Afternoon Exercises
Thursday, June 26

(1) (a) Compute the Schur polynomial $s_{(2,1)}(x_1, x_2, x_3)$ using the definition in terms of semistandard Young tableaux. List all of the tableaux and their monomials.
(b) Compute the determinant:

$$\frac{\det\left(x_i^{\lambda_j+k-j}\right)}{\det\left(x_i^{k-j}\right)}$$

for $\lambda = (2, 1, 0)$. Verify that this gives the Schur polynomial $s_{(2,1)}(x_1, x_2, x_3)$.

(2) (a) Find the value of $c^{\nu}_{\lambda,\mu}$ when $\lambda = (2, 1, 1)$, $\mu = (2, 2)$, and $\nu = (3, 3, 2)$.
(b) Find the value of $c^{\nu}_{\lambda,\mu}$ when $\lambda = (2, 1)$, $\mu = (2, 1, 1, 1)$, and $\nu = (3, 3, 2)$.
(c) Use the Littlewood-Richardson rule to expand $\left(s_{(2,1)}(x_1, x_2, x_3)\right)^2$ in the Schur basis.
(d) Explain how the Pieri rule follows as a special case of the Littlewood-Richardson rule.

(3) Consider the Vandermonde determinant.

$$\det \begin{bmatrix} x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1} \\ x_1^{k-2} & x_2^{k-2} & \cdots & x_k^{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

(a) Explain why $x_i - x_j$ divides the determinant whenever $i < j$.
Hint: What happens when you swap columns i and j in the matrix?
(b) Use the previous part to show that the Vandermonde determinant is equal to

$$\prod_{1 \le i < j \le n} (x_i - x_j).$$

(c) Explain using the bialternate formula for Schur polynomials why they are symmetric.

(4) When λ is a horizontal or a vertical strip, the Schur polynomial s_λ has a particularly nice form.
(a) Describe the monomials that appear in s_λ when $\lambda = (a)$.
(b) Describe the monomials that appear in s_λ when $\lambda = 1^b = (1, 1, \ldots, 1)$, i.e., it's a tuple so that 1 appears b times.
(c) Define $h_a = s_{(a)}$. We call h_a a **complete homogeneous symmetric polynomial**. The **Jacobi-Trudi** formula says that we can express s_λ as the determinant of a matrix whose entries are h_i's. Explicitly,

$$s_\lambda = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & \cdots & h_{\lambda_k} \end{vmatrix}$$

Verify the formula works for $s_{21}(x_1, x_2)$.
(d) Explain how, in principle, knowing the Jacobi-Trudi formula and the Pieri rule allows you to expand the product of any two Schur polynomials in the Schur basis (without using the full Littlewood-Richardson rule). Demonstrate your argument with the example $s_{21}(x_1, x_2)$.
(e) Check your answer by using the Littlewood-Richardson rule. Which method seems nicer?

(5) Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition with all distinct parts, that is $\lambda_1 > \lambda_2 > \cdots > \lambda_k$. Use the bialternate formula for Schur polynomials to prove the rule for multiplying $s_\lambda(x_1, \ldots, x_k) \cdot s_\square(x_1, \ldots, x_k)$.