Lecture Series A: Problem Set 1

1. Recall that for $I \in \binom{[n]}{k}$, $\mathcal{U}_I \subset \mathbb{P}^{\binom{n}{k}-1}$ denotes the subset where $p_I \neq 0$. Show that $\mathcal{U}_I \cap \mathrm{Gr}_{k,n} \cong \mathbb{C}^{k(n-k)}$.

2. Show that the Plücker relations hold. That is, for $I \in \binom{[n]}{k-1}$, $J \in \binom{[n]}{k+1}$, show that for any $V \in \mathrm{Gr}_{k,n}$,

$$\sum_{j \in J} \mathrm{sgn}(j, I, J)\ p_{I \cup \{j\}}(V)\ p_{J \setminus \{j\}}(V) = 0$$

where $\mathrm{sgn}(j, I, J) = (-1)^{|J \cap [j-1]| + |I \cap [j+1,n]|}$.

3. In lecture, we saw that for $\mathrm{Gr}_{2,4}$, there is exactly one Plücker relation, which has three terms. For arbitrary $\mathrm{Gr}_{k,n}$, how should you choose $I \in \binom{[n]}{k-1}$, $J \in \binom{[n]}{k+1}$ so that in the corresponding Plücker relation, only three terms are nonzero?

4. (Supplemental) In the morning lectures, we'll discuss the combinatorics of two different bases for $\mathbb{C}[\mathrm{Gr}_{k,n}]$. In this exercise, you'll investigate a third, called the *standard monomial basis*, in the case $k = 2$.

For $I = \{i_1 < i_2\}, J = \{j_1 < j_2\} \subset [n]$, we write $I \leq J$ if $i_1 \leq j_1$ and $i_2 \leq j_2$. We call a collection $\{I_b\}_{b=1}^r \subset \binom{[n]}{2}$ *standard* if $I_1 \leq I_2 \leq \cdots \leq I_r$. In other words, $\{I_b\}_{b=1}^r$ is standard if the tableau of shape $2 \times r$ whose bth column is I_b is a semi-standard Young tableau. A monomial $\prod_{b=1}^r p_{I_b}$ in Plücker coordinates is *standard* if the collection of index sets $\{I_b\}_{b=1}^r$ is standard.

a.) Show that any non-standard monomial in $\mathbb{C}[\mathrm{Gr}_{2,n}]$ can be written as a sum of standard monomials.

b.) Show that the standard monomials are linearly independent in $\mathbb{C}[\mathrm{Gr}_{2,n}]$, and hence form an additive basis.

c.) Show the standard monomial basis has integral structure constants but does not have positive structure constants. That is, show that if you multiply two standard monomials and write the result as a linear combination of standard monomials, the coefficients are integers but are not always positive.