CAP in ∞-dim) Solving a periodic Solution Periodic Sol in Sequence Space.

Recall Duffing Eq

(*) $x'' + \varepsilon x' + \alpha x + \beta x^3 + \gamma \cos t = 0$

Fourier transform $\mathcal{F}: C^\omega(A_\rho) \to \ell^1_{\nu,\mathbb{Z}} = \ell^1_\nu$

$\mathcal{F}(x) = \{a_n\}_{n\in\mathbb{Z}}$, $\mathcal{F}^{-1}(a) = \sum_{n\in\mathbb{Z}} a_n e^{int}$

Pseudo Diff Op $K: \ell^1_{\nu,\mathbb{Z}} \to \ell^1_{\nu',\mathbb{Z}}$ $\nu > \nu' \geq 0$.

$$(Ka)_k = ika$$

Convolution Product

$$(a*b)_n = \sum_{k\in\mathbb{Z}} a_{n-k} b_k$$

B-algebra

$$\|a*b\|_{\ell^1_{\nu,\mathbb{Z}}} \leq \|a\|_{\ell^1_{\nu,\mathbb{Z}}} \|b\|_{\ell^1_{\nu,\mathbb{Z}}},\quad \nu \geq 1.$$

Def $F_N = \pi_N \circ F \circ \pi_N$

Spz $\bar{a} \in X_N$ & $\|F_N(\bar{a})\| << 1$.

Want) Define Quasi-Newton op

$$T(a) = a - AF(a)$$

w/ $A \approx DF(\bar{a})^{-1}$ & show

$T: \overline{B_r(\bar{a})} \to \overline{B_r(\bar{a})}$ is a contraction

First) How to define A?

Frechet derivative

$$DF(a)\cdot h = (K^2 + \varepsilon K + \alpha)h + 3\beta a*a*h$$
$$= Lh + N(a)\cdot h$$

Define (densely defined) function

$F: \ell^1_\nu \to \ell^1_\nu$ as

$$F(a) = (K^2 + \varepsilon K + \alpha)a + \beta a*a*a + \gamma b$$

where $b_n = \frac{\gamma}{2}\hat{e}_{-1} + \frac{\gamma}{2}\hat{e}_1$.

Galerkin Truncation. Def subspaces

$$\ell^1_{\nu,\mathbb{Z}} = X_N \times X_\infty$$
$$X_N = \{a \in \ell^1_{\nu,\mathbb{Z}} \mid a_n = 0 \text{ if } |n| \geq N+1\}$$
$$X_\infty = \{a \in \ell^1_{\nu,\mathbb{Z}} \mid a_n = 0 \text{ if } |n| \leq N\}$$

w/ projections

$\pi_N: \ell^1_\nu \to X_N$ & $\pi_\infty: \ell^1_\nu \to X_\infty$

Def $L = (K^2 + i\varepsilon K + \alpha)$

$$(L^{-1}h)_k = \frac{1}{-k^2 + i\varepsilon k + \alpha} h_k$$

$N(a)\cdot h = 3\beta a*a*h$

Note: L is unbounded on X_∞ cannot approx by finite.

$h \mapsto 3\beta a*a*h$ is bounded

Plan: Ignore convolution when approx inverting $DF(a)$ on X_∞

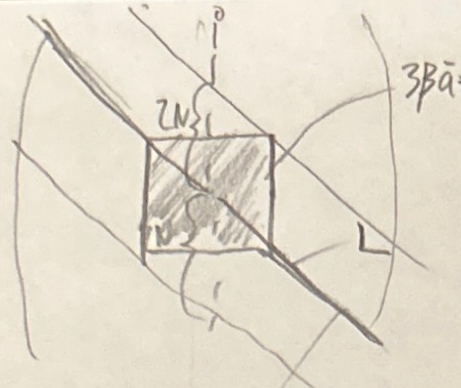
Keep On Board

Approx DF) $A^\dagger = \underbrace{\pi_N DF(\bar{a})\pi_N}_{\text{think} \to A^\dagger_N \in M_{2N+1}(\mathbb{C})} + L\pi_\infty$

Approx $(A^\dagger)^{-1}$) Numerically def $A_N \approx (A^\dagger_N)^{-1}$,

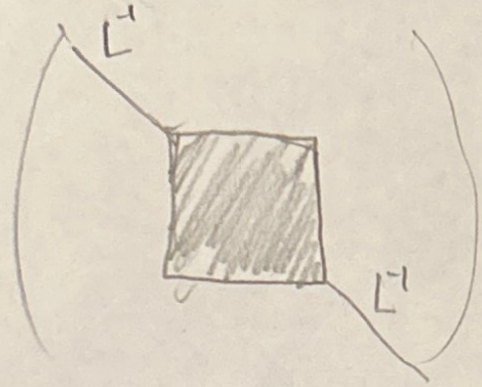
$A := A_N\pi_N + L^{-1}\pi_\infty$ ↑ thin

$DF(\bar a) =$



$A^\dagger =$

$A =$



Newton-Kantorovich Th'm

Fix $F\in C^1(X,Y)$, $\bar x\in X$, $A^\dagger\in B(X,Y)$

$A\in B(Y,X)$, H. Spz

Keep on Board

$Y_0 \geq \|AF(\bar x)\|_X$

$Z_0 \geq \|I - AA^\dagger\|_{B(X)}$

$Z_1 \geq \|A[DF(\bar x) - A^\dagger]\|_{B(X)}$

$Z_2(r) \geq \sup_{c\in B_r(\bar x)} \|A[DF(c) - DF(\bar x)]\|_{B(X)}$

$X = \ell^1_\nu$, $Y = \ell^1_{\nu'}$

$\bar x = \bar a$ $\quad \nu > \nu'$

Def rad. poly

$$p(r) = (Z_0 + Z_1 + Z_2(r))r - r + Y_0$$

If $\exists\, r_0 > 0$ st $p(r_0) < 0$ then $\exists!$ $\tilde x \in \overline{B_{r_0}(\bar x)}$ s.t. $F(\tilde x) = 0$.

Y_0 | $\|AF(\bar a)\|_{\ell^1_\nu} = \|A\{(F(\bar a))_n\}^{3N}_{n=-3N}\|$

$= \|A_N F_N(\bar a)\|_{\ell^1_\nu} + \|L^{-1}\pi_\infty F(a)\|_{\ell^1_\nu}$

$= \|A_N F_N(\bar a)\|_{\ell^1_\nu} + \sum_{N<|n|\leq 3N} \left|\frac{F(\bar a)_n}{n^2 + i\epsilon n + \alpha}\right| \nu^{|n|}$

$=: Y_0$

Z_0 | $\|I - (A_N\pi_N + L^{-1}\pi_\infty)(A_N^\dagger\pi_N + L\pi_\infty)\|_{B(\ell^1_\nu)}$

$= \|\cancel{(\pi_N + \pi_\infty)} - (A_N A_N^\dagger \pi_N + \cancel{L^{-1}L\pi_\infty})\|_{B(\ell^1_\nu)}$

$= \|\underbrace{\pi_N - A_N A_N^\dagger \pi_N}_{Mat_{2N+1}(\mathbb{F})\text{ - norm}}\|_{B(\ell^1_\nu)}$

Z_2 | $\cancel{\sup_{c\in B_r(\bar a)} \|A[DF(c) - DF(\bar a)]\|_{B(\ell^1_\nu)}}$

$= \sup_{c\in B_r(\bar a)} \|A[$

$\boxed{Z_2(r)}$ $\sup_{c \in B_r(\bar a)} \| A[DF(c) - DF(\bar a)] \|_{B(\ell^1_\nu)}$

$\le \sup_{c \in B_r(\bar c)} \sup_{\|h_1\|=1} \| A[(Lh_1 + 3\beta c * c * h_1) - (Lh_1 + 3\beta \bar a * \bar a * h_1)\|_{\ell^1_\nu}$

$\le \sup_{c \in B_r(\bar a)} \sup_{\|h_1\|=1} 3\beta \|A\| \cdot \| [(c * c - \bar a * \bar a) * h_1]\|$, $\quad c = \bar a + h_2$

$\le \sup_{h_2 \in B_r(0)} \sup_{\|h_1\|=1} 3\beta \|A\| \, \| (\bar a + h_2) * (\bar a + h_2) - \bar a * \bar a \|_{\ell^1_\nu} \|h_1\|_{\ell^1_\nu}$

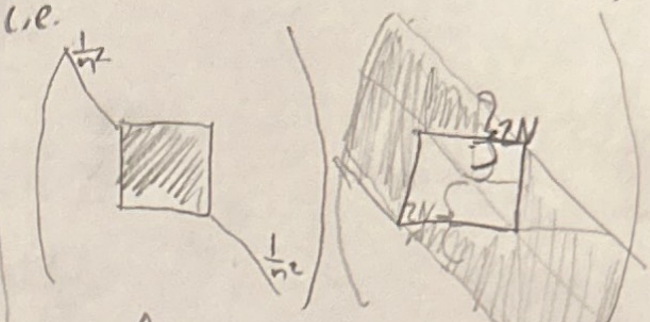
$= \sup_{h_2 \in B_r(0)} 3\beta \|A\| \, \| \cancel{\bar a * \bar a} - 2\bar a * h_2 + h_2 * h_2 - \cancel{\bar a * \bar a} \|$

$\le 3\beta \|A\| \left(2\|\bar a\| r + r^2 \right)$,

$\cancel{M = DF(\bar a) - A^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}}$

$\boxed{Z_1}$ $\| A[DF(\bar a) - A^+] \|_{B(\ell^1_\nu)}$

i.e.



A $\qquad$ $DF(\bar x) - A^+$

Problem! Both Matrices can be large!

$\boxed{\text{E.g.}}$ Work with toy basis $\hat e_0, \hat e_1, \hat e_2$

$DF(\bar a) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 8 & 2 \\ 1 & 2 & 100 \end{pmatrix}$, $A^+ = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 0 \\ 0 & 0 & 100 \end{pmatrix}$, $A = \begin{pmatrix} 2 & -\frac12 & 0 \\ -\frac12 & \frac14 & 0 \\ 0 & 0 & \frac{1}{100} \end{pmatrix}$

$M = DF(\bar a) - A^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Note $\|A\|_1, \|M\|_1, \|AM\|_1 \ge 1$

$AM = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{100} & \frac{2}{100} & 0 \end{pmatrix} = B$

Use weighted operator norm!

Lemma $\|B\|_{\ell^1_\nu} = \max_j \frac{1}{\nu^{|j|}} \sum_i |b_{ij}| \nu^{|i|}$

$\|AM\|_{\ell^1_\nu} = \max \left\{ \frac{1}{\nu^1} \frac{\nu^3}{100}, \frac{1}{\nu^2} \frac{2\nu^3}{100}, \frac{1}{\nu^3} \nu \right\}$ $\sim N$

$= \max \left\{ \frac{\nu^2}{100}, \frac{2\nu}{100}, \frac{1}{\nu^2} \right\}$

ν	$\|AM\|_{\ell^1_\nu}$	
1	1	Not enough
2	1/4	just Right
10	1	too Much

$\underline{Z_1}$ | Fix $h \in \ell^1_{\nu,\mathbb{Z}}$, $\|h\|_{\ell^1_\nu} \le 1$

Write $h = \pi_N h + \pi_\infty h =: h^{(N)} + h^\infty$

$z = [DF(\bar a) - A^\dagger] \cdot h =: z^{(N)} + z^\infty$

$w = Az$

$= A_N z^{(N)} + L^{-1} z^\infty$

$=: w^{(N)} + w^\infty$

Want: $\|w^{(N)}\| \le Z_1^a$, $\|w^\infty\|_{\ell^1_\nu} \le Z_1^b$

Recall $DF(\bar a)\cdot h = Lh + N(\bar a)\cdot h$

$A^\dagger \cdot h = Lh + \pi_N N(\bar a) \cdot \pi_N h$

$$z = \underbrace{\pi_N N(\bar a) \cdot \pi_\infty h}_{z^{(N)}} + \underbrace{\pi_\infty N(\bar a) \cdot h}_{z^\infty}$$

$\|w^\infty\|_{\ell^1_\nu} = \|L^{-1}(\pi_\infty N(\bar a)\cdot h)\|_{\ell^1_\nu}$

$\le \|L^{-1}\pi_\infty\|_{op} \cdot \|3\beta \bar a * \bar a * h\|_{\ell^1_\nu}$

$\le \|L^{-1}\pi_\infty\|_{op} \cdot 3\beta \|\bar a\|^2_{\ell^1_\nu} \cdot \|h\|_{\ell^1_\nu}$

1

For $b \in \ell^1_\nu$,

Note $(L^{-1}\pi_\infty b)_n = \begin{cases} 0 & \text{if } |n| \le N \\ \dfrac{b_n}{-n^2 + i\varepsilon n + \alpha} & \text{if } |n| \ge N+1 \end{cases}$

Then $\|L^{-1}\pi_\infty\| \le \dfrac{1}{N^2 - \alpha}$

$$\|w^\infty\|_{\ell^1_\nu} \le \frac{1}{N^2 - \alpha} \cdot 3\beta \|\bar a\|^2_{\ell^1_\nu} = Z_1^b$$

~~Z_1^a Plan | For diagonal Matrix~~

~~$(\Gamma_\nu^{(N)})_{kj} = \delta_{kj}\, \nu^{-2(N+1-|k|)}$~~

~~$w^{(N)} = A_N z^{(N)}$~~

~~$= A_N \Gamma_\nu^{(N)} \Gamma^{-1} z^{(N)}$~~

~~$\|w^{(N)}\| \le \|A_N \Gamma\| \cdot \|\Gamma^{-1} z^{(N)}\|$~~

Note, $h \mapsto z^{(N)} = \pi_N N(\bar a) \cdot \pi_\infty h$

$= 3\beta \pi_N (\bar a * \bar a * h^\infty)$

Note $\bar a = \pi_N \bar a$, $\bar a * \bar a = \pi_{2N}(\bar a * \bar a)$

If $\pi_{3N} h^\infty = 0 \Rightarrow \pi_N(\bar a * \bar a * h^\infty) = 0$

$\Longrightarrow$

$z^{(N)} = 3\beta \pi_N (\bar a * \bar a * (1 - \pi_N) h)$

$= 3\beta \pi_N (\bar a * \bar a * (\pi_{3N} - \pi_N) h)$

Hence $h \mapsto z^{(N)}$ can be rep by finite Matrix

$$Z_1^a = 3\beta \sup_{N+1 \le |j| \le 3N} \frac{1}{\nu^{|j|}} \|A_N \cdot \pi_N (\bar a * \bar a * e_j)\|$$