Lecture Series A: Problem Set 3

1. Let Σ be a seed and k, ℓ mutable indices. Prove the following statements.

a.) $\mu_k(\mu_k(\Sigma)) = \Sigma$.

b.) If there are no arrows between k, ℓ, then $\mu_k(\mu_\ell(\Sigma)) = \mu_\ell(\mu_k(\Sigma))$.

2. If Q is a quiver, let Q^{op} denote the quiver obtained by reversing all arrows. Show that $\mathcal{A}(\mathbf{x}, Q) = \mathcal{A}(\mathbf{x}, Q^{op})$.

3. We saw in lecture that $\mathbb{C}[\mathrm{Gr}_{2,n}]$ is a cluster algebra whose seeds are in bijection with triangulations of an n-gon. We also saw that a cluster algebra has finitely many seeds if and only if in some seed, the mutable part of the quiver is an orientation of a type ADE Dynkin diagram. Given an orientation $\vec{P}$ of a path P (= type A Dynkin diagram), find a triangulation T so that the mutable part of Q_T is $\vec{P}$.

4. Let τ be a labeled tree. Prove that any two orientations of τ are mutation-equivalent to each other. (It is also true that if orientations of trees τ, τ' are mutation-equivalent then $\tau = \tau'$. But the only known proof uses quiver representations.)

5. (Supplemental) Cluster algebras were motivated by *total positivity*; in this exercise, we explore total positivity for $\mathrm{Gr}_{2,n}$.

A point $V \in \mathrm{Gr}_{2,n}$ is *positive* (resp. nonnegative) if for some representative matrix all of its Plücker coordinates are positive (resp. nonnegative) real numbers.

a) Let $\mathbf{x}_T$ be a cluster for $\mathbb{C}[\mathrm{Gr}_{2,n}]$. Show that $V \in \mathrm{Gr}_{2,n}$ is positive if and only if $p_I(V) > 0$ for all $p_I \in \mathbf{x}_T$. Use this to show that

$$\{V \in \mathrm{Gr}_{2,n} : V \text{ totally positive}\}$$

is homeomorphic to an open ball.

b) Consider $V = [A] \in \mathrm{Gr}_{2,n}$ where A is a matrix with real entries. Consider the columns of A as vectors $A_1, \dots, A_n$ in the plane $\mathbb{R}^2$. What does this configuration of vectors look like if V is positive? Nonnegative?

c) The nonnegative points of $\mathrm{Gr}_{2,n}$ can be divided into strata according to which Plücker coordinates are positive and which are zero. That is, for $\mathcal{M} \subset \binom{[n]}{2}$, we can define the stratum

$$S_{\mathcal{M}} := \{V \in \mathrm{Gr}_{2,n} : p_I(V) > 0 \text{ if } I \in \mathcal{M},\ p_I(V) = 0 \text{ if } I \notin \mathcal{M}\}.$$

Characterize the collections $\mathcal{M}$ for which the stratum $S_{\mathcal{M}}$ is nonempty. Hint: use b) to give some set of combinatorial objects which is in bijection with nonempty strata (it might not be very pretty). Then explain how to obtain $\mathcal{M}$ from this object.

d) Part a) showed that every cluster $\mathbf{x}_T$ is a *membership test* for the stratum $S_{\binom{[n]}{2}}$.

Let $S_{\mathcal{M}}$ be a stratum where exactly one Plücker coordinate is zero. Find a sub-cluster which is a membership test