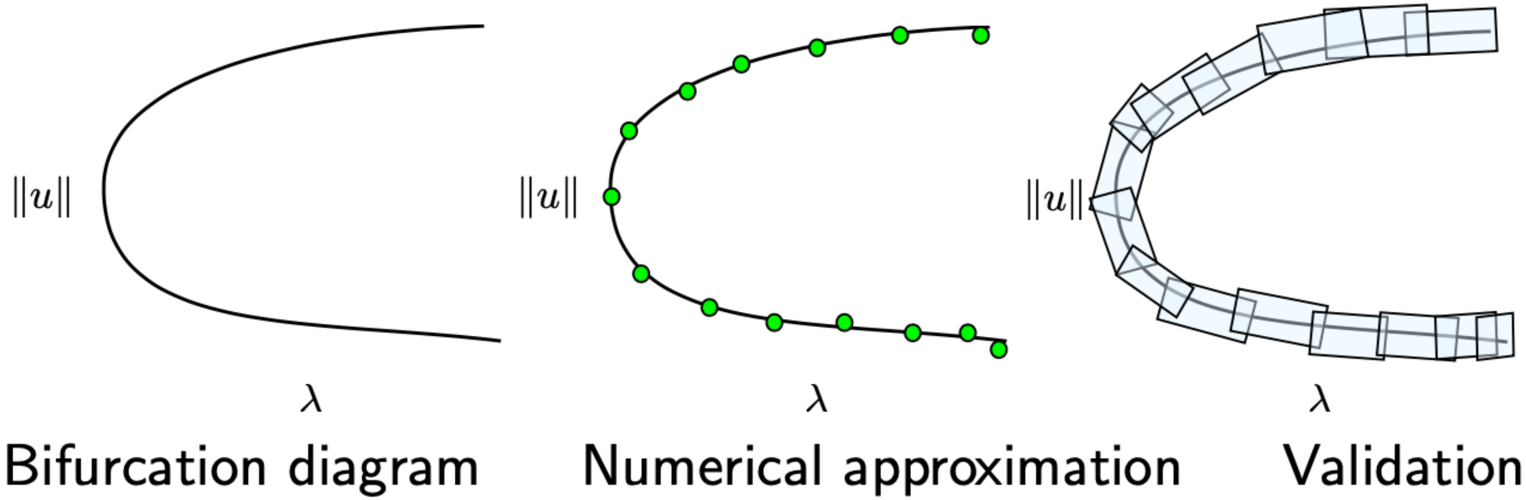
Summary/Looking Forward: Current Trends in the Field



Computer-Assisted Proofs in Applied Mathematics

Organizers

Jonathan Jaquette
New Jersey Institute of Technology

Evelyn Sander
George Mason University

Teaching Assistants

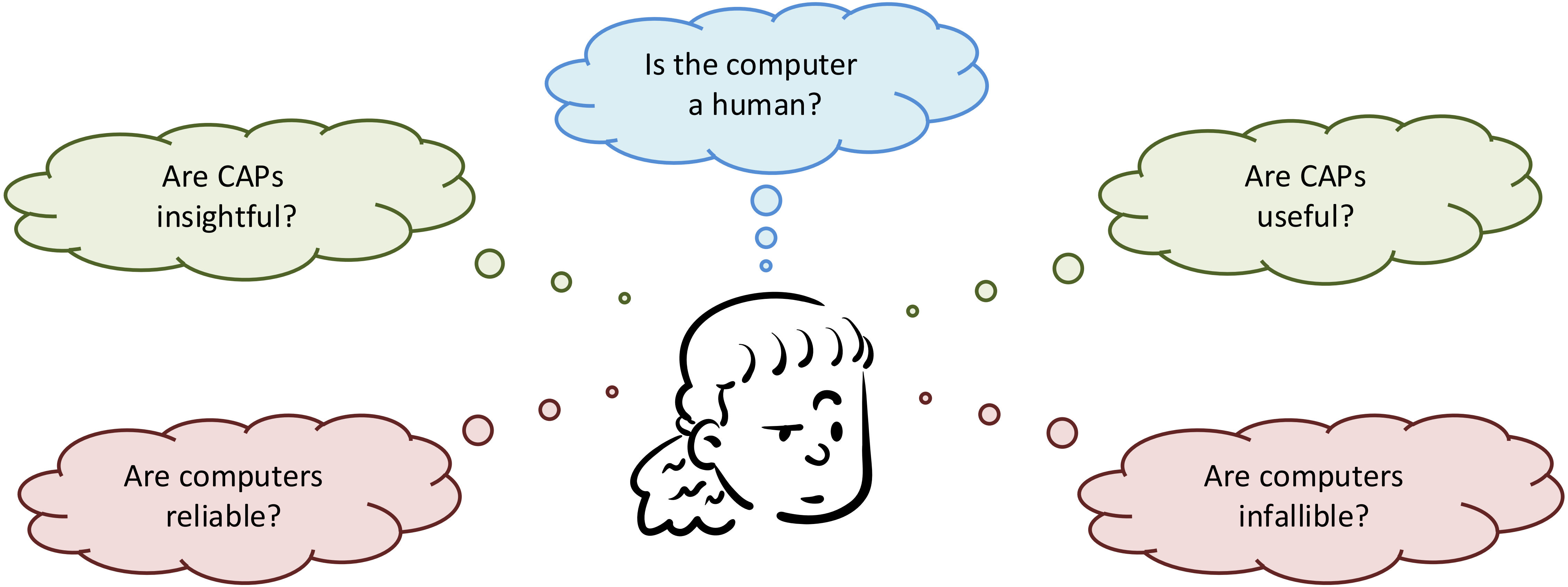
Alanna Haslam-Hyde
Boston University

Michael Storm
New Jersey Institute of Technology

What is a Computer Assisted Proof?

My Definition: *A proof involving computations.*

e.g. 109 is prime; $9 < \pi^2 < 10$



Stages of the CAP

Neumaier's definition:

Stage 1
- Qualitative understanding reduces the problem to a finite number of subproblems

Stage 2
- Reduce each subproblem to a finite number of equations and/or inequalities to be checked

Stage 3
- A specialized computational algorithm is executed

Periodic orbits in Duffing oscillator

Stage 1
- Use Fourier analysis to convert the problem to a $F(x) = 0$

Stage 2
- Derive formulas for ϵ, δ, γ bounds so that a solution exists if $\exists\, r_* > 0$ s.t.
$$\epsilon + \delta r_* + \gamma(r_*) r_*^2 < r_*$$

Stage 3
- Compute everything with interval arithmetic

Typical CAP of $f(x) = 0$

Approximate Solution

- Asymptotic expansion
- Standard numerics
- Machine learning

+

Explicit Theorem

- Proof checks a FINITE # of conditions / inequalities
- Not " … $\exists \epsilon > 0$ … "

→

Computer Assisted Proof

Week 1

- Essential Methods
  - Interval arithmetic, definite integrals, matrix algorithms
- Types of problems we'll solve
  - $\min_{x \in X} f(x)$
  - $f(x) = 0$
- Applications:
  - Trefethen's 100 digit challenge
  - Nonlinear ODEs
- Representing functions
  - Taylor series, Fourier series

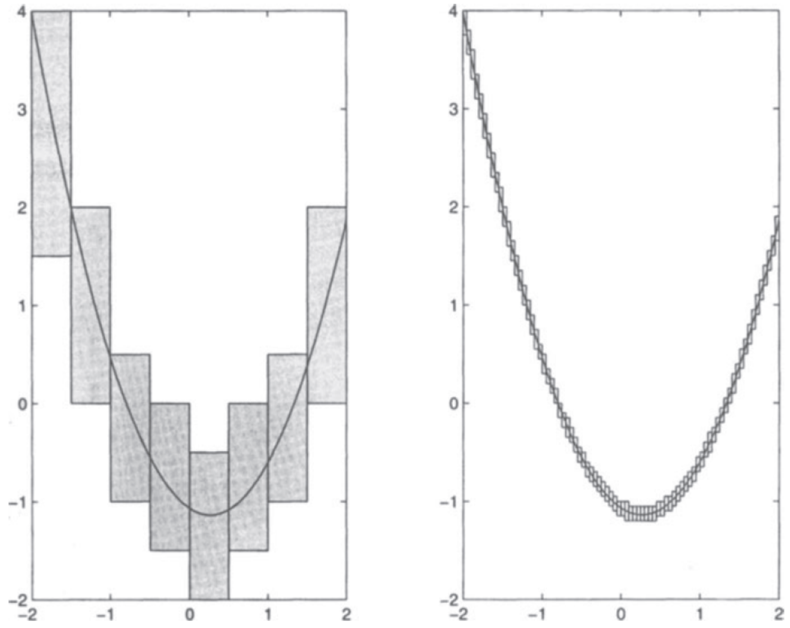


Fig. 5.3. Graphs of the multivalued approximation to $f(x) = (3x - 4)(5x + 4)/15$ obtained by means of interval arithmetic based on different basic lengths: 0.5 for the left graph and 0.05 for the right graph.

Interval Arithmetic

- There is a theory of interval arithmetic in one and higher dimensions.
  - Gives enclosures of the exact answer, but can have overestimation
- Implementation: Intlab
- Other implementations:
  - CAPD (Computer Assisted Proofs in Dynamics) http://www.capd.ii.uj.edu.pl
  - Julia interval arithmetic package and more specifically RadiiPolynomial
  - C++ interval arithmetic library

Norm bounds, integration and root finding

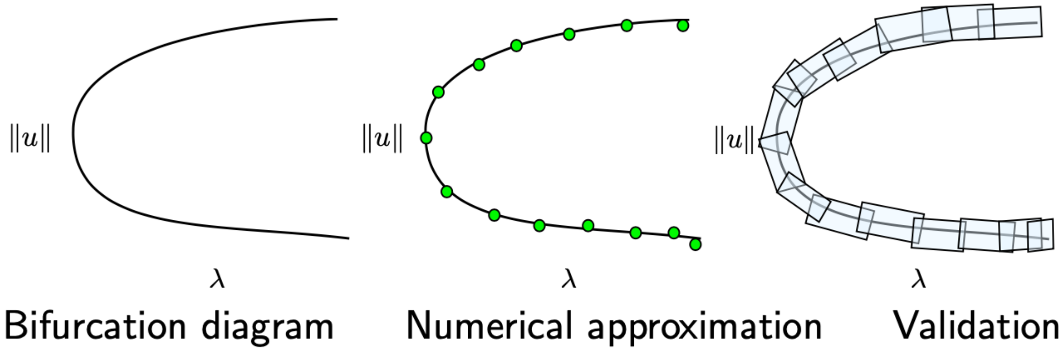
- Matrix challenge problem:
  - Bound a norm of a truncated matrix, and then bound the tail of the missing pieces
- Integral challenge problem:
  - Use rigorous integration to get bounds on a (crazy complicated) integral
- Period three implies chaos:
  - find periodic orbits for the logistic map by rewriting it as a root finding problem and rigorously solving

Series solutions and their tails

- Henon Heiles equation:
  - Represent integration of an ODE using a (truncated) Taylor series and then bound the tail
- Duffing equation:
  - Represent a periodic orbit as a truncated Fourier series and then bound the tail

Week 2

- Additional Topics:
  - Infinite dimensional CAPs
  - Continuation, Bifurcation, PDEs
- Group Projects
  - Water waves, pattern formation, stability, bifurcations, blowup, chaos



Continuation, bifurcation, BVP, Chebyshev

- Rewrite parametrized problems as root finding problems and validate
- Rewrite bifurcation problems as extended systems which are root finding problems, along with a few nondegeneracy conditions
- Solutions to boundary value problems and PDEs can be represented as a truncated series, and we bound the tails
- Chebyshev polynomials are another type of series: trigonometric polynomials in disguise, can be validated

A few non-systematic recent examples

- Disclaimer of bias: We give a few examples of interesting papers (and some pretty pictures).
- This not a systematic survey!!

Rigorous numerical integration

Asked 8 years, 8 months ago Active 8 years, 8 months ago Viewed 1k times

12

3

I need to evaluate some (one-variable) integrals that neither SAGE nor Mathematica can do symbolically. As far as I can tell, I have two options:

(a) Use GSL (via SAGE), Maxima or Mathematica to do numerical integration. This is really a non-option, since, if I understand correctly, the "error bound" they give is not really a guarantee.

(b) Cobble together my own programs using the trapezoidal rule, Simpson's rule, etc., and get rigorous error bounds using bounds I have for the second (or fourth, or what have you) derivative of the function I am integrating. This is what I have been doing.

Is there a third option? Is there standard software that does (b) for me?

na.numerical-analysis

Share Cite Edit Follow Flag

asked Mar 5 '13 at 23:03

H A Helfgott

15.6k 2 35 103

[Submitted on 30 Dec 2013 (v1), last revised 17 Jan 2014 (this version, v2)]

The ternary Goldbach conjecture is true

H. A. Helfgott

The ternary Goldbach conjecture, or three-primes problem, asserts that every odd integer n greater than 5 is the sum of three primes. The present paper proves this conjecture.
Both the ternary Goldbach conjecture and the binary, or strong, Goldbach conjecture had their origin in an exchange of letters between Euler and Goldbach in 1742. We will follow an approach based on the circle method, the large sieve and exponential sums. Some ideas coming from Hardy, Littlewood and Vinogradov are reinterpreted from a modern perspective. While all work here has to be explicit, the focus is on qualitative gains.
The improved estimates on exponential sums are proven in the author's papers on major and minor arcs for Goldbach's problem. One of the highlights of the present paper is an optimized large sieve for primes. Its ideas get reapplied to the circle method to give an improved estimate for the minor-arc integral.

By Cauchy-Schwarz, this is at most

$$\sqrt{\frac{1}{2\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\left|\frac{L'(s,\chi)}{L(s,\chi)}\cdot\frac{1}{s}\right|^2|ds|}\cdot\sqrt{\frac{1}{2\pi}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}|G_\delta(s)s|^2\,|ds|}$$

By (4.12),

$$\begin{aligned}\sqrt{\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\left|\frac{L'(s,\chi)}{L(s,\chi)}\cdot\frac{1}{s}\right|^2|ds|} &\leq \sqrt{\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\left|\frac{\log q}{s}\right|^2|ds|}\\ &\quad+\sqrt{\int_{-\infty}^{\infty}\frac{\left|\frac{1}{2}\log\left(\tau^2+\frac{9}{4}\right)+4.1396+\log\pi\right|^2}{\frac{1}{4}+\tau^2}d\tau}\\ &\leq \sqrt{2\pi}\log q+\sqrt{226.844},\end{aligned}$$

where we compute the last integral numerically.[4]

[4]By a rigorous integration from $\tau = -100000$ to $\tau = 100000$ using VNODE-LP [Ned06], which runs on the PROFIL/BIAS interval arithmetic package[Knü99].

Current topics in the field:

- Dynamical systems
  - Maps ODEs PDEs DDEs Integrodifferential Equations
- What to study/validate
  - Equilibria and stability
  - Solving initial value problems
  - Periodic solutions
  - Quasi periodic solutions
  - Chaos
  - Stable/unstable manifold, heteroclinic orbit, homoclinic orbit,
  - Blow up of solutions

Current topics in the field:

- PDEs
  - Type of Solution
  - Type of Domain
    - Fourier - Periodic, Neumann boundary conditions,
    - Other spectral Methods : Spherical, radial, unbounded
    - With finite elements, anything!
  - Type of equation
    - Semi-linear
    - Emerging work on quasi linear

Manifolds using parameterization method

2017, Volume 4: 21-70. Doi: 10.3934/jcd.2017002

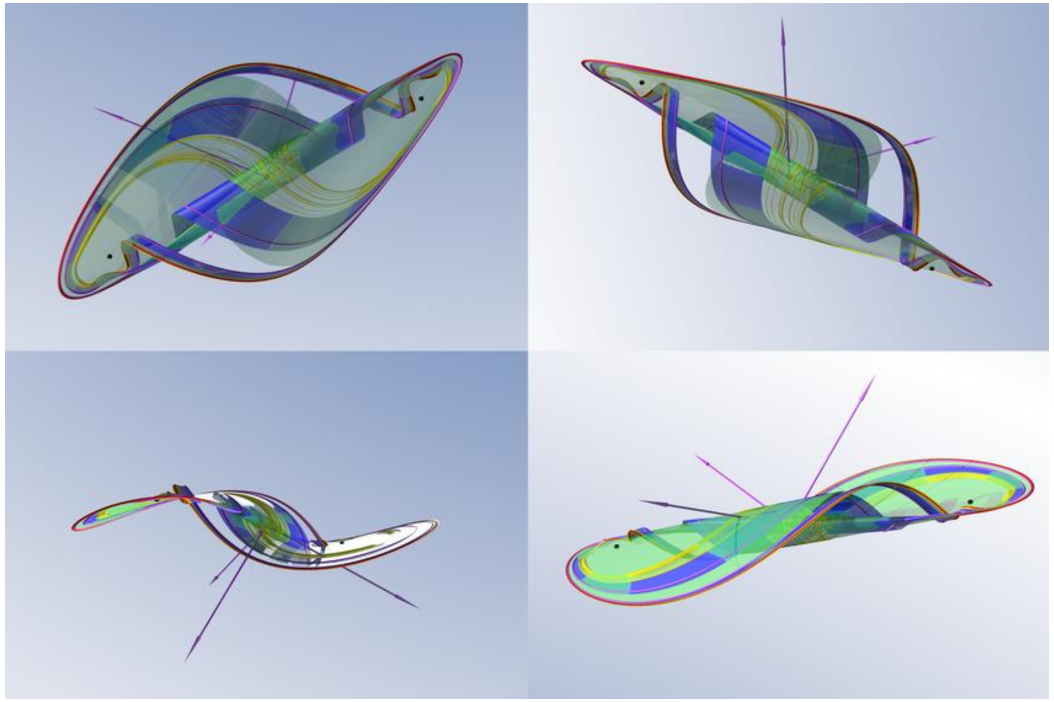
This issue < Previous Article Next Article >

Parameterization method for unstable manifolds of delay differential equations

C. M. Groothedde[1] and J. D. Mireles James[2]

1. Vrije Universiteit Amsterdam, Department of Mathematics, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands
2. Florida Atlantic University, Department of Mathematical Sciences, 777 Glades Road, Boca Raton, FL 33431, USA

Published: March 2018

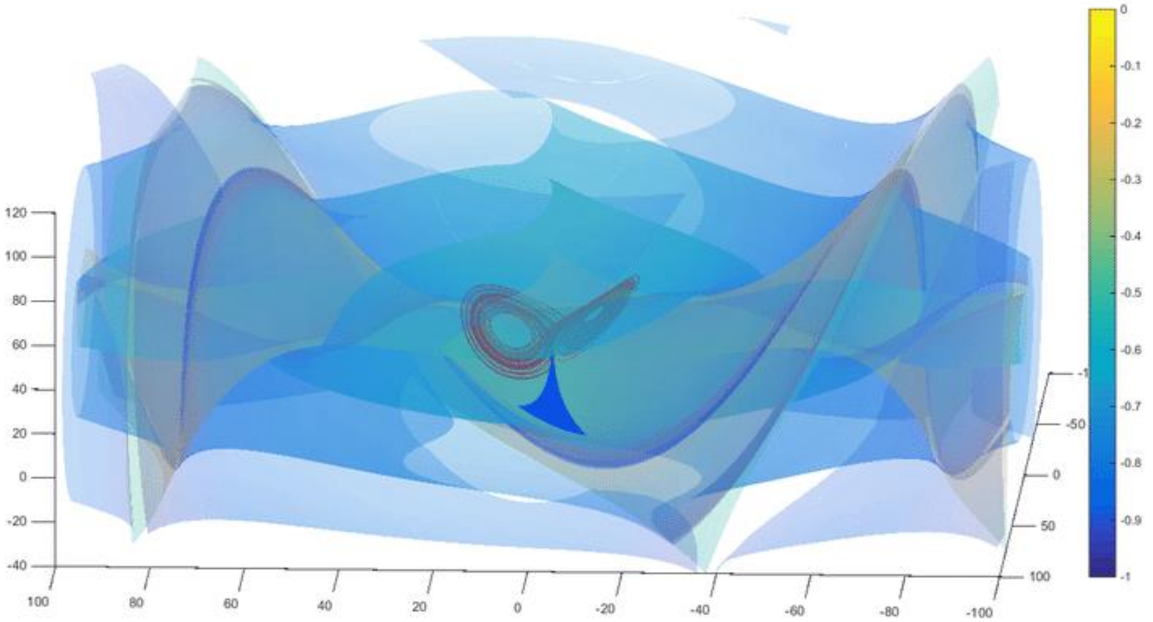


Use series solutions to describe manifolds

SIAM J. Applied Dynamical Systems
Vol. 17, No. 1, pp. 157–202

Analytic Continuation of Local (Un)Stable Manifolds with Rigorous Computer Assisted Error Bounds*

William D. Kalies†, Shane Kepley†, and J. D. Mireles James†



Chaos via geometric horseshoes or manifolds

Some classic papers

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, January 1995

CHAOS IN THE LORENZ EQUATIONS: A COMPUTER-ASSISTED PROOF

C. R. Acad. Sci. Paris, t. 328, Série I, p. 1197-1202, 1999
Systèmes dynamiques/*Dynamical Systems*

The Lorenz attractor exists

Warwick TUCKER

Recent work

Symbolic dynamics for the Kuramoto-Sivashinsky PDE on the line: connecting orbits between periodic orbits

Daniel Wilczak and Piotr Zgliczyński[1]

Jagiellonian University, Faculty of Mathematics and Computer Science,
Łojasiewicza 6, 30–348 Kraków, Poland
e-mail: {Daniel.Wilczak,Piotr.Zgliczynski}@uj.edu.pl

March 25, 2025

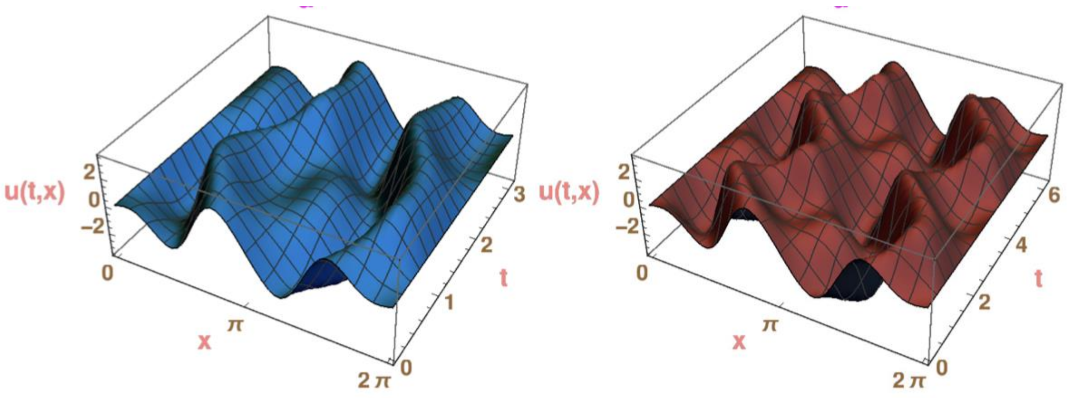


Figure 1: Two approximate time-periodic orbits u^1 and u^2.

Celestial mechanics

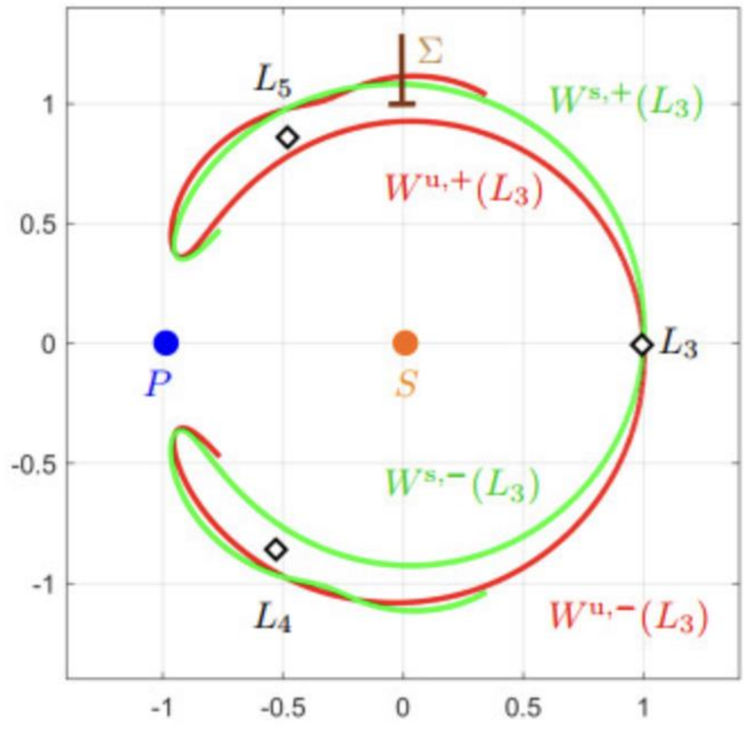
- A larger literature of celestial mechanics
- Of interest to very applied people (eg. NASA, JPL)

2025, Volume 45, Issue 1: 56-88. Doi: 10.3934/dcds.2024086 This issue < Previous Article Next Articl

Breakdown of homoclinic orbits to L_3: Nonvanishing of the Stokes constant

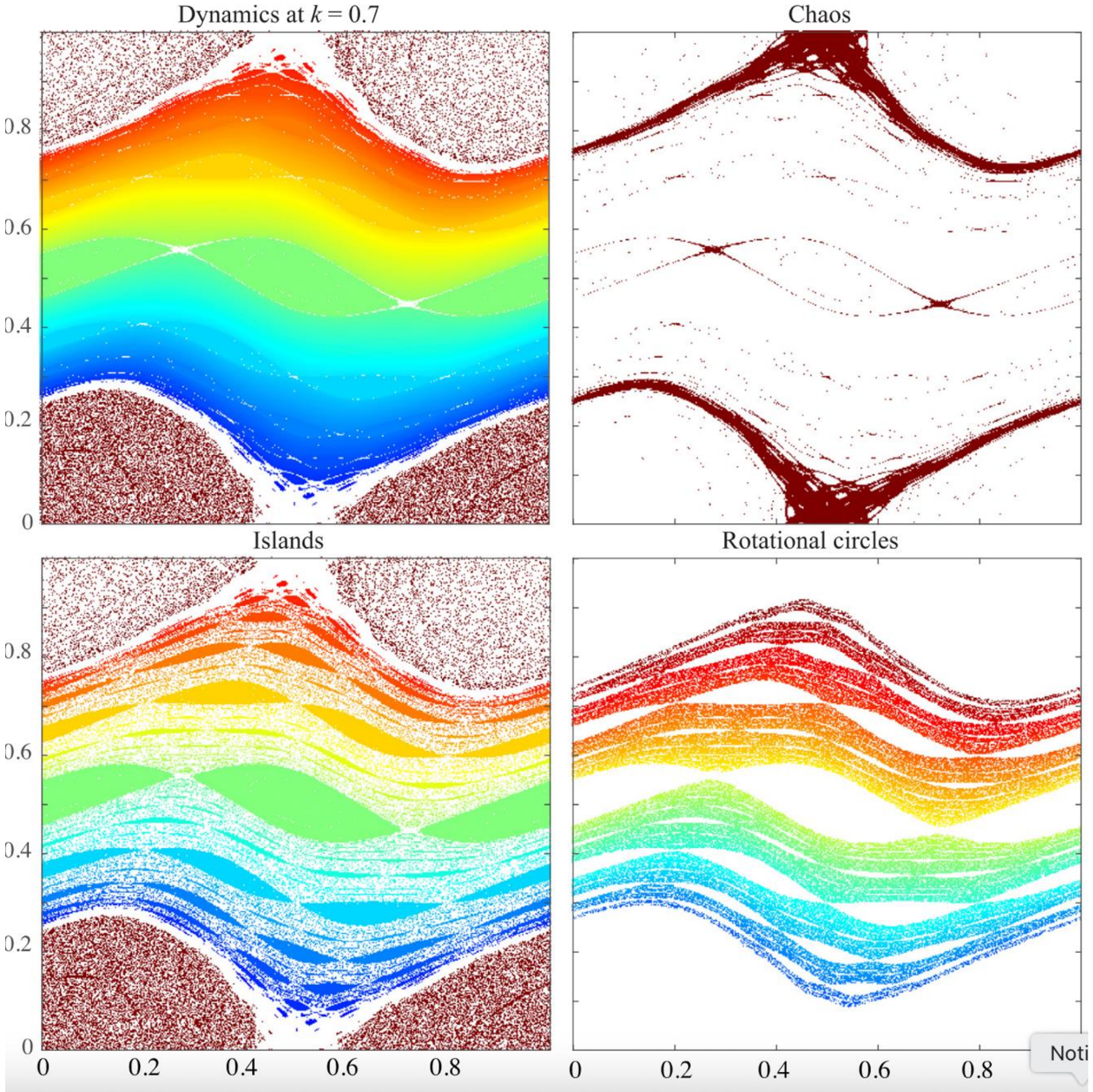
Inmaculada Baldomá[1, 5], Maciej J. Capiński[2], Mar Giralt[3] and Marcel Guardia[4, 5]

1. Departament de Matemàtiques & IMTECH, Universitat Politècnica de Catalunya, Diagonal 647, 08028



KAM theory

Moore Prize Winner 2018



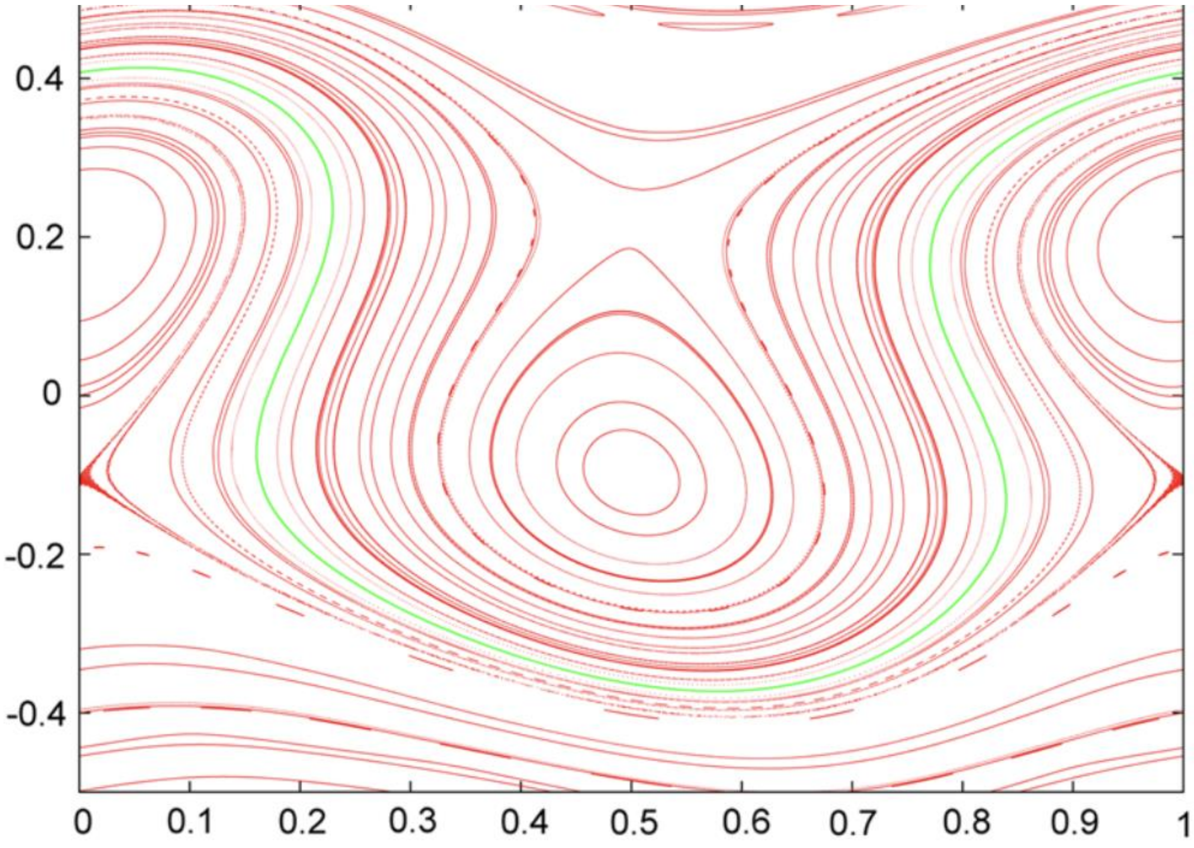
Found Comput Math (2017) 17:1123–1193
DOI 10.1007/s10208-016-9339-3

FOUNDATIONS OF COMPUTATIONAL MATHEMATICS
The Journal of the Society for the Foundations of Computational Mathematics

CrossMark

Rigorous Computer-Assisted Application of KAM Theory: A Modern Approach

J.-Ll. Figueras[1] · A. Haro[2] · A. Luque[3]



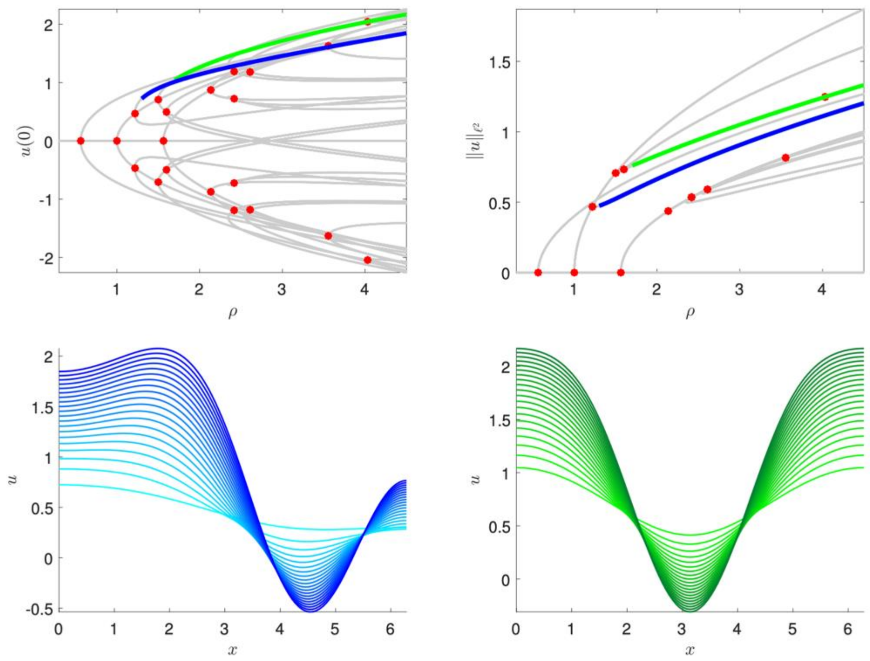
Continuation methods alternatives

SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 22, No. 2, pp. 765–801

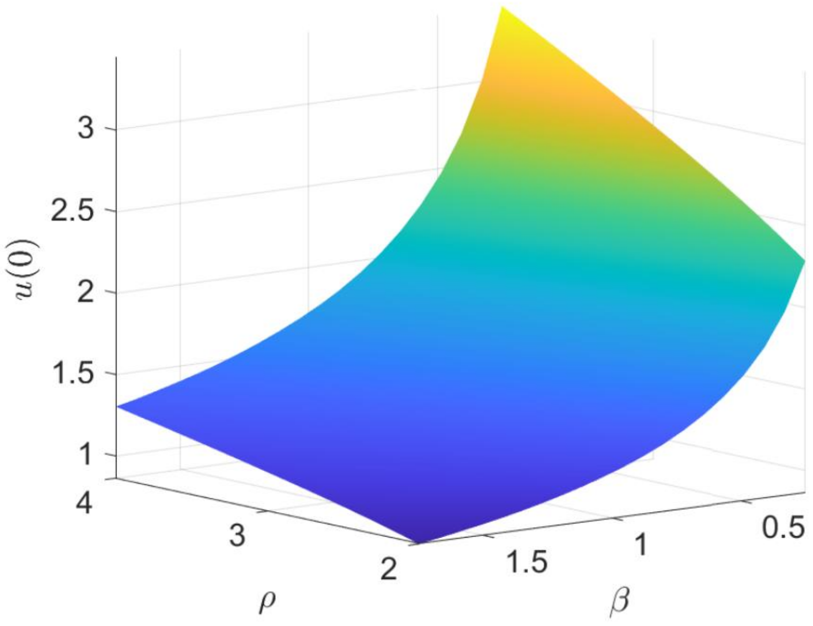
Red herring!

A Posteriori Validation of Generalized Polynomial Chaos Expansions*

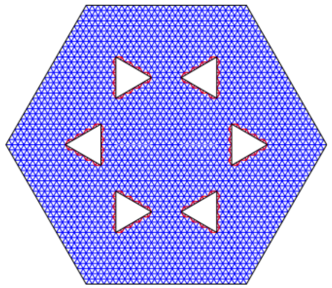
Maxime Breden†



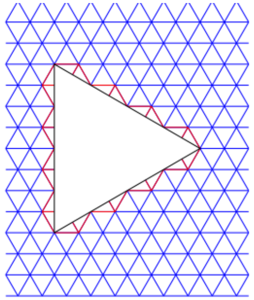
Using (Chebyshev) series for continuation in one and two parameters



Finite Element method



(a) Triangulation of the domain.



(b) Triangulation of the domain around the holes.

- The ‘finite element method’ offers another way to represent solutions, and bound error
  - Applicable to general domains
  - Less accurate / slower convergence

A counterexample to Payne’s nodal line conjecture with few holes

Joel Dahne[a], Javier Gómez-Serrano[b,c,*], Kimberly Hou[d]

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[c] Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes, 585, Barcelona, 08007, Spain
[d] Department of Mathematics, Princeton University, Fine Hall, Washington Rd, Princeton, NJ 08544, USA

ARTICLE INFO

Article history:
Available online 10 July 2021

Keywords:
Nodal line conjecture
Spectral theory
Computer-assisted proof

ABSTRACT

Payne conjectured in 1967 that the nodal line of the second Dirichlet eigenfunction must touch the boundary of the domain. In their 1997 breakthrough paper, Hoffmann-Ostenhof, Hoffmann-Ostenhof and Nadirashvili proved this to be false by constructing a counterexample in the plane with many holes and raised the question of the minimum number of holes a counterexample can have. In this paper we prove it is at most 6.

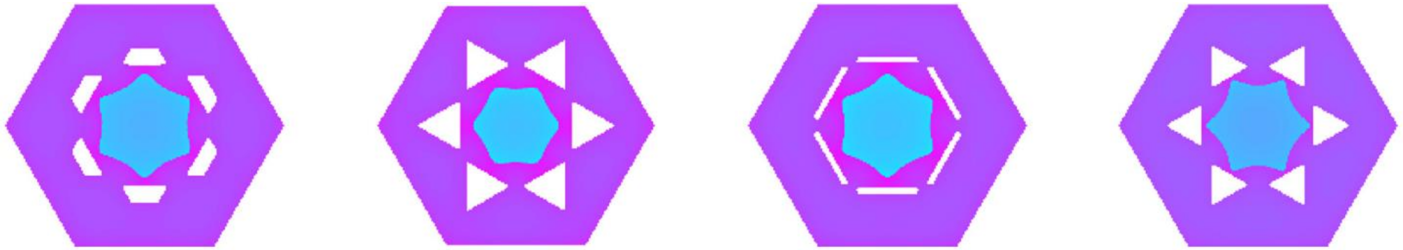


Fig. 2. Four domains for which the nodal line seems to be closed. Similarly to Fig. 1b we plot sign(u) log($|u|$). The rightmost one is the one we finally chose and the only one for which we have proved that the nodal line indeed is closed.

Blowup in PDEs

- Blow up solutions can often be renormalized through a suitable change of variables
- Study the dynamics of the new system

Forum of Mathematics, Pi (2025), **Vol. 13**:e6 1–139
doi:10.1017/fmp.2024.12

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Smooth imploding solutions for 3D compressible fluids

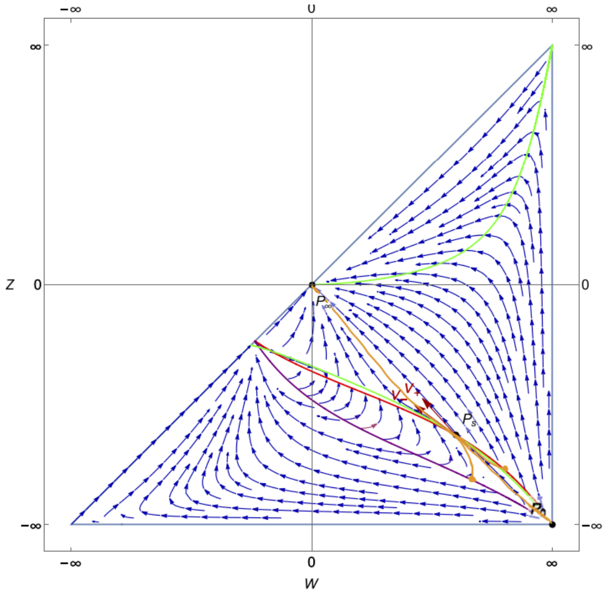
Tristan Buckmaster[1], Gonzalo Cao-Labora[2] and Javier Gómez-Serrano[3]

In this paper, we construct self-similar imploding solutions to the 3D isentropic compressible Euler equations

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= 0, \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \end{aligned} \tag{1.1}$$

$$\partial_\xi W = \frac{-(r + \frac{1}{2}((1+2\alpha)W + (1-\alpha)Z))W + \frac{\alpha}{2}Z^2}{1 + \frac{1}{2}(W + Z + \alpha(W - Z))}$$

$$\partial_\xi Z = \frac{-(r + \frac{1}{2}((1-\alpha)W + (1+2\alpha)Z))Z + \frac{\alpha}{2}W^2}{1 + \frac{1}{2}(W + Z - \alpha(W - Z))}$$



Machine Learning

- How to get approximate solution?
  - Asymptotic analysis, standard numerics
  - Machine learning, divination
- Neural Networks themselves can be viewed as a dynamical system
  - Exhibits periodic orbits, chaos

Asymptotic Self-Similar Blow-Up Profile for Three-Dimensional Axisymmetric Euler Equations Using Neural Networks

PDF

Share

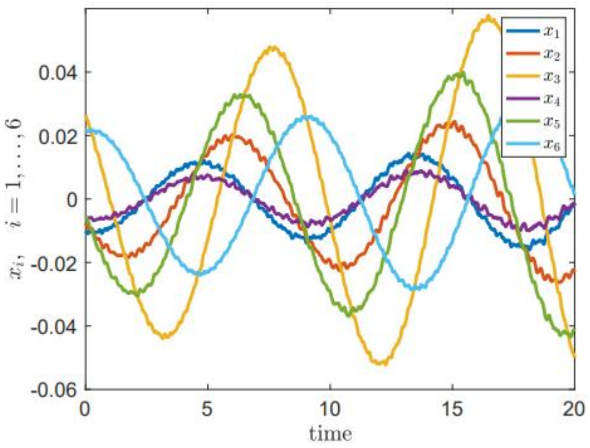
Y. Wang[1], C.-Y. Lai[1], J. Gómez-Serrano[2,3,4], and T. Buckmaster[5,6,*]

Computer Validation of Neural Network Dynamics: A First Case Study

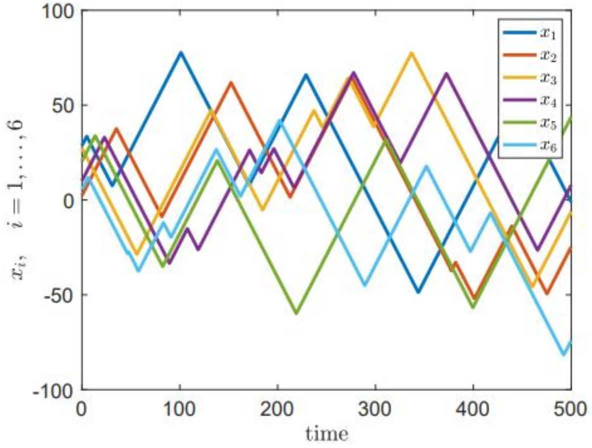
17 Pages • Posted: 22 Aug 2023

Elena Queirolo
Technische Universität München (TUM)

Christian Kuehn
Technische Universität München (TUM)

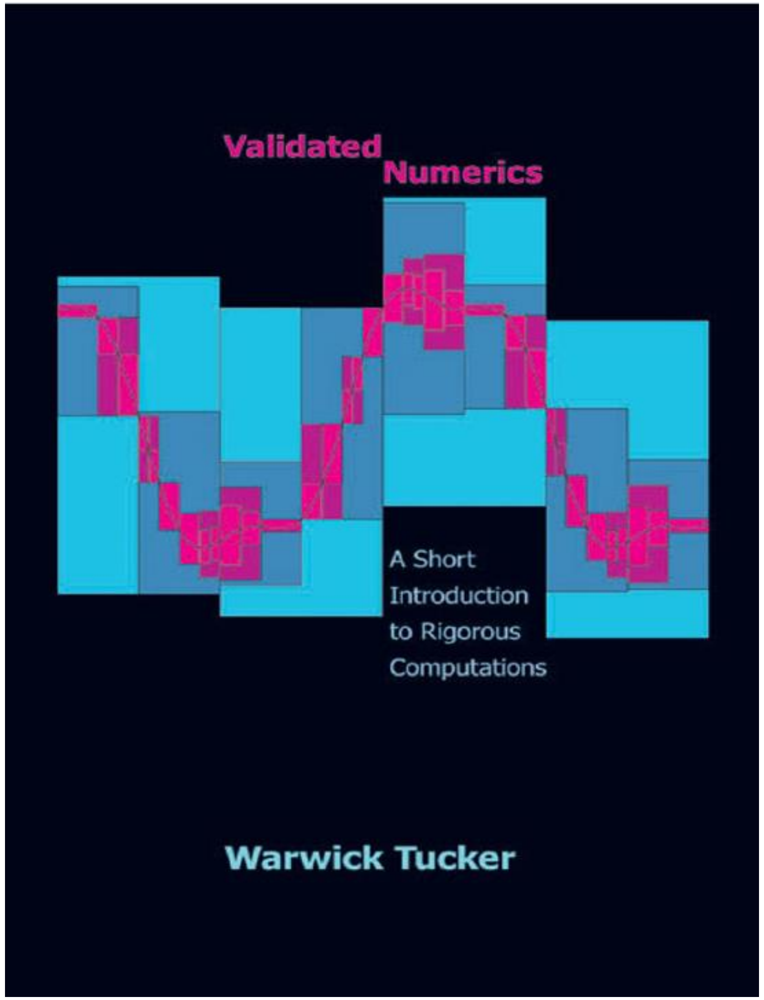


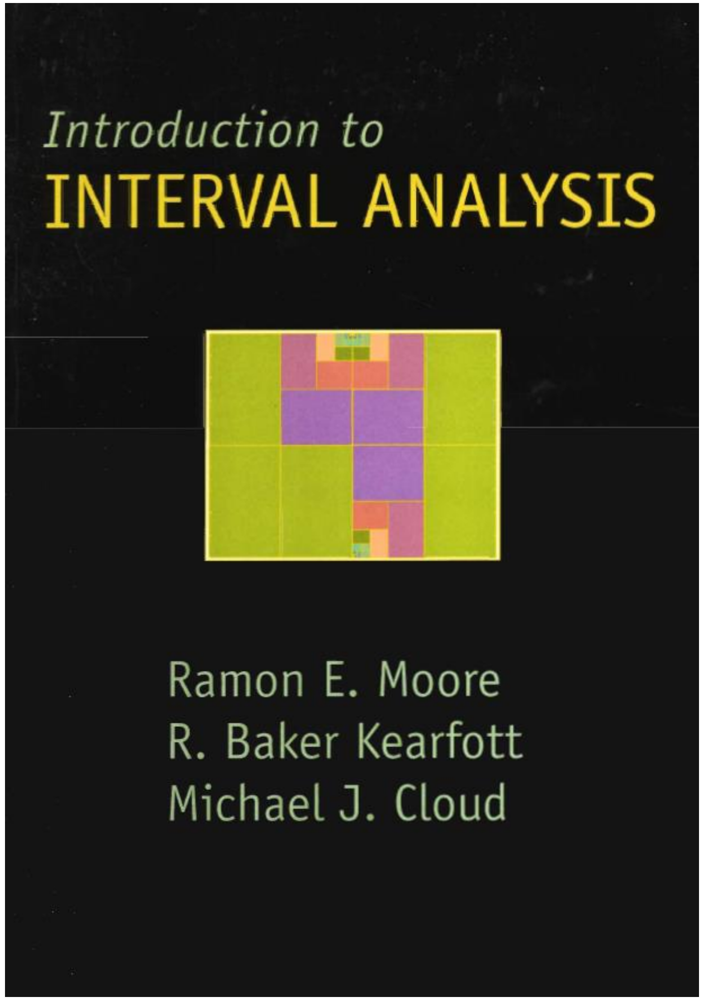
(e) Transient dynamics in 400 dimensions, $\gamma = 0.201686$

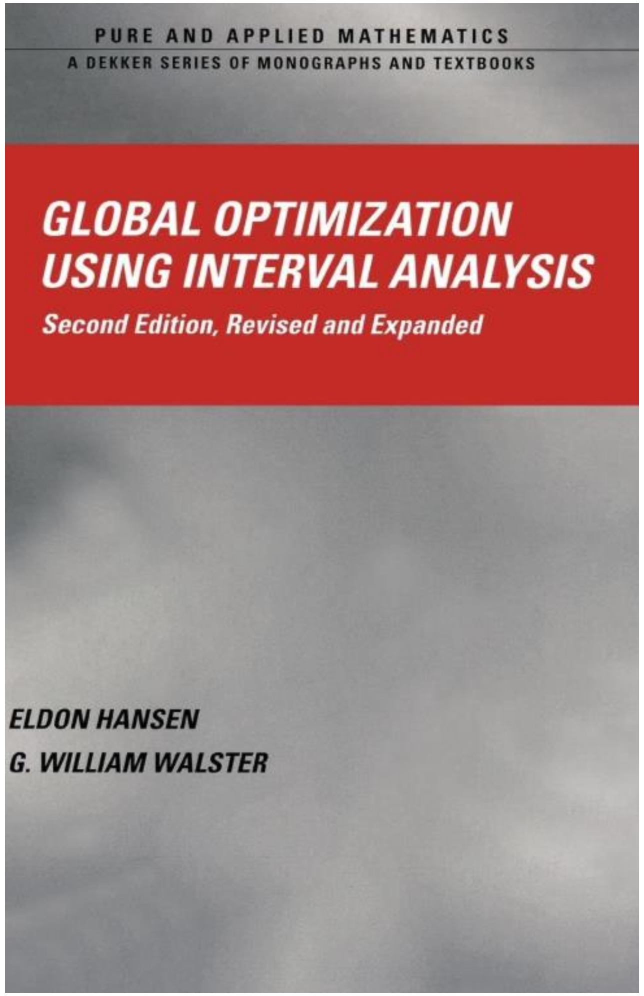


(f) Another orbit in 400 dimensions for the same parameter value

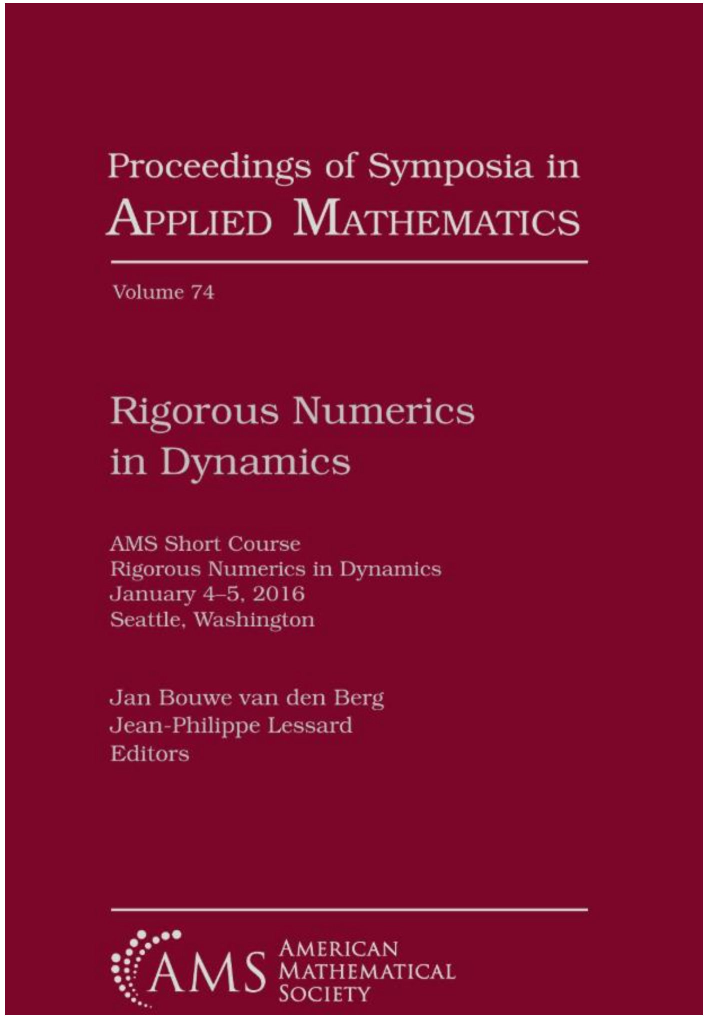
Fundamentals of Interval Arithmetic

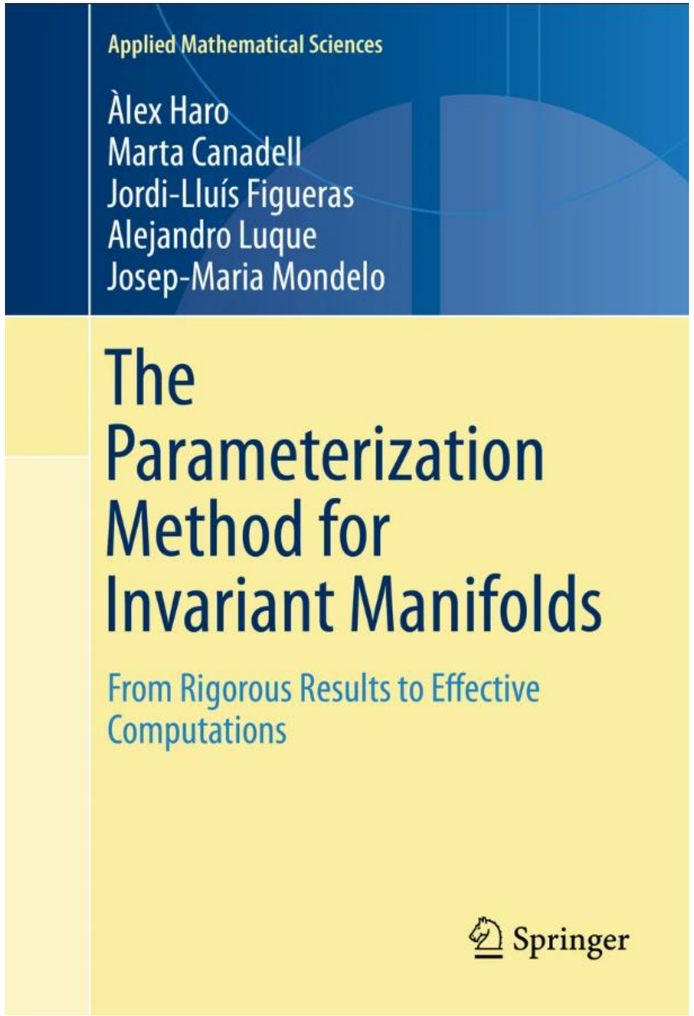


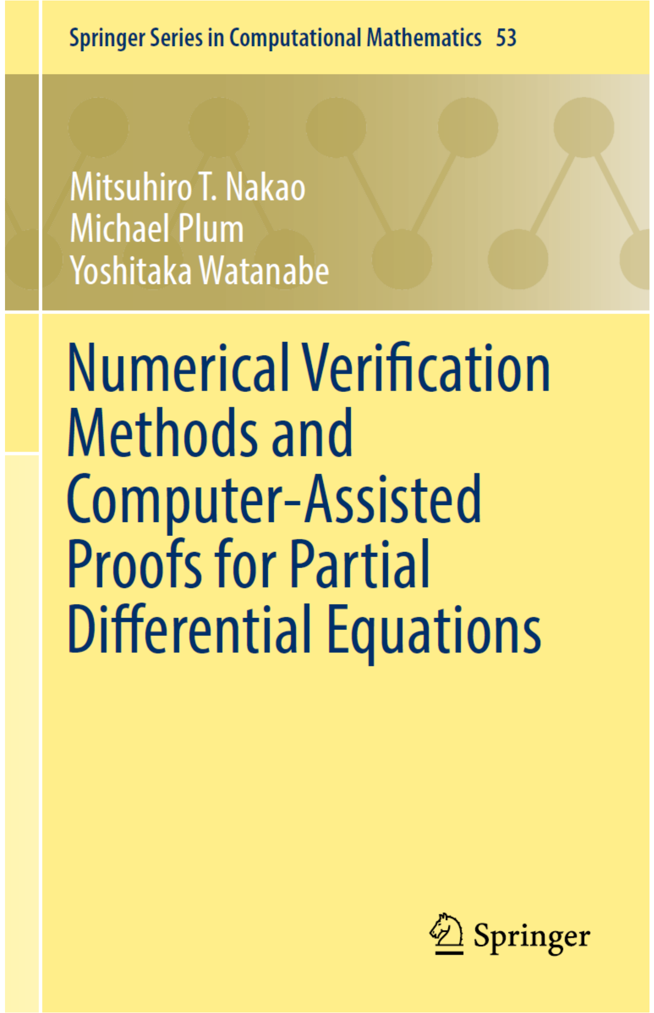




Applications to solving differential equations





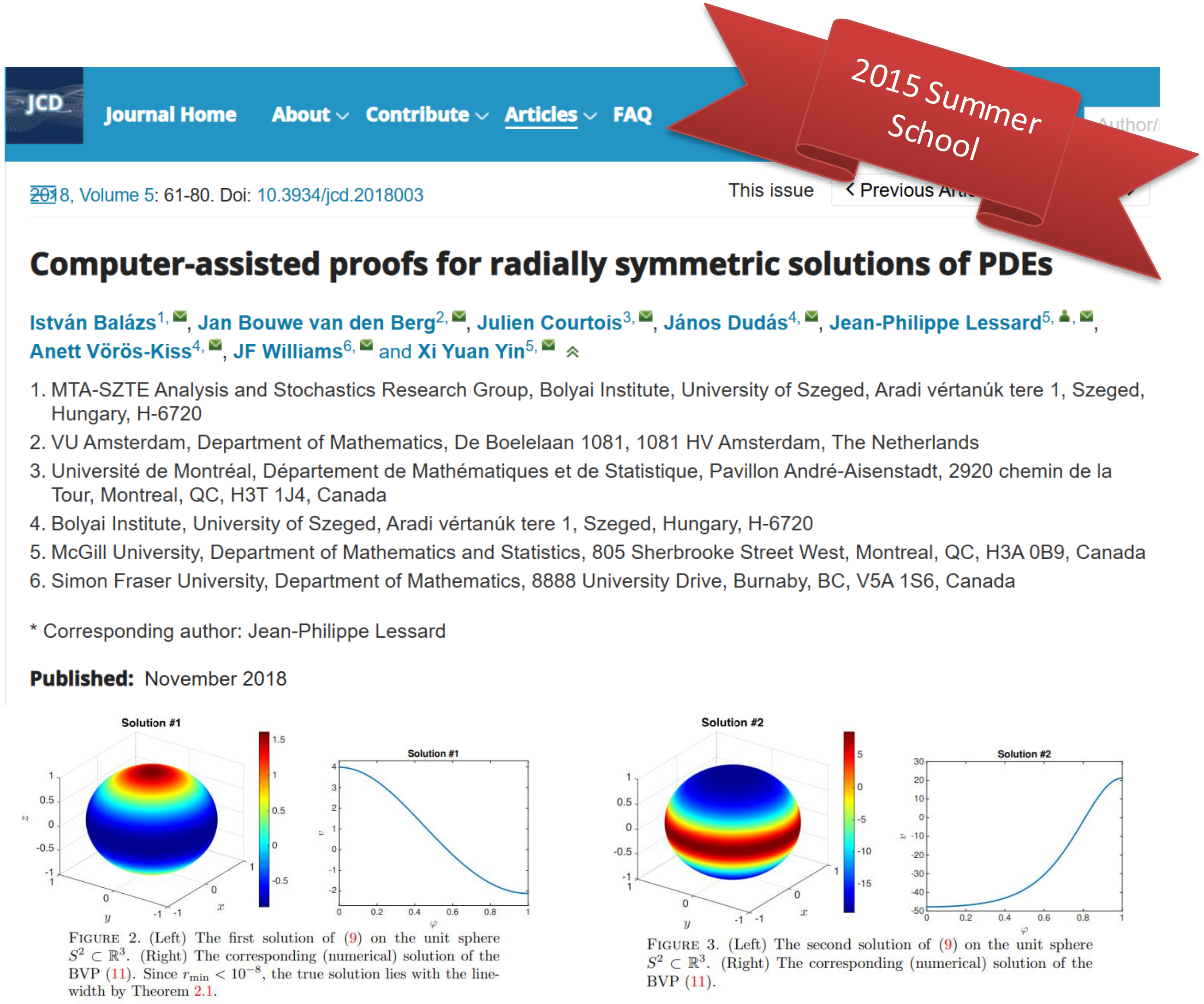


Resources Beyond

- See bibliography available on summer school website: https://www.slmath.org/summer-schools/1107

Project to Publication Pipeline

- Past summer school projects have led to publications
- Stages of writing a math paper
  - Formulating question
  - Proving results
  - Write up / frame in broader lit.
- How long does it take to publish a paper?
  - 1,3,5,∞ years
- Continuing your project, or not
  - Authorship / Acknowledgements



2018, Volume 5: 61-80. Doi: 10.3934/jcd.2018003

Computer-assisted proofs for radially symmetric solutions of PDEs

István Balázs[1], Jan Bouwe van den Berg[2], Julien Courtois[3], János Dudás[4], Jean-Philippe Lessard[5], Anett Vörös-Kiss[4], JF Williams[6] and Xi Yuan Yin[5]

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* Corresponding author: Jean-Philippe Lessard

Published: November 2018

FIGURE 2. (Left) The first solution of (9) on the unit sphere $S^2 \subset \mathbb{R}^3$. (Right) The corresponding (numerical) solution of the BVP (11). Since $r_{\min} < 10^{-8}$, the true solution lies with the line-width by Theorem 2.1.

FIGURE 3. (Left) The second solution of (9) on the unit sphere $S^2 \subset \mathbb{R}^3$. (Right) The corresponding (numerical) solution of the BVP (11).

Continuing Collaborations Interest Form

This form is to indicate to the organizers of the SL Math CAP Summer School whether you are interested in continuing to collaborate on your project beyond the dates of the summer school.

Organizing future collaborations.

Please indicate your commitment level. *

- ◯ I am definitely committed to a long-term collaboration with the goal of submitting a paper.
- ◯ I am committed to meeting in the next month or two to discuss further collaboration.
- ◯ I am interested, but I am unsure of my commitment level or time availability.
- ◯ I am interested in staying in touch, but I cannot commit to a long-term collaboration.

If you are committed to meeting in the future, would you be willing to organize/schedule future meetings?

- ◯ Yes. I could schedule meetings for future collaboration.
- ◯ Maybe. I am willing, but would prefer if someone else signed up for this job.
- ◯ No. This type of organization is not my strength.

One of two surveys
Please fill out SLMath survey later this morning

https://forms.gle/wDiRuU2cj55kGYmC9