Lecture 2: First examples of discriminants (Simon)

The most famous example of a discriminant comes from the quadratic equation $a\,x^2 + b\,x + c = 0$ with $a \neq 0, b, c \in \mathbb{C}$. There are two solutions $x \in \mathbb{C}$, unless the coefficients satisfy $\Delta = b^2 - 4\,ac = 0$.

Exercise 1. *Prove the above statement. Show that the quadratic equation has two real solutions if $\Delta > 0$ and no real solutions if $\Delta < 0$. In this sense, the discriminant* discriminates *between these qualitatively different behaviors over the real numbers.*

In general, and as a slogan, a discriminant characterizes non-generic behavior of the solution to a parametric problem. In our context, the space of parameters is an irreducible algebraic variety Z, and a property is said to hold *generically* if it holds for all parameter values $z \in Z$ not contained in a closed subvariety $\nabla \subsetneq Z$. That closed subvariety ∇ is called the *discriminant variety*. The precise definition of ∇ depends on the specific context. Here are some examples.

Example 1. For the quadratic equation $f = a\,x^2 + b\,x + c = 0$, the space of parameters is $Z = \mathbb{C}^3 \setminus \{a = 0\}$, with coordinates a, b, c. The property "f has two solutions" holds generically: it fails to hold when $\Delta = b^2 - 4ac = 0$. The equation $\Delta = 0$ defines a quadratic (toric) surface in Z. That surface is the discriminant variety ∇.

Example 2. We now present a more "grown-up" version of Example 1, which is more in line with our general set-up and notation. The binary quadric $f = z_1\,x_0^2 + z_2\,x_0x_1 + z_3\,x_1^2$ has two roots on the projective line $\mathbb{P}^1$, unless $\Delta = z_2^2 - 4\,z_1z_3 = 0$. Since Δ is a homogeneous polynomial, it is natural to think of its zero locus ∇ as a curve in $\mathbb{P}^2$ instead. This mirrors the fact that scaling the coefficients z_1, z_2, z_3 (previously a, b, c) leaves the roots of f unchanged.

The tools from elimination theory which you learned in the first lecture often come in handy for computing the defining equation(s) of a discriminant variety. We illustrate this by extending Example 2 to binary cubics $z_1x_0^3 + z_2x_0^2x_1 + z_3x_0x_1^2 + z_4x_1^3$. Consider the variety in $\mathbb{P}^3 \times \mathbb{P}^1$ defined by

$$3\,z_1\,x_0^2 + 2\,z_2\,x_0x_1 + z_3\,x_1^2 \;=\; z_2\,x_0^2 + 2z_3\,x_0x_1 + 3z_4\,x_1^2 \;=\; 0. \tag{1}$$

Its projection to $\mathbb{P}^3$ consists of all coefficients $z = (z_1 : z_2 : z_3 : z_4)$ for which our binary cubic has a double root. The algebraic counterpart of "projecting to $\mathbb{P}^3$" is "eliminating x_0 and x_1". Doing this naively, i.e., eliminating x_0, x_1 from the ideal generated by our two equations, we obtain the wrong answer. Indeed, the restriction of the projection $\mathbb{C}^4 \times \mathbb{C}^2 \to \mathbb{C}^4$ is dominant. Each fiber contains $(0, 0)$, which has no corresponding point in $\mathbb{P}^1$. First, we must saturate the ideal generated by our two equations by the irrelevant ideal $\langle x_0, x_1 \rangle$ of $\mathbb{P}^1$. After that, eliminating gives

$$\Delta \;=\; z_2^2z_3^2 - 4\,z_1z_3^3 - 4\,z_2^3z_4 + 18\,z_1z_2z_3z_4 - 27\,z_1^2z_4^2. \tag{2}$$

Example 3. Discriminants are not necessarily irreducible. The binary quadric f from Example 2 generically has two roots in $\mathbb{P}^1$, both of whose projective coordinates x_0, x_1 are nonzero. That is, generically f has two roots in the algebraic torus $\mathbb{C}^* \subset \mathbb{P}^1$. This fails when $\Delta = z_1(z_2^2 - 4\,z_1z_3)z_3 = 0$.

Example 4. Consider an $n \times n$ matrix A whose entries are n^2 projective coordinates on $Z = \mathbb{P}^{n^2-1}$. We denote these entries by $z_{11}, z_{12}, \ldots, z_{nn}$. The property "the linear system $A\,x = 0$ has no solutions $x \in \mathbb{P}^{n-1}$" holds generically. It fails if and only if $z \in Z$ lies on the hypersurface $\nabla \subset Z$ defined by $\Delta = \det A = 0$. The discriminant has degree n and its equation Δ is multilinear.

Exercise 2. *In this exercise you rediscover the determinant using the tools you learned in Lecture 1. Consider the variety given by the* n *equations* $Ax = 0$ *in* n^2+n *variables* $z_{11}, z_{12}, \ldots, z_{nn}, x_1, \ldots, x_n$*. This lives naturally in* $\mathbb{P}^{n^2-1} \times \mathbb{P}^{n-1}$*. Show that the image of the projection of this variety to* $\mathbb{P}^{n^2-1}$ *is precisely* ∇ *from Example 4. For* $n = 3$*, compute the principal elimination ideal using a computer algebra system, and observe that it is generated by the determinant.*

Example 5. *Resultants* generalize determinants to nonlinear equations. Consider two quadratic equations $z_1 x_0^2 + z_2 x_0x_1 + z_3 x_1^2 = z_4 x_0^2 + z_5 x_0x_1 + z_6 x_1^2 = 0$. Generically, these equations have no solutions. There is a solution if and only if the *resultant polynomial* vanishes:

$$\Delta = \det \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 & 0 \\ 0 & z_4 & z_5 & z_6 \end{pmatrix} = 0.$$

This defines a hypersurface $\nabla \subset Z = \mathbb{P}^5$. The "only if" direction is an easy exercise.

Example 6. The discriminant (2) is the resultant of (1). It is given by the following determinant:

$$\det \begin{pmatrix} 3\,z_1 & 2\,z_2 & z_3 & 0 \\ 0 & 3\,z_1 & 2\,z_2 & z_3 \\ z_2 & 2\,z_3 & 3\,z_4 & 0 \\ 0 & z_2 & 2\,z_3 & 3\,z_4 \end{pmatrix}.$$

Example 7. We now consider *three* quadratic equations

$$z_1 x_0^2 + z_2 x_0x_1 + z_3 x_1^2 = z_4 x_0^2 + z_5 x_0x_1 + z_6 x_1^2 = z_7 x_0^2 + z_8 x_0x_1 + z_9 x_1^2 = 0$$

These have no solutions, unless the following four determinants vanish:

$$\det \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 & 0 \\ 0 & z_4 & z_5 & z_6 \end{pmatrix}, \det \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \\ z_7 & z_8 & z_9 & 0 \\ 0 & z_7 & z_8 & z_9 \end{pmatrix}, \det \begin{pmatrix} z_4 & z_5 & z_6 & 0 \\ 0 & z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 & 0 \\ 0 & z_7 & z_8 & z_9 \end{pmatrix}, \det \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{pmatrix}.$$

(Exercise: prove this.) These equations define an irreducible discriminant variety ∇ of dimension 6 in $Z = \mathbb{P}^8$. Its radical ideal has seven generators, one of degree three and six of degree four.

Example 8. We consider a general ternary quadric with coefficients $z \in \mathbb{P}^5$:

$$f = z_1 x_0^2 + z_2 x_0x_1 + z_3 x_0x_2 + z_4 x_1^2 + z_5 x_1x_2 + z_6 x_2^2.$$

The conic $V(f) \subset \mathbb{P}^2$ is generically smooth. This fails precisely when

$$\Delta = \det \begin{pmatrix} 2z_1 & z_2 & z_3 \\ z_2 & 2z_4 & z_5 \\ z_3 & z_5 & 2z_6 \end{pmatrix} = 0.$$

The discriminant variety is a cubic four-fold in $Z = \mathbb{P}^5$. The proof is an exercise.

Example 9. Consider the irreducible hypersurface in $\mathbb{P}^{n^2-1}$ defined by $\det A = 0$, where A is a matrix with entries z_{ij}. In Example 4, we identified this as a discriminant. Here we use this singular hypersurface as our parameter space Z. For a generic point $z \in Z$ the matrix A has rank $n-1$. The discriminant is the variety $\nabla \subset Z$ of points for which A has rank $\leq n-2$. This coincides with the singular locus of ∇, and is defined by the vanishing of the $(n-1)$-minors of A. For $n=3$, ∇ is the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ (and hence toric). What is the dimension of ∇ for $n=4$?

Example 10. Lines in $\mathbb{P}^3$ are parametrized by the Grassmannian $Z = \mathrm{Gr}(2,4)$. The twisted cubic curve C in $\mathbb{P}^3$ is the (toric) curve parameterized by $t \mapsto (1 : t : t^2 : t^3)$. A generic line $L \subset \mathbb{P}^3$ does not intersect C. The discriminant is a hypersurface $\nabla \subset Z$ called the *Chow hypersurface* of C. It consists of all lines in $\mathbb{P}^3$ for which $L \cap C \neq \emptyset$. Exercise: find its equation in Plücker coordinates. Substituting p_{ij} by the (i,j) minor of

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

in the defining equation of the Chow form, we obtain the resultant of two binary cubics $a_0x_0^3 + a_1x_0^2x_1 + a_2x_0x_1^2 + a_3x_1^3, b_0x_0^3 + b_1x_0^2x_1 + b_2x_0x_1^2 + b_3x_1^3$. This is a 6×6 Sylvester determinant.