Boundary value problems

This lecture covers methods for covering boundary value problems. These methods are numerical but will lead to validation in a future lecture.

- Differential equation for $u(x)$ on interval $x \in \Omega = (a,b)$ (mostly I use $(0,1)$)
- Boundary condition for u.

Dirichlet: $u(0), u(1)$ given

Neumann: $u'(0), u'(1)$ given

Robin: combo

Spectral method: Represent as a series of trigonometric polynomials $\{\varphi_k(x)\}$

1. **complete** ie all functions in $L^2(\Omega)$ can be arbitrarily well approximated with linear combinations of these functions
2. **orthogonal** in $L^2(\Omega)$ ie $\int_\Omega \varphi_j(x)\varphi_k(x)\,dx = 0 \quad j \neq k$
3. respect the boundary conditions

We illustrate with a series of examples

Example

$u'' + u = f$ on $(0,1)$ — Second order linear

$u'(0) = u'(1) = 0$ — Homogeneous Neumann BC

Where f also satisfies the B.C.

Fact $\{\cos(l\pi x)\}_{k=0}^{\infty}$

is a complete orthogonal set on $(0,1)$. which each have homogeneous Neumann boundary conditions.

$$\boxed{\|\cos(l\pi x)\| = \left(\int_0^1 (\cos(l\pi x))^2\,dx\right)^{\frac{1}{2}} = c_l = \begin{cases} 1 & l = 0 \\ \frac{1}{\sqrt{2}} & l \neq 0 \end{cases}}$$

Write $f(x) = \sum_{l=0}^{\infty} \beta_l \cos(l\pi x)$

$$\int_0^1 f(x)\cos(k\pi x)\,dx = \int \left(\sum_{l=0}^{\infty} \beta_l \cos(l\pi x)\right)\cos(k\pi x)\,dx$$

$$= \sum \beta_l \int_0^1 \cos(l\pi x)\cos(k\pi x)\,dx$$

$$= \beta_k \quad c_k^2$$

$$\boxed{\beta_k = \frac{\langle f, \cos k\pi x\rangle}{c_k^2}}$$

- How about numerics?

For $x_k = \frac{2k+1}{2N}$ $\quad k = 0, \dots, N-1$

Interpolation:

$$f(x_k) = \sum_{l=0}^{N-1} \beta_l \cos(l\pi x_k) \quad \forall k.$$

and we can use the dct to find β_l, using appropriate padding to counter aliasing.

- Assume $u(x) = \sum_{l=0}^{\infty} a_l \cos(l\pi x)$

$$\Rightarrow u''(x) = \sum_{l=0}^{\infty} -(l\pi)^2 a_l \cos(l\pi x)$$

- The differential equation becomes

$$\sum_{l=0}^{\infty} \left[-(l\pi)^2 a_l + a_l\right] \cos(l\pi x) = \sum_{l=0}^{\infty} \beta_l \cos(l\pi x)$$

$$\Rightarrow \left(-(l\pi)^2 + 1\right) a_l = \beta_l \quad \text{by orthogonality}$$

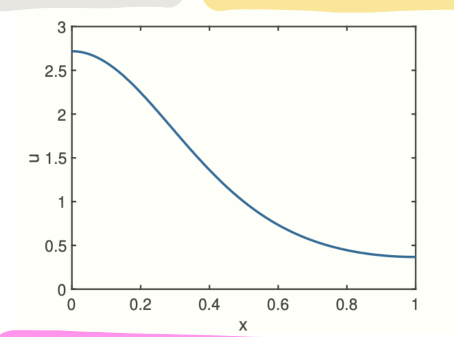
$$\boxed{a_l = \frac{\beta_l}{1-(l\pi)^2}}$$

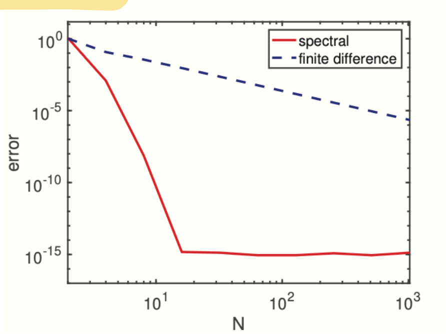
- Numerically, we would only go up to N.

- **Test case**

$$f(x) = \left(1 + \pi^2 \sin^2(\pi x) - \pi^2 \cos(\pi x)\right) e^{\cos(\pi x)}$$

Exact answer $u(x) = e^{\cos(\pi x)}$





- **Unknown answer**

$$f(x) = e^{\cos(3\pi x)}$$

Issues with method

f must satisfy BC for fast convergence
u' would not be representable using this series as it doesn't satisfy same BC.

- assume $Lu = u'' + u$

$$Lu + f = 0 \quad \& \quad Lu_N + f_N = 0$$

$$L(u - u_N) + f - f_N = 0$$

$$u - u_N = -L^{-1}(f - f_N)$$

- We can actually view this as an infinite-dimensional linear algebra problem $(I-A)x=b$ where

$$A = \pi^2 \begin{pmatrix} 0^2 & & & \\ & 1^2 & & 0 \\ 0 & & 2^2 & \\ & & & \ddots \end{pmatrix} \qquad x = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \end{pmatrix} \qquad b = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \end{pmatrix}$$

where the norm is the 2-norm.

We have found a solution to $(I-A_{NN})u=b_N$

$A_{NN} = \left. N \right\{ \underbrace{\begin{pmatrix} & \\ & \end{pmatrix}}_{N}$, we have no off diagonals

$$\|u-u_N\|_2 = \|x-x_N\| = \|(I-A_N)^{-1} b_N\|.$$

Bounding the tail $\ni$ bounding the ∞-dim'l piece

$$I - A_{\infty\infty} = \begin{pmatrix} 1-\pi^2(N+1)^2 & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

$$(I-A_{\infty\infty})^{-1} = \mathrm{diag}\left(\frac{1}{1-\pi^2 k^2}\right) \quad k=N+1,\dots$$

$$\|(I-A_{\infty\infty})^{-1}\|_2^2 \le \left(\sum_{l=N+1}^{\infty}\left(\frac{1}{1-\pi^2k^2}\right)^2\right)$$

$(1-\pi^2k^2)^2 \ge \frac{\pi^4k^4}{2}$ (when $k>1$) $\Rightarrow$

$$\le \frac{1}{\pi^4}\sum_{l=N+1}^{\infty}\frac{2}{k^4}$$

$$\le \frac{2}{\pi^4}\int_N^{\infty}\frac{1}{x^4}\,dx$$

$$= \frac{2}{\pi^4}\left(\frac{-3}{x^3}\right)\Bigg|_N^{\infty}$$

$$= \frac{2}{\pi^4}\;\frac{3}{N^3} = \frac{6}{\pi^4N^3}$$

$$\|(I-A_{\infty\infty})^{-1}\|_2 \le \frac{\sqrt{6}}{\pi^2N^{3/2}}$$

additionally we need

$$\|(I-P_N)f\|_2 = \left\|f-\sum_{l=0}^{N}\beta_l\cos(l\pi x)\right\|_2$$

Thus $\boxed{\|u-u_N\|_2 \le \frac{\sqrt{6}}{\pi^2N^{3/2}}\|(I-P_N)f\|_2}$

Beyond linear cases

a similar but nonlinear BVP is the stationary Allen-Cahn equation

$$u'' + \lambda(u - u^3) = 0 \qquad u'(0) = u'(1) = 0$$

We start in the same way as before

$$u = \sum_{\ell=0}^{\infty} a_\ell \cos(\ell \pi x)$$

$$u'' = \sum_{\ell=0}^{\infty} -(\ell\pi)^2 a_\ell \cos(\ell\pi x)$$

$$u^3 = \left(\sum_{\ell=0}^{\infty} a_\ell \cos(\ell\pi x)\right)^3 = \sum_{\ell=0}^{N} \beta_\ell \cos(\ell\pi x)$$

Thus we have an implicit definition of $\{a_\ell\}$

$$0 = \sum_{\ell=0}^{\infty}\left[-(\ell\pi)^2 a_\ell + \lambda(a_\ell - \beta_\ell(a))\right]\cos(\ell\pi x)$$

By the orthogonality of $\{\cos(\ell\pi x)\}$, this implies

$$G_\ell(a) = -(\ell\pi)^2 a_\ell + \lambda(a_\ell - \beta_\ell(a)) = 0 \qquad \ell = 0, 1, 2, \ldots$$

Our numerical goal is to evaluate

$$\boxed{u_N = \sum_{l=0}^{N-1} a_l \cos(l\pi x)}$$

and then establish a tail bound.

Computing β_l from a_l

If a_l are known, we can compute β_l:

Padding Fix N. We know that to find $\beta_0, \dots, \beta_{N-1}$ from $a_0, \dots, a_{N-1}$, we require $M=3N$ spatial points.

Let $u_k = \sqrt{M}\,\mathrm{idct}(a_0, \dots, a_{N-1}, \overbrace{0, \dots, 0}^{M-N})$

Let $y_k = u_k^3$, the cube of interpolant at $\frac{2k+1}{2(3M)}$

Find $\frac{1}{\sqrt{M}}\mathrm{idct}(y_0, \dots, y_{M-1})$, ← scale constant

giving $\beta_0, \dots, \beta_{N-1}$ as the first N terms.

Now we can solve 6 using zero finding.

Error bounds

The error bounds depend very much on the spaces under consideration. We will give accuracy & uniqueness information in the analytic setting as have been discussed previously. This does not give information on either accuracy or uniqueness on the standard Sobolev spaces usually used in PDEs. For this, there are references provided.