Afternoon Exercises
Monday, June 23

(1) Let's start by thinking about how we can represent complete flags as matrices.
  (a) Let $n = 3$. Find a matrix that represents each of these flags.
    (i) $\langle e_1 \rangle \subseteq \langle e_1, e_2 + e_3 \rangle \subseteq \langle e_1, e_2 + e_3, e_1 + e_3 \rangle$
    (ii) $\langle e_1 + e_2 + e_3 \rangle \subseteq \langle e_1, e_2 + e_3 \rangle \subseteq \langle e_1, e_2, e_3 \rangle$
  (b) Explain why every complete flag has some matrix that represents it.
  (c) Let $M, N \in \mathrm{GL}(n)$. Show that M and N represent the same complete flag if and only if there exists some invertible, lower triangular matrix $b \in B_-$ so that $M = bN$.
  (d) Use the previous two parts to conclude that $\mathrm{FL}(n) = B_- \backslash \mathrm{GL}(n)$.

(2) Now let's think about Schubert cells.
  (a) Pick a "random" invertible 3×3 matrix and let $F.$ be the flag it represents. Try to have it not be "special" at all. What Schubert cell is $F.$ in? Try another random invertible matrix. What Schubert cell do you get?
  (b) Explain using column operations on matrices how by starting from the permutation matrix for w, you can get to the special representative of any flag in X_w^o by acting by an invertible, upper triangular matrix $b \in B_+$.
  (Recall, the left action is by matrix multiplication on the right!)
  (c) Use the previous part to conclude that $X_w^o = B_+ \cdot F_w$, where F_w is the permutation flag associated to w.

(3) Recall, given $w \in S_n$,
$$\ell(w) = \{1 \leq i < j \leq n : w(i) > w(j)\}.$$
The **Rothe diagram** of $w \in S_n$ is the picture you get by plotting dots in an $n \times n$ grid at positions $(i, w(i))$ for all $i \in [n]$, and then striking out the cells to the right and below each dot. Write $D(w)$ for the cells that are not struck out.
  (a) Explain why $\ell(w) = \#D(w)$.
  (b) Let $w_0 = n\, n-1 \ldots 1$ be the longest permutation in S_n. How does $\#D(w)$ relate to $\#D(w_0 w)$? Hint: left multiplication by w_0 reverses the columns of the permutation matrix.
  (c) Use these observations to explain why X_w^o has dimension $\binom{n}{2} - \ell(w)$.

(4) Recall that the (strong) Bruhat order on S_n is defined by covering relations of the form $w < wt_{i,j}$ with $\ell(w) + 1 = \ell(wt_{i,j})$.
  (a) If $w < wt_{i,j}$, explain how you can find the permutation flag for $wt_{i,j}$ as the limit of a sequence of flags in X_w^o.
  (b) Show that if $w \leq v$ in Bruhat order, then $r_w(i,j) \geq r_v(i,j)$ for all $i, j \in [n]$. (The converse holds as well!)

(5) **Extra!** There's a whole theory of **opposite** Schubert varieties. They are very useful when computing in cohomology. Let $Y_w^0 = B_- \cdot F_w$ be the **opposite** Schubert cell, and $Y_w = \overline{Y_w^0}$ the **opposite** Schubert variety.
  (a) Describe a way to put matrix representatives for points in Y_w^0 in a "special form" that is similar to what we did for X_w^0. What are you doing at the level of matrices?
  (b) Describe what Y_w is in terms of the Bruhat order and opposite Schubert cells.
  (c) What happens when you take $X_w \cap Y_w$?
  (d) What does the previous part tell you about $[X_w] \cdot [Y_w] \in H^*(\mathrm{FL}(n))$?
  (e) Express the class of $[Y_w]$ in terms of Schubert classes. Hint: Think about acting on Y_w by a special element of $\mathrm{GL}(n)$.