Functional Framework of Chebyshev

$$\begin{tikzcd}
\text{Functions on } [-1,1] \arrow[r, shift left, "\mathcal{C}:\text{Chebyshev transform}"] \arrow[d, "\text{evaluate}"] & \text{Sequence Space} \arrow[l, shift left, "\mathcal{C}^{-1}"] \arrow[d, "\text{proj}^n"] \\
\text{Values on Grid} \arrow[r, shift left, "\text{fft}"] & \text{Finite Sequence} \arrow[l, shift left, "\text{ifft}"]
\end{tikzcd}$$

New: For CAP, it is better to integrate than differentiate.

Def| Chebyshev Transforms

$\mathcal{C} : L^2([-1,1],\mathbb{R}) \to S_{\mathbb{N}}(\mathbb{R})$,

$f \xrightarrow{\mathcal{C}} \{a_n\}_{n=0}^{\infty}$ $\qquad a_n = \dfrac{\langle T_n, f\rangle}{\langle T_n, T_n\rangle}$

Inverse Cheby. trans

$\mathcal{C}^{-1} : \ell^1_{1,\mathbb{N}} \to C^0([-1,1],\mathbb{R})$

$a \xrightarrow{\mathcal{C}^{-1}} f \qquad f(t) = \sum_{n=0}^{\infty} a_n T_n(t)$

proof| $\sup_{t\in[-1,1]} |T_n(t)| = 1$

& $\{a_n\}$ converges absolutely.

Def| Chebyshev Poly.

$$T_0(t) = 1$$
$$T_1(t) = t$$
$$T_{k+1}(t) = 2t\,T_k(t) - T_{k-1}(t).$$

Orthogonal Poly: w/r inner product $\langle\ ,\ \rangle$ for $f, g \in L^2([-1,1],\mathbb{R})$

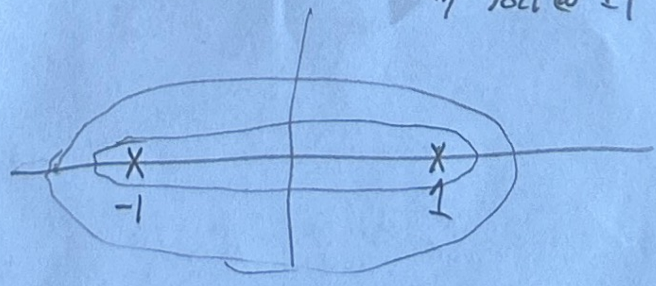
$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(t)\,g(t)}{\sqrt{1-t^2}}\,dt$$

$$\langle T_k, T_j\rangle = \begin{cases} 0 & \text{if } j \neq k \\ \pi & \text{if } k = j = 0 \\ \pi/2 & \text{if } k = j > 0 \end{cases}$$

Thm| If f is Real analytic, $\mathcal{C}(f) \in \ell^1_r$, w/ $r > 1$

If $a \in \ell^1_r$, $r > 1$, then $\mathcal{C}^{-1}(a)$ is Real Analytic

Domain of Analyticity: Bernstein Ellipse w/ foci @ ± 1



Less affected by singularity in $\mathbb{C}$!

Goal | Translate $\dot{x}=f(x)$ to eq. on seq

Derivatives

$$n \geq 2$$

$$2T_n(x) = \frac{1}{n+1}\frac{d}{dx}T_{n+1}(x) - \frac{1}{n-1}\frac{d}{dx}T_{n-1}(x)$$

$$\frac{d}{dx}T_{n+1}(x) = 2(n+1)T_n(x) + 2\frac{n+1}{n-1}\left[\frac{d}{dx}T_{n-1}(x)\right]$$

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$$\left[2(n-1)T_{n-2}(x) + \frac{d}{dx}\cdots\right]$$

Derivative Matrix in Sequence Space

$$\mathcal{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ & 0 & 4 & 0 & \\ & & & 6 & \\ & 0 & & & \end{bmatrix}$$

Equivalent to Integral on Higher modes $k \geq 2$

$$\langle T_k, \int C^{-1}(a)\,dx \rangle = \langle T_k, C^{-1}(f(a)) \rangle$$

Diff Eq | $x'(t) = f(x(t))$, $x(t_0) = x_0$

Integral Eq | $x(t) = x_0 + \int_{t_0}^{t} f(x(s))\,ds$

☆ turn page

Nonlinearity | Chebyshev ~ Fourier in disguise

$$T_k(\cos(\theta)) = \cos(k\theta)$$

$$k \geq j$$

$$T_k(\cos\theta)\,T_j(\cos\theta) = \cos(k\theta)\cos(j\theta)$$

$$= \frac{1}{2}\left[\cos((k+j)\theta) + \cos((k-j)\theta)\right]$$

$$= \frac{1}{2}$$

$\mathcal{D}$ is a (full ish) upper triangular Matrix

Hard to invert in ∞-dim.

(in finite dim, also need boundary cond),

Integrate!

$$\int 2T_n(x)\,dx = \int\left(\frac{1}{n+1}\frac{d}{dx}T_{n+1}(x) - \frac{1}{n-1}\frac{d}{dx}T_{n-1}(x)\right)dx$$

$$= \frac{1}{n+1}T_{n+1}(x) - \frac{1}{n-1}T_{n-1}(x) + C$$

Def Shift $(\sigma^+(a))_{n+1} = a_n$, $(\sigma^-(a))_{n-1} = a_n$

Def Pseudo Diff $(Ka)_k = k a_k$

Def Pseudo Integral $\mathcal{L}a = \frac{1}{2}K^{-1}(\sigma^+ - \sigma^-)$

$$= \frac{1}{2}\left[T_{k+j}(\cos\theta) + T_{k-j}(\cos\theta)\right]$$

$\boxed{x = \cos\theta}$

$$T_k(x)\,T_j(x) = \frac{1}{2}\left(T_{k+j}(x) + T_{k-j}(x)\right)$$

Def | Product $*: \ell^1_{\nu,\mathbb{N}} \times \ell^1_{\nu,\mathbb{N}} \to \ell^1_{\nu,\mathbb{N}}$

For $a, b \in \ell^1_{\nu,\mathbb{N}}$ as

$$(a*b)_k = \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \in \mathbb{Z}}} a_{|k_1|}\,b_{|k_2|}$$

Takeaway | We can evaluate products in Chebyshev using FFT

A $x(-1) = x_L$ or $x(1) = x_R$?

Note $T_n(1) = 1$

$T_n(-1) = (-1)^n$

$$x(\pm 1) = \sum_{n=0}^{\infty} a_n T_n(\pm 1)$$

$$= \sum_{n=0}^{\infty} a_n (\pm 1)^n$$

Def: linear Map $\Gamma_{L/R} : \ell^1_{\nu,\mathbb{N}} \to \mathbb{R}$

$\cancel{R}$ $\Gamma_L(a) = \sum_{n=0}^{\infty} (-1)^n a_n$, $\Gamma_R(a) = \sum_{n=0}^{\infty} a_n$

Similar to phase condition in e-val / periodic orbit, Banach Algebra?

$\|a\|_{\ell^1_{v,\mathbb{N}}} = \sum_{k=0}^{\infty} |a_k| \, v^k$

$R: \ell^1_{v,\mathbb{N}} \to \ell^1_{v,\mathbb{Z}}$

$\{a_k\}_{k=0}^{\infty} \to \{a_{|k|}\}_{k\in\mathbb{Z}}$

$S: \ell^1_{v,\mathbb{Z}} \to \ell^1_{v,\mathbb{N}}$

$\{a_k\}_{k\in\mathbb{Z}} \longrightarrow \{a_k\}_{k=0}^{\infty}$

$\|R\| = 2$, $\|S\| = 1$

$*_{Ch} \sim$ Chebyshev conv | $*_F \sim$ Fourier conv

$\|a *_{Ch} b\|_{\ell^1_{v,\mathbb{N}}} = \|S((Ra)*(Rb))\|_{\ell^1_{v,\mathbb{N}}}$

$\le \|(Ra)*(Rb)\|_{\ell^1_{v,\mathbb{Z}}}$

$\le \|Ra\|_{\ell^1_{v,\mathbb{Z}}} \|(Rb)\|_{\ell^1_{v,\mathbb{Z}}}$

$\boxed{\|a*b\|_{\ell^1_{v,\mathbb{N}}} \le 4 \|a\|_{\ell^1_{v,\mathbb{N}}} \|b\|_{\ell^1_{v,\mathbb{N}}}}$

Essentially a Banach Algebra

(We could Redefine the norm to make it a true B-algebra.

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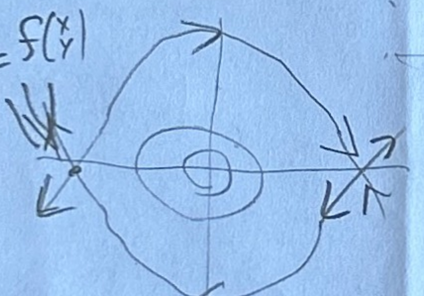
Converting BVP → Sequence Eq

Consider

$\begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix} y \\ \frac{-x+x^3}{2}\end{pmatrix} = f\begin{pmatrix}x\\y\end{pmatrix}$



Eq $\tilde{x} = \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}$

@ $\tilde{X} = \begin{pmatrix}\pm 1\\0\end{pmatrix}$ Eval, or

$Df(X) = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\lambda = \pm 1$, $v = \begin{pmatrix}\pm 1\\1\end{pmatrix}$

$W^{s,u}_{\tilde{x}}(\tilde{X})$ = stable/unstable manifolds of $\tilde{X}$

$= \left\{ \begin{pmatrix}x\\y\end{pmatrix} \;\middle|\; \begin{pmatrix}x(t)\\y(t)\end{pmatrix} \to \tilde{X} \right.$ as $t \to \infty$ (s), $t \to -\infty$ (u)

Stable/Unstable Evect are 1^{st} order approx of $W^{s,u}(\tilde{X})$

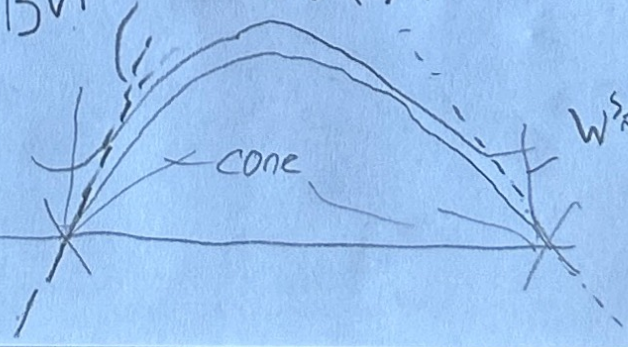
HeteroClinic Orbit:

A Solution $X(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}$ s.t.

$X(t) \to \begin{pmatrix}1\\0\end{pmatrix}$ as $t \to +\infty$

$X(t) \to \begin{pmatrix}-1\\0\end{pmatrix}$ as $t \to -\infty$

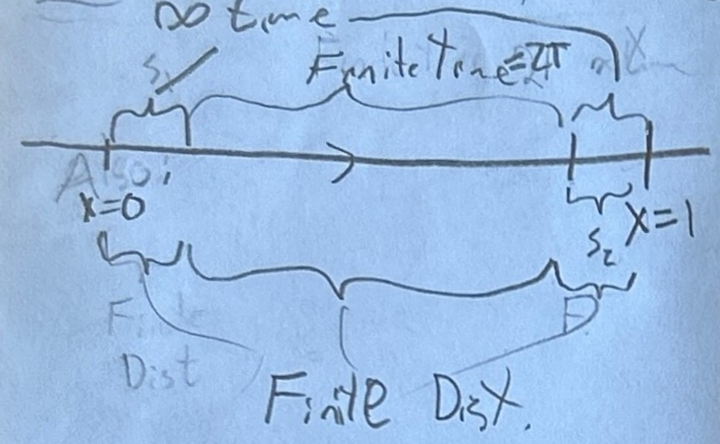
BVP $\approx W^u = \begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} s_1$ $s_1 \in \mathbb{R}$



$W^s \approx \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-1\\1\end{pmatrix} s_2$

What is time of flight? ..... ∞!

Also $x' = x(1+x)$, $x(t) = \frac{1}{1+e^{-t}}$



Set up BVP

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ \frac{-x + x^3}{2} \end{pmatrix}$$

$$\begin{pmatrix} x(-T) \\ y(-T) \end{pmatrix} = W^u(s_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_1$$

$$\begin{pmatrix} x(T) \\ y(T) \end{pmatrix} = W^s(s_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} s_2$$

Computational (hyper) parameters

T, s_1, s_2

Need to fix 2, solve for 1

Fix	Solve
s_1, s_2	T
s_1, T	s_2
...	

Spz $s_1 = s_2 = \varepsilon$, say 10^{-5}.

$$\vec{W}^u = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \varepsilon \in \mathbb{R}^2$$

$$\vec{W}^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \varepsilon \in \mathbb{R}^2$$

Chebyshev is on $[-1,1]$

BVP is on $[-T,T]$

Rescale Time $\tau \mapsto Tt$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = T \begin{pmatrix} y \\ \frac{-x+x^3}{2} \end{pmatrix}, \quad t \in [-1,1]$$

$$\begin{pmatrix} x(-1) \\ y(-1) \end{pmatrix} = \vec{W}^u$$

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \vec{W}^s$$

In sequence Space

$\mathcal{C}(x) = a$, $\mathcal{C}(y) = b \in \ell^1_{\nu,N}$

$$\begin{pmatrix} x(-1) \\ y(-1) \end{pmatrix} = \begin{pmatrix} \sum a_n T_n(-1) \\ \sum b_n T_n(-1) \end{pmatrix} = \begin{pmatrix} \sum a_n (-1)^n \\ \sum b_n (-1)^n \end{pmatrix} \overset{\text{want}}{=} \vec{W}^u$$

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} \sum a_n \\ \sum b_n \end{pmatrix} \overset{\text{want}}{=} \vec{W}^s$$

Integral Eq

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = T \int^t \begin{pmatrix} y(s) \\ \frac{-x(s) + x(s)^3}{2} \end{pmatrix} ds$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{T}{2} \frac{1}{2} K^{-1} (\sigma^+ - \sigma^-) \begin{pmatrix} b \\ \frac{-a + a*a*a}{2} \end{pmatrix}$$