}^{\infty} \left|\frac{1}{n(n+1)} h_n\right| r^n$$

$$\leq \sup_{\|h\|=1} \sum_{n=N+1}^{\infty} \left|\frac{h_n}{(N+1)N}\right| r^n$$

$$= \frac{1}{(N+1)N} \sup_{\|h\|=1} \sum_{n=N+1}^{\infty} |h_n| r^n$$

$\|h\|_{\ell^1_r}$

$$\boxed{= \frac{1}{(N+1)N}}$$

Hence

$$\sup_{a_\infty \in B_r(0)} \|DT_\infty(a_\infty)\|_{B(\ell^1_\nu)}$$

$$\le \|\pi_\infty L\|_{B(\ell^1_\nu)} \|\sigma^2\|_{B(\ell^1_\nu)} \left(1+2\|\hat a\|_{\ell^1_\nu} + 2r\right)$$

$$\le \underbrace{\frac{\nu^{-2}}{N(N+1)}\left(1+2\|\hat a\|_{\ell^1_\nu}+2r\right)}_{=: Z(r)}$$

Recall $Y_0 = \frac{|\hat a_{N-1}|}{(N+1)N} + \frac{|\hat a_N|}{(N+2)(N+1)} + \sum_{n=N+1}^{2(N+1)} \frac{(\hat a * \hat a)_{n-2}\,\nu^n}{n(n-1)}$

Define Rad poly

$p(r) = Z(r)r - r + Y_0$

If $\exists\, r_0 > 0$ s.t. $p(r_0) < 0$

then $\exists!\, \hat a_\infty \in B_{r_0}(0)$ s.t. $T_\infty(\hat a_\infty) = \hat a_\infty$

& $T(\hat a + \hat a_\infty) = \hat a + \hat a_\infty$

$\Rightarrow z = \varphi^{-1}(\hat a + \hat a_\infty)$ solves the IVP

$$\|z(\nu) - z^{(N)}(\nu)\|_{C^0} \le \sum_{n=0}^{N} \mathrm{rad}(A_n)\nu^n + \overbrace{\sum_{n=N+1}^{\infty} |(\hat a_\infty)_n|\nu^n}^{\|\hat a_\infty\|_{\ell^1_\nu}}$$

$$= \left[\sum_{n=0}^{N} \mathrm{rad}(A_n)\nu^n\right] + r_0$$

Show Code if Time Allows