Lecture Series A: Problem Set 7

1. When we defined the plabic seed Σ_G, we labeled each face using the *sources* of trips. One could alternately label each face using the *targets* of trips; call this seed Σ_G^T. What is the relationship between Σ_G and Σ_G^T? What is the relationship between $\mathcal{A}(\Sigma_G)$ and $\mathcal{A}(\Sigma_G^T)$?

2. For $J = \{j_1, \ldots, j_k\} \in \binom{[n]}{k}$, let $J + 1 := \{j_1 + 1, \ldots, j_k + 1\}$, where addition is modulo n. Show that the map

$$\alpha : \mathbb{C}[\mathrm{Gr}_{k,n}] \to \mathbb{C}[\mathrm{Gr}_{k,n}]$$
$$p_J \mapsto p_{J+1}$$

is a *cluster automorphism*, meaning that it is a ring automorphism which sends cluster variables to cluster variables and clusters to clusters.
Bonus: Describe an automorphism $\psi : \mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k,n}$ so that $p_J \circ \psi = \alpha(p_J)$. Be careful with signs!

3. Show that if G is a $\mathrm{Gr}_{k,n}$ graph, the face labels of G are weakly separated.

4. Let $\mathcal{C} \subset \binom{[n]}{k}$ be a maximal-by-inclusion weakly separated collection and let $\mathcal{T}$ be the corresponding plabic tiling. Let G be the bicolored graph dual to $\mathcal{T}$. Show that G is a $\mathrm{Gr}_{k,n}$-graph, that is, has trip permutation $\pi_{k,n}$ and is reduced.
Hint: the faces of G have inherited some labels from $\mathcal{T}$. If these labels have any hope of being the actual face labels of G, which edges must the trip with source a use? Consider that subset of edges, and show that it is in fact a trip with source a and target $\pi_{k,n}(a)$. Then show that the trips satisfy the criteria for reducedness.